SLAC – PUB – 3572
February 1985
(T/E/A)

QUANTUM EFFECTS IN LINEAR COLLIDER SCALING LAWS*

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ABSTRACT

Compared to classical calculations, quantum corrections greatly reduce the radiation emitted when the e^+ and e^- beams collide in a linear collider. This allows a given luminosity to be obtained with much lower beam powers by making the beam size smaller.

INTRODUCTION

In the design of high energy, high luminosity e^+e^- linear colliders there are several constraints which must be satisfied by the beam parameters[1] . These beam parameters and their nominal values at SLC are:

$2\sigma_r$	:	the radius of the beam (in this paper we will assume uniform cylindrical beams)	$(2 \times 1.8\ \mu)$
$2\sqrt{3}\sigma_z$	:	the length of the bunch (The numerical factors in the definitions of σ_z and σ_r are a convention to help make formulas for cylindrical beams similar to those for Gaussian ones.)	$(2\sqrt{3} \times 1\ \text{mm})$
N	:	the number of particles per bunch	(5×10^{10})
f	:	the repetition rate of the machine	(180 Hz)
γ	:	E/mc^2 (where E is the beam energy).	(1×10^5)

From these five basic beam parameters, four other parameters can be derived. There are constraints on these four parameters placed by the needs of High Energy Physics and by beam dynamics. The derived parameters are

- The total beam power

$$P = fN\gamma mc^2 \quad (= 74 \text{ KW at SLC}) \tag{1}$$

has no strict constraints on it but small beam powers are preferred as the total AC power consumed by the accelerator and therefore its cost scales with P.

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

Contributed to the 2nd International Workshop on Laser Acceleration of Particles, Los Angeles, California, January 7–18, 1985.

- The luminosity

$$\mathcal{L} = \frac{N^2 f}{4\pi\sigma_r^2} \quad (= 6 \times 10^{30}\text{cm}^{-2}\text{sec}^{-1} \text{ at SLC}^{[2]}) \tag{2}$$

must increase with the energy of the beams in order to keep a constant event rate. Since the interaction cross section falls as $1/\gamma^2$ we must have $\mathcal{L} = \mathcal{L}_0 \frac{\gamma^2}{\gamma_0^2}$ where e.g., $\frac{\mathcal{L}_0}{\gamma_0^2} = \frac{10^{30}}{(10^5)^2}$.

- The disruption parameter

$$D = \frac{N r_e \sigma_z}{2\gamma\sigma_r^2} \tag{3}$$

is related to how many oscillations an electron will go through as it passes through the magnetic field caused by the other beam. If it is too large there is an instability which will increase the beam size and reduce the luminosity. One must require $D < 20$.

- The beamsstrahlung parameter

$$\delta_{classical} = \frac{r_e^3}{3\sqrt{3}} \frac{N^2\gamma}{\sigma_r^2\sigma_z} \tag{4}$$

tells what fraction of a beams energy is lost due to synchrotron radiation in the magnetic field of the other beam. The subscript indicates the equation comes from a classical calculation. To do a clean high energy physics experiment, one wants the spread in the center-of-mass energy to be less than about 10%. This spread is $\delta/\sqrt{12}$, hence, one must require $\delta < 0.3$.

There are a total of 9 parameters: σ_r, σ_z, N, f, γ, P, $\mathcal{L}$, D and δ and 4 equations relating them. So one can specify 5 parameters and then solve for the other 4. For example, specifying γ, $\mathcal{L}$, f, P and D one can solve for σ_r, σ_z, N and δ. In particular

$$N = \frac{P}{fmc^2\gamma} \tag{5}$$

$$\sigma_z = \frac{DP}{2\pi mc^2 r_e \mathcal{L}} \tag{6}$$

$$\sigma_r = \frac{P}{\gamma mc^2\sqrt{4\pi f\mathcal{L}}} \tag{7}$$

$$\delta_{classical} = \frac{(4\pi)^2 r_e^4 \gamma \mathcal{L}^2 mc^2}{6\sqrt{3} fDP}. \tag{8}$$

Plugging in reasonable numbers for a 5 TeV linac[3] :

$$\begin{aligned} \gamma &= 10^7 \\ \mathcal{L} &= 10^{34}\,\mathrm{cm}^{-2}\mathrm{sec}^{-1} \\ f &= 5\,\mathrm{kHz} \\ D &= 2 \\ P &= 10\,\mathrm{MW} \end{aligned} \tag{9}$$

gives

$$\begin{aligned} N &= 2.4 \times 10^9 \\ \sigma_z &= 0.14\,\mathrm{mm} \\ \sigma_r &= 0.0049\,\mu\mathrm{m} = 49\,\text{Å} \\ \delta &= 76 \end{aligned} \tag{10}$$

It is unphysical to have each electron lose 76 times more energy than it has.

QUANTUM CORRECTIONS TO BEAMSSTRAHLUNG

The cause of this unphysical result is that $\delta_{classical}$ was derived with classical formulas for the radiation of a moving charge in a magnetic field. It turns out that we are in the quantum regime. This can be seen as follows. Consider a particle at radius r in the uniform cylindrical bunch. For the numerical examples $r = 2\sigma_r$ (a particle at the very outside of the uniform cylindrical bunch) is used. It sees a magnetic field

$$B = \frac{N_r e}{\sqrt{3} r \sigma_z} \tag{11}$$

where N_r is the number of electrons inside a circle of radius r. B is 4.8×10^7 Gauss for our example. There is an equal force coming from the electric field. The radius of curvature of the electron's trajectory in these fields is

$$\rho = \frac{\gamma mc^2}{2eB} = \frac{\sqrt{3}\gamma mc^2 r\sigma_z}{2N_r e^2} = \frac{\sqrt{3}\gamma r\sigma_z}{2N_r r_e} \tag{12}$$

This is 177 cm for our example. Now classically the synchrotron radiation spectrum peaks at $\omega \approx \frac{1}{3}\omega_c$ where $\omega_c = \frac{3c\gamma^3}{2\rho}$ [4] . The critical energy is the energy of these photons

$$E_c = \hbar\omega_c = \frac{3}{2}\hbar c\frac{\gamma^3}{\rho} = \frac{\sqrt{3}\hbar c\gamma^2 N_r r_e}{r\sigma_z}. \tag{13}$$

This is 268 ergs = 168 TeV for our example. This is much greater than the 5 TeV beam energy so clearly such photons can not be radiated.

Obviously the classical calculation is invalid and a proper quantum mechanical calculation must be used. A full quantum treatment of synchrotron radiation was done in 1952 by Sokolov, Ternov and Klepikov[4] . The results are illustrated in Fig. 1[5] . When $E/E_c \gg 1$ the classical calculation is correct. When $E/E_c \ll 1$, the power spectrum follows the classical curve and then drops exponentially at the electron's energy. Here we will use the approximation that the radiated power follows the classical curve until the photons energy equals the electron's energy where it sharply drops to zero. This approximation is always greater than the exact solution and is accurate to a few percent for $E/E_c < 0.1$.

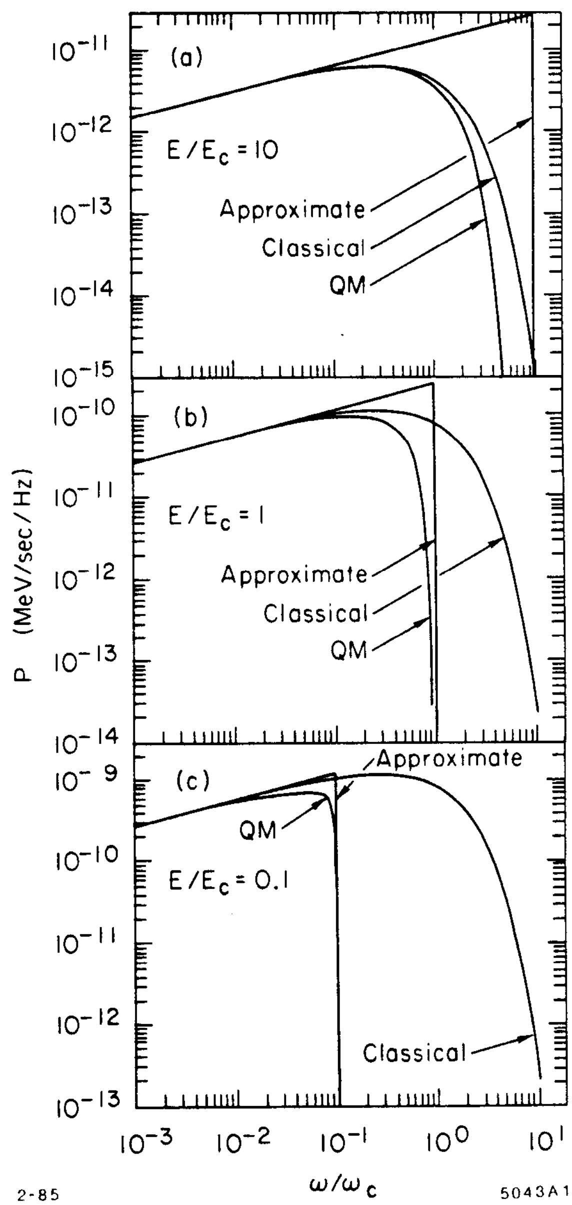

Fig. 1. Differential power spectra of the radiation emitted by 5 TeV electrons for several values of E/E_c. Parts a–c are in the classical, intermediate and quantum regimes respectively. Shown are the classical calculation, the exact quantum calculation and the approximation used here.

The classical synchrotron radiation formula for the power radiated as a function of frequency is[4]

$$P(\omega) = \frac{2r_e c\gamma^4 mc^2}{3\omega_c \rho^2} S\left(\frac{\omega}{\omega_c}\right) \tag{14}$$

where S contains the integral of a modified Bessel function. For $\omega/\omega_c \ll 1$; $S(\omega/\omega_c) \approx \frac{4}{3}(\omega/\omega_c)^{1/3}$. Hence, using this approximation, the total power radiated is

$$P_{tot} = \int_0^{E/\hbar} P(\omega) d\omega = \frac{2r_e c\gamma^4 mc^2}{3\rho^2} \left(\frac{E}{E_c}\right)^{4/3} \tag{15}$$

this power is radiated for the time it takes a particle to pass through the other bunch, namely $2\sqrt{3}\sigma_z/c$. Hence δ which is the fraction of the electrons energy which is radiated is

$$\delta_{QM} = \frac{16r_e^3\gamma N_r^2}{3\sqrt{3}r^2\sigma_z} \left(\frac{E}{E_c}\right)^{4/3} = \frac{16r_e^3\gamma N_r^2}{3\sqrt{3}r^2\sigma_z} \left(\frac{r\sigma_z mc}{\sqrt{3}\hbar\gamma N_r r_e}\right)^{4/3} \tag{16}$$

$$\text{for} \quad \frac{E}{E_c} = \left(\frac{r\sigma_z mc}{\sqrt{3}\hbar\gamma N_r r_e}\right) \ll 1. \tag{17}$$

Putting in numbers for the outside particle in the above example gives

$$\delta_{QM} = 2.7. \tag{18}$$

Note that the outside particle sees the largest magnetic field and loses the most energy. Averaging over radius for uniform cylindrical beams gives another factor of 3/4,

$$\delta_{QM} \leq \frac{r_e^3\gamma N^2}{\sqrt{3}\sigma_r^2\sigma_z} \left(\frac{2\sigma_r\sigma_z mc}{\sqrt{3}\hbar\gamma N r_e}\right)^{4/3}. \tag{19}$$

Note that at small radius the extreme quantum limit does not apply, but the approximation used here is always greater than both the exact classical and quantum formulas. Hence an upper bound on δ_{QM} has been calculated. A more accurate answer will require a computer simulation which is in progress.

EFFECTS ON SCALING LAWS OF LINEAR COLLIDERS

It is interesting to express these quantum formulas in terms of γ, $\mathcal{L}$, f, P and D instead of σ_r, σ_z, N and δ. Plugging equations (5)–(7) into (17) and (19) gives

$$\frac{E}{E_c} = \frac{P f^{1/2} D}{2\sqrt{3}(\pi \mathcal{L})^{3/2} r_e^2 \hbar c \gamma} \tag{20}$$

$$\delta_{QM} = \frac{8mc^2}{\sqrt{3}} \left(\frac{r_e}{2\sqrt{3}\hbar c}\right)^{4/3} \left(\frac{DP}{\gamma f}\right)^{1/3} . \tag{21}$$

For our example 5 TeV machine $E/E_c = 0.03$ which is truly much less than one. In fact from the form of equation (20) it is clear that for any high luminosity, high energy collider quantum effects in beamsstrahlung are important. This is because $\mathcal{L} \sim \gamma^2$ so the denominator increases like γ^4 and there are practical limits on how large P, f and D in the numerator can be made. This strong γ dependence is why at SLC quantum corrections are unimportant but at 5 TeV they are very large.

For our example 5 TeV machine the corrections were not quite large enough. We still have an average δ_{QM} of 2.1[7] which is greater than the 0.3 needed by high energy physics. Referring to equation (21) one sees that if D and P are both reduced by a factor of 20 then $\delta_{QM} = 0.29$ which is acceptable. Using equations (5)–(8) one gets the following consistent set of parameters for a 5 TeV machine

$$\begin{aligned}
\gamma &= 10^7 \\
\mathcal{L} &= 10^{34}\,\mathrm{cm}^{-2}\,\mathrm{sec}^{-1} \\
f &= 5\,\mathrm{kHz} \\
D &= 0.1 \\
P &= 0.5\,\mathrm{MW} \\
N &= 1.2 \times 10^8 \\
\sigma_z &= 0.4\,\mu\mathrm{m} \\
\sigma_r &= 2.5\,\text{Å} \\
E/E_c &= 7.5 \times 10^{-5} \quad \rightarrow \quad \text{quantum regime} \\
\delta_{classical} &= 30{,}000 \\
\delta_{QM} &= 0.29
\end{aligned} \tag{22}$$

Comparing equations (8), (20) and (21) one sees there are two possible regimes of beam parameters for a high energy (> 5 TeV), high luminosity e^+e^- linear collider.

1. One can use very high power ($> 100\,\mathrm{MW}$) and high repetition-rate beams. For this case the beam size is relatively large and the classical formulation of beamsstrahlung is valid. However if the accelerator is *e.g.* 3% efficient then the power consumption is 3 GW and the accelerator is obviously very expensive.
2. One can use low power beams. For this case the beam size is very small and the quantum formulation of beamsstrahlung must be used. Attaining such a small beam size will be difficult to say the least.

Note how different the scaling laws are for classical and quantum beamsstrahlung. To keep $\delta_{classical}$ small f, D and P must be large. To keep δ_{QM} small, f must be large while D and P are small.

In conclusion, quantum corrections to the beamsstrahlung are often important. Their inclusion extends the set of linear collider beam parameters which are useful for high energy physics.

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