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AXIAL AND PARITY ANOMALIES AND VACUUM CHARGE: A DIRECT APPROACH*

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ABSTRACT

We study the axial and parity anomalies in abelian gauge theories using the direct yet intuitive approach of counting the relative number of states of one chirality with respect to the other. A fundamental gauge invariant quantity, the determinantal ratio, is introduced to this purpose. We find that the number of states is conserved and that the gauge fields differentially phase shift states of opposite chirality at infinite energies. This implies a relative change of the density of states at infinite energies which must be compensated by a rearrangement of the density of states at finite energies. We then derive a sum rule which yields two alternative formulae for the index of a Dirac operator. One expresses the index in terms of its high energy behavior, and the other in terms of the low energy properties; these are the "zero modes" of definite chirality. Two examples are worked out in detail to clarify our general result.

The physics of the axial anomaly is shown to translate into that of the parity anomaly in $(2+1)$ dimensions, in which parity and chirality have interchanged roles. We also analyze the vacuum charge in regard to its high and low energy origin. The possibility of spectral flow is formulated and briefly discussed. In short, we provide a physical interpretation of certain mathematical indices, relate them to an extended version of Levinson's theorem of potential scattering, and simplify their evaluation.

1. Axial Anomaly, density of states and indices

A) Introduction: The Problem and Strategy:

The axial anomaly arises as the violation of the classical conservation law for axial currents at the quantum level. In quantum theory this violation is seen to arise from the interaction of Fermi fields with background gauge fields (hereafter referred to as b.f.). Therefore we are led to study the axial currents induced by these b.f.'s, and the resulting conservation laws. Ever since their discovery[1] they have played a fundamental role in the understanding of gauge theories as well as being important in certain physical reactions.

Originally they were associated with ultraviolet divergences of the theory, whose regularization presented a conflict between gauge and chiral invariance. Anomalies severely restrict the fermion content of axial-vector theories.[2] The existence of anomalies has been related via the Atiyah-Singer index theorem[3] to the possible zero eigenvalues of the Dirac operator. These "zero modes" make the functional integral vanish when the fermions interact with a topologically non-trivial gauge field (such as instantons, vortices, etc.)[4]

In this note we will discuss the anomaly and its physical origin by making extensive use of the generalization of Levinson's theorem[5,6] that was developed in our earlier papers on fractional charge[7] (hereafter called BB) and indices in one dimension.[8] In BB we gave a complete method for the calculation of the charge of the vacuum state. Our approach yielded a simple and direct way of computing the fractional part of the charge. In addition, by introducing a suitable comparison Hamiltonian, we also gave another "Index", the $(\eta - \eta_V)$ of the concluding section, which we showed was always an even integer measuring the spectral flow (energy levels crossing zero). This splitting allows a clear separation of the global from the local properties of the b.f. This method will allow a clear understanding of the role played by the two dimensional anomaly in the physics of the vacuum charge in $(2+1)$ dimensions.

The anomaly has already been studied from several different perspectives: index theorems,[9] simple models,[10] lattice formulations,[11] etc. It was Fujikawa[12,13] that realized that the fermionic measure in the functional integral was not invariant under chiral transformations, with the Jacobian being related to the anomaly. On the other hand, by using a dispersive analysis[14,15] and unitarity[16] it was found that the axial anomaly induces a singularity at threshold in certain amplitudes. More recently, methods of differential geometry were introduced to discuss the anomaly.[17]

We feel that there are still questions in need of answers: why is the anomaly reflected both as a high energy (triangle diagram) and as a low energy (zero modes, threshold singularity) phenomena? why not any old energy range? what is the role of regulators since the anomaly is finite and independent of the regulators? what is the physical interpretation and content of index theorems? Recently a new effect was discovered in $(2+1)$ dimensions, that of a parity anomaly[18–21] in which the vacuum currents induced by the b.f. have abnormal parity. Again observed as a conflict between gauge invariance and parity,[20] a discrete symmetry, its existence was related to the chiral anomaly[19] in $(1+1)$ dimensions and high energy behavior. However, it was also argued that the effect arose from the existence of "zero modes"[22] —a new realization of the interplay of high and low energy features.

This paper is a modest attempt to try to answer the above by providing an understanding of anomalies using simple concepts and methods. Our aim is to clarify the interplay of high and low energies and to offer a physical explanation of some of the content of index theorems.

The strategy is very simple- count the number of positive minus the number of negative chirality states in the presence of a b.f. A net difference breaks chiral invariance. To this end we provide an extension of Levinson's theorem so familiar from potential scattering.[5] The standard form of Levinson's theorem is a (completeness) comparison of two Hamiltonians, usually the free particle problem compared to the scattering case with a potential. Our extension of the theorem allows one to also compare the positive and negative energy parts of the spectrum of the same Hamiltonian. This device has several advantages, amoung which are ease of calculation and the fact that no explicit regulator is necessary. Since we shall be discussing long range potentials, special care is required because many of the results of scattering theory do not hold.[23] One cannot just make a cursory reference to standard treatments of scattering theory in the cases of interest. For example, for a soliton-type potential, the divergence of certain volume integrals of the potential violate one of the existence conditions for the standard Jost function. The manner for avoiding this problem was given in Ref. 8.

After a general presentation of our formalism, we will apply it to two specific models in two dimensions. The first is a strip problem of finite width in the y-direction. The treatment of this classic problem that the reader may find most useful for purposes of comparison is that of Michael Stone[24] , where additional references can be found. The second example that we treat is the vortex problem. In this problem there is a centralized cylinder or bar of flux. For comparison the reader should consult the paper by Joe Kiskis[25] on the low energy aspects of this problem. Both of these examples possess an anomaly and our approach will precisely relate the high and low energy features and will yield a detailed physical picture of the underlying phenomena.

Our main purpose in presenting these two examples is to develop a more detailed understanding of the anomaly. The expert reader will find our presentation too tedious, and others may find it too detailed, but all may find it a useful exercise. We have tailored the presentation so that it should be understandable to a wide audience.

In the next section contact is made with the work of Fujikawa (Ref. 12) and a bridge is established with the dispersive analysis of Refs. 14,15,16. After a discussion of the well known results of perturbation theory in $(3+1)$ dimensions, the treatment is generalized and the phase shift is evaluated at infinite energies in terms of the standard anomaly. In the following section the vacuum charge and parity anomalies are studied in $(2+1)$ dimensions in the abelian theory. The treatment relies heavily upon the techniques developed in BB. Finally we summarize our conclusions. Two Appendices are provided that give some useful technical details and formulae.

B) Review:

In order to make this exposition self-contained and thus readable, and to introduce our notation and language, let us first turn to a review which the reader will find familiar.[26]

The axial-vector current is written as

$$\langle J^5_\mu(x)\rangle = i\mathrm{Tr}\left[\gamma^5\gamma^\mu S(x,x)\right], \tag{1.1}$$

where $S(x,y)$ is the fermion propagator in the presence of the b.f. and satisfies

$$(i\not{D} + im)\, S(x,y) = \delta(x-y) \tag{1.2}$$

and $i\not{D} = i\gamma^\mu(\partial_\mu - iA_\mu)$. We introduce a spectral representation of $S(x,y)$ in terms of the eigenvectors and eigenvalues of $i\not{D}$. The massless Dirac equation

$$i\not{D}\psi_\lambda(x) = \lambda\psi_\lambda(x) \tag{1.3}$$

then leads immediately to the relation

$$S(x,y,m) = \sum_\lambda \frac{\psi_\lambda(x)\, \psi_\lambda^+(y)}{\lambda + im} \;, \tag{1.4}$$

where $\sum_\lambda$ is understood as a sum over discrete states and an integral over states in the continuous part of the spectrum.

Using these last two equations one can write the divergence of the axial vector current as

$$\partial_\mu \langle J_\mu^5 \rangle = -2\, im \sum_\lambda \frac{\psi(x)\gamma^5\psi^+(x)}{\lambda + im} + 2\sum_\lambda \psi(x)\gamma^5\psi^+(x)\;. \tag{1.5}$$

Clearly the second term on the right is the familiar and ambiguous $(0 \times \infty)$ and it must be regulated.

As usual we introduce a Pauli-Villars (gauge invariant) regulator and write

$$\partial_\mu \langle J_\mu^5 \rangle = -2im\, J^5(x,m) + 2i \lim_{M\to\infty} M\, J^5(x,M) \tag{1.6}$$

where

$$J^5(x,m) = \sum_\lambda \frac{\psi(x)\gamma^5\psi^+(x)}{\lambda + im}\;. \tag{1.7}$$

Integrating over Euclidean space one achieves

$$\int \partial_\mu \langle J^5_\mu \rangle \, d^d x = 2\Delta(m) + 2A\,, \tag{1.8}$$

where the anomaly is identified as

$$A = \lim_{M\to\infty} iM \,\mathrm{Tr}\left[\frac{\gamma_5}{i\not{D} + iM}\right], \tag{1.9}$$

and the trace is performed on space and Dirac labels. We propose to study the quantity

$$\Delta(m) = -im\,\mathrm{Tr}\left[\frac{\gamma^5}{i\not{D} + im}\right], \tag{1.10}$$

which has been extensively studied in the literature.[27] Since as we shall see shortly, $\Delta(m)$ contains considerable information about the spectrum of $i\not{D}$.

We will study this quantity in $(1+1)$ Euclidean dimensions ($d = 2$) and for the matrices will use the representation

$$\gamma_1 = \sigma_1 \qquad \gamma_2 = \sigma_2 \qquad \gamma_5 = \sigma_3\,. \tag{1.11}$$

The σ's are the usual Pauli matrices. In this representation, the Dirac equation is written as

$$i\not{D}\psi = \begin{pmatrix} 0 & L \\ L^+ & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \lambda \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}\,. \tag{1.12}$$

The chiral structure of the theory is now evident. After a short calculation one finds

$$\Delta(m) = m^2 \mathrm{Tr}\left[\frac{1}{L^+L + m^2} - \frac{1}{LL^+ + m^2}\right]\,. \tag{1.13}$$

From the Dirac equation we see that the wave function $\psi_+(\psi_-)$ is an eigenvector of LL^+ and (L^+L) with eigenvalue λ^2. Therefore it is clear that $\Delta(m)$ has information on the relative spectrum of LL^+ and L^+L, whose eigenvectors have positive and negative chirality.

C) General Formulation:

Motivated by our previous work, we introduce the quantity[8]

$$J_R(-z) = \det \left[\frac{H^- + z}{H^+ + z}\right] \tag{1.14}$$

which gives a direct comparison of the positive and negative chirality spectrum, and where

$$H^- = L^+L \ , \qquad H^+ = LL^+ \tag{1.15}$$

and

$$\Delta(z) = z \frac{d}{dz} \ell n J_R(-z) \ . \tag{1.16}$$

Therefore the anomaly defined by (1.9) can be written as

$$A = - \lim_{z \to \infty} \Delta(z) \ . \tag{1.17}$$

the z-derivative of the logarithm of the Jost ratio $J_R(z)$ is the difference of the resolvents of the operators H^- and H^+ at an energy $E = -z$. This observation allows us to derive the following general sum rule.

Consider the quantities

$$J_R(E) = J_R(-z = E + i\eta) \tag{1.18}$$

and

$$\frac{d}{dE} \ell n J_R(E) = \mathrm{Tr} \left[\frac{1}{E - H^-} - \frac{1}{E - H^+}\right] \ . \tag{1.19}$$

In general, the spectrum of $H^{\pm}$ consists of discrete bound eigenvalues (here assumed to be isolated) and a continuum. Thus the second of the above equations will have poles at the bound state energies and continuum branch cuts. The discontinuity across such cuts is directly related to the relative density of states. This simple but important fact is the basic reason that we do not forced to regulate the sums and integrals in this approach. The infinite density of states in the free (no b.f.) case is precancelled. One finds by direct calculation that

$$\Delta\rho(E) = \rho^-(E) - \rho^+(E) = -\frac{1}{\pi}\,\mathrm{Im}\,\frac{d}{dE}\,\ell n J_R(E + i\eta), \tag{1.20}$$

where $\Delta\rho(E)$ is the relative density of states with $+$ and $-$ chirality.

Integrating (1.19) in the complex E-plane along the contour shown in Figure 1, we find a <u>sum rule</u> that connects low and high energies

$$n_B^- - n_B^+ - I_\epsilon = I_\delta + I_\Gamma \ , \tag{1.21}$$

where

$$I_\epsilon = \frac{1}{2\pi i}\int_\epsilon \frac{d}{dE}\,\ell n J_R(E)\, dE$$

$$I_\delta = -\int_{E_T}^{\Gamma} \Delta\rho(E)\, dE \tag{1.22}$$

$$I_\Gamma = \frac{1}{2\pi i}\oint_\Gamma \frac{d}{dE}\,\ell n J_R(E)\, dR \ .$$

The contour ϵ is a circle of radius ϵ around the left edge of the continuum. The contour Γ is the large circle whose radius will ultimately be taken to ∞. The sum rule expressed in (1.21) relates the difference in the number of isolated bound states of H^- and H^+ to the three I integrals (i.e. threshold, the continuum and infinity behavior).

From (1.12) it is seen that the eigenvalues of H^- and H^+ have the familiar pairing property. If ψ^- is an eigenvector of H^- then $L\psi^-$ is an eigenvector (not normalized) of H^+ with the same eigenvalue. This is the usual argument that ensures that the number of bound states with E not zero are the same for H^- and H^+. This pairing argument is known to fail in the continuum , essentially because the density of continuum states is different for the two Hamiltonians.

When the variation in the background gauge fields is compact (i.e. $F_{\mu\nu} \xrightarrow[x,y\to\infty]{} 0$), the difference between H^- and H^+ is vanishingly small at spatial infinity (Euclidean space) and the Jost ratio J_R can be shown to exist (introduce a suitable comparison Hamiltonian as in Ref. 7 and 8). Also notice that

$$H^+ - H^- \sim \epsilon_{\mu\nu}\partial_\mu A_\nu \ . \tag{1.23}$$

Therefore an interpretation of the basic sum rule is that the total number of states is the same in H^+ and H^-. Indeed it will be easy to see in the two dimensional examples given in the next section that the difference between the two Hamiltonians is compact and is proportional to B, the magnetic field, which we take to vanish sufficiently fast at infinity. It is essential to keep in mind that we will always be comparing two systems or operators that differ by a compact (i.e. localized in space) potential.

If the continuum does not extend to zero energies, that is, a gap exists in the energy spectrum, we may expect that the integral I_ϵ arising from the small circle at the lower edge of the continuum will vanish. Indeed, for isolated bound states in the gap between zero and threshold, the pairing of chirality states ensures that their net contribution to the sum rule vanishes. Thus from continuity one might expect that the same cancellation will occur for those states just at the edge of the continuum. We are then led to expect that the integral I_ϵ will be different from zero only when the continuum extends to zero energy and indeed there are no isolated states in this case. These statements will be proved in the next section for the models considered.

Now the total contribution from the $E = 0$ states is known as the index of the operator $i\not{D}$. Therefore from (1.21) we find two fundamental expressions for the index, thereby clearly exposing its low and high energy features:

$$\mathrm{Index}(i\not{D}) = \left[n_B^- - n_B^+ - \frac{1}{2\pi i} \int_\epsilon \frac{d}{dE} \ell n J_R(E)\, dE \right] \tag{1.24}$$

or

$$\mathrm{Index}(i\not{D}) = - \int_{E_T+0}^{\Gamma} \left[\rho^-(E) - \rho^+(E) \right] + \frac{1}{2\pi i} \oint_\Gamma \frac{d}{dE} \ell n J_R(E)\, dE \tag{1.25}$$

We can use either of these formulae to evaluate the index in specific cases. Their consistency will be checked in the two examples discussed in the next section. The reader should be warned that the above index may differ in sign from some of the literature.

If the threshold were to approach zero, and 'pinch' the discrete spectrum then it is necessary to rewrite the low energy relation in the form

$$\mathrm{Index}(i\not{D}) = -\frac{1}{2\pi i} \int_e \frac{d}{dE} \ell n J_R(E)\, dE \ , \tag{1.26}$$

where the contour e (shown in Figure 1) starts at ϵ to the right of threshold and circles enclosing the origin. One can use this equivalent form to show that *the order of limits does not matter* in our later discussion of examples.

From the previous analysis we have seen that the fundamental quantity that we wish to study in different contexts is the ratio $J_R(E)$. Now let us turn to a characterization of the behavior of the Jost ratio $J_R(E)$ and its consequences for the sum rule.

For energies above threshold, the Jost ratio is complex and its phase is defined as

$$J_R(E) = |J_R(E)|\, e^{-i\delta_R(E)} \tag{1.27}$$

where the relative phase shift of the scattering states is

$$\delta_R(E) = \delta^-(E) - \delta^+(E) \tag{1.28}$$

and from the previous definitions it follows that

$$\Delta\rho(E) = \frac{1}{\pi}\frac{d}{dE}\,\delta_R(E) \tag{1.29}$$

and thus

$$I_\delta = \frac{1}{\pi}[\delta_R(0+) - \delta_R(\Gamma)]\,, \tag{1.30}$$

where $\delta_R(0+)$ arises from the lower limit of the integration and is the sum of the eigenphases, each evaluated at its own threshold.

Now as E approaches threshold, the behavior of the Jost ratio can be characterized as

$$J_R(E) \sim (E - E_T)^q \tag{1.31}$$

which leads immediately to the result $I_\epsilon = -q$. In the subsequent sections q will be shown to be the total magnetic flux in the two examples that are discussed there.

The existence of the anomaly, i.e. a non-trivial limit for $\Delta(z)$ for $z \to \infty$, has direct consequences for the sum rule. Suppose that the anomaly is $A = F$, then by equation (1.17), we find

$$\lim_{z \to \infty} \Delta(z) = -F \tag{1.32}$$

which implies

$$\frac{d}{dE} \ell n J_R(E) \xrightarrow[E \to \infty]{} -\frac{F}{E} \tag{1.33}$$

and therefore

$$I_\Gamma = \frac{1}{2\pi i} \oint_{\Gamma \to \infty} \frac{d}{dE} \ell n J(E)\, dE = -F\ . \tag{1.34}$$

Therefore the contribution from the large circle Γ in the sum rule is determined by the anomaly. In addition, the value of the phase shift at infinite energy is also completely determined by the anomaly; if the above three relations are satisfied, then by integration

$$J_R(z) \xrightarrow[z \to \infty]{} J_0 \times z^{-F} \tag{1.35}$$

where J_0 is a constant, and thus by continuation around the large semicircle,

$$J_R(E + i\eta) \xrightarrow[E \to \infty]{} J_0 \times (-E - i\eta)^{-F}\ , \tag{1.36}$$

The phase can now be read off as

$$\delta_R(\infty + i\eta) = -\pi F. \tag{1.37}$$

There is a (2π) branch ambiguity in this phase. We have made the natural choice above, but our final results will be independent of the branch.

This result has a very deep physical meaning. It implies that the background fields cause a relative phase shift in states with opposite chirality at very high energies. One then interprets this result as meaning that the anomaly is produced when the density of infinite energy states of one chirality is enhanced relative to ones of opposite chirality. The sum rule (i.e. the conservation of states) implies that the states being removed at high energies, appear elsewhere in the relative spectrum. Using the results from (1.34) and (1.37), the sum rule of (1.21) can be written as:

$$\mathrm{Index}(i\not{D}) = \frac{1}{\pi}\left[\delta_R(0+) - \delta_R(\infty)\right] + I_\Gamma \ , \tag{1.38}$$

where $\delta_R(0+)$ is the phase shift (at threshold) defined following eqn. (1.30) .

In terms of the eigenvectors and eigenvalues of $i\not{D}$. The index of the operator $(i\not{D})$ now reads

$$\mathrm{Index}(i\not{D}) = \frac{\delta_R(0+)}{\pi}. \tag{1.39}$$

We now see that this result is nothing but an extension of the familiar Levinson's theorem of potential scattering.[5]

To close this section we briefly summarize those results which we expect to be general before turning to two more specific examples in the following sections. We assumed that the variations in the b.f.'s were compact and hence the Jost ratio of determinants ,$J_R(-z)$, exists. The total number of states of each chirality is the same. The existence of the anomaly implies that at very high energies, states of opposite chirality are differentially phase shifted and there is a relative deficit in the density of states. This is the high energy aspect of the anomaly. This deficit of states at high energies must *then be compensated by a rearrangement of states in the finite part of the spectrum.*

In the next section we study two problems that can be cast in the above framework and will then clarify and illuminate the physics of these concepts.

2. Two Problems in two dimensions

Let us now proceed to implement this approach to selected problems in two dimensions (1+1 Euclidean). As was mentioned in the introductory remarks, the physics of anomalies presents several puzzles. Anomalies are customarily considered to be related to the high energy (short distance) behavior of the theory. On the other hand, index theorems[3,9] and recent dispersion relation analyses[14,15,16] indicate that they are also a low energy property of the theory, and that the high and low energy aspects are completely equivalent. This brings us to the questions posed in the introduction: Why is only the lowest and highest energies relevant (and not intermediate values)? What is the "magic" that produces the anomaly in the first place? To explore these questions and the physical interpretation of this behavior, we now turn to two quite general examples in two dimensions. We shall explicitly show that the sum rule and the two alternative expressions for the index are valid.

A) Stone's Strip:

In this section we will study the problem in which space (Euclidean) is finite in one direction, y, with width L, and the gauge field is configured as

$$A_x = 0 \qquad A_y = A_y(x) \qquad F_{xy} = \partial_x A_y(x) \ . \tag{2.1}$$

This problem has been studied from different points of view in several recent papers, and perhaps the reader would like to compare our treatment with the clear paper of Michael Stone[24,28] which originally sparked our interest in this problem. Since the y-direction is a "free" problem, we impose periodic boundary conditions (now the topology is that of a cylinder of radius L and infinite length in x). The Dirac spinors can then be written as

$$\psi(x,y) = \psi(x)\, e^{ik_n y} \tag{2.2}$$

with

$$k_n = \frac{2\pi n}{L} \qquad n = 0, \pm 1, \pm 2, \ldots \;. \tag{2.3}$$

Therefore the problem now becomes an (infinite) collection of one-dimensional modes each labeled by an integer n. For each fixed n, the corresponding Dirac equation reads

$$i\not{D}_n(\psi(x) = \begin{bmatrix} 0 & \partial_x + \phi_n \\ \partial_x - \phi_n & 0 \end{bmatrix} \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} = \lambda\, \psi(x) \tag{2.4}$$

where

$$\phi_n(x) = k_n - A_y(x) \;. \tag{2.5}$$

The $\lambda = 0$ states ("zero modes") can be found in the usual way. They are

$$\psi_-(x) \sim \exp\left(- \int^x \phi_n(x')\, dx' \right) \tag{2.6}$$

or

$$\psi_+(x) \sim \exp\left(+ \int^x \phi_n(x')\, dx' \right) \tag{2.7}$$

Now ψ_- is normalizable when

$$(n - F^+) > 0 \; ; \quad (n - F^-) < 0 \tag{2.8}$$

whereas ψ_+ is normalizable when

$$(n - F^+) < 0 \; ; \quad (n - F^-) > 0, \tag{2.9}$$

and we have defined

$$F^\pm = \frac{LA_y^\pm}{2\pi} \tag{2.10}$$

and

$$A_y^\pm = \lim_{x \to \pm\infty} A_y(x) \; . \tag{2.11}$$

The total flux passing thru the two-dimensional system is $(F^+ - F^-)$. We shall treat the case of positive F ($F = F^\pm$) in order to simplify the notation. From the discussion in the previous section we expect that

$$n_B^- - n_B^+ \sim ([F^-] - [F^+]) \; , \tag{2.12}$$

where $[F]$ stands for the nearest integer just below F. As mentioned above, for fixed n in (2.3) the problem is one-dimensional and this fact brings up another puzzle: for each mode there is no anomaly since $\Delta(z)$ vanishes for large z for each n value; how then does the anomaly arise? The answer arises of course from the infinite number of n values (modes).For each n value we can define the quantity $\Delta^n(z)$ just as in (1.13). This quantity has been extensively studied in the literature;[29,30,8] it is given by

$$\Delta^n(z) = \frac{1}{2}\left[\frac{\phi_+^n}{\sqrt{z+(\phi_+^n)^2}} - \frac{\phi_-^n}{\sqrt{z+(\phi_-^n)^2}}\right] \tag{2.13}$$

with

$$\phi_\pm^n = \frac{2\pi}{L}\left(n - F^\pm\right) . \tag{2.14}$$

Since the n modes are multiplicative in the Jost ratio and thus additive in Δ, we introduce the quantity

$$\Delta(z,F) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\frac{(n-F)}{\sqrt{x^2+(n-F)^2}} - \frac{n}{\sqrt{x^2+(n)^2}}\right] \tag{2.15}$$

which is an odd function of F, with

$$x^2 = z\left(\frac{L}{2\pi}\right)^2 \tag{2.16}$$

and finally write

$$\Delta(z) = \Delta(z,F^+) - \Delta(z,F^-) . \tag{2.17}$$

We have added and subtracted the $F = 0$ terms to achieve (2.15). First the problem of evaluating $\Delta(z,F)$ as given by (2.15) will be discussed. Note that this $\Delta(z,F)$ corresponds to a problem in which the gauge field vanishes at one endpoint. The corresponding Dirac operator will be denoted by $(i\not{D}_0)$. The results for $F = F^\pm$ will then be combined to demonstrate the validity of the sum rule.

If F is an integer, one may be tempted to shift the sum index n in (2.15) and conclude that $\Delta(z, F = \text{integer})$ is identically zero. However this is an illegitimate shift since each of the two terms in (2.15) is linearly divergent. Note that this is the analog of the "routing of momenta"[1] problem in the Feynman diagrams that determine the anomaly. However this divergence in each term is independent of F and z. Therefore we can compute unambiguously that

$$\frac{\partial\Delta(z,F)}{\partial F} = -\frac{1}{2}x^2 \sum_{n=\infty}^{\infty} \frac{1}{[x^2 + (n-F)^2]^{3/2}} \cdot \tag{2.18}$$

This sum is convergent and the reader will recognize this as the analog of the usual procedure for computing the "surface terms" for the routing problem in the anomalous diagrams.

We have studied this sum and refer the reader to appendix A for details. The expression (2.18) can easily be evaluated for any F in the limit of large z ($x^2 \to \infty$); approximating the sum by an integral in this limit one finds that

$$\lim_{z\to\infty} \Delta(z,F) = -F \ . \tag{2.19}$$

As shown in appendix A the correction to Δ in this limit is exponential in x. The anomaly is therefore $A = F$.

Hence we see that the puzzle is resolved–the "conspiracy" between infinitely many one-dimensional modes produce a non-trivial limit. Indeed, for $x^2 \to \infty$ the values of n that dominate the sum are of order x^2. Now using the results of Section 1, we find

$$I_\Gamma = -F \qquad \text{and} \qquad \frac{1}{\pi}\delta_R(\infty + i\eta) = -F \ . \tag{2.20}$$

This nonzero value of the relative phase shift means that states have been removed at high energies as explained in Section 1. To see where they went study the density of continuum states and the bound spectrum. Using (2.15) the density of states can be computed from (1.20) to be

$$\Delta\rho(E,F) = \Delta\rho_b + \Delta\rho_c \; , \tag{2.21}$$

where

$$\Delta\rho_b = \frac{1}{2}\left\{\sum_{-\infty}^{\infty}[\epsilon(n-F) - \epsilon(n)]\right\}\delta(E) \tag{2.22}$$

and

$$\Delta\rho_c = -\frac{1}{2}\sum_{-\infty}^{\infty}[\{n-F\} - \{n\}] \; . \tag{2.23}$$

The subscripts b and c stand for the bound and continuum contributions to $\Delta\rho(E,F)$. We have introduced the notation

$$\{j\} \equiv \frac{(j)\theta\left(E\,(L/2\pi)^2 - (j)^2\right)}{E\sqrt{E\,(L/2\pi)^2 - (j)^2}} \tag{2.24}$$

and

$$\epsilon(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \qquad \theta(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases} \tag{2.25}$$

For fixed E the sum in $\Delta\rho_c$ is cutoff by the θ functions and is finite. Thus the sum over the $F = 0$ term, involving $\{n\}$, vanishes identically. The remaining term has thresholds at $E_T = (2\pi/L)^2(n-F)^2$. Therefore when F is an integer, one of the thresholds is at $E = 0$ and the continuum extends down to the origin. It is in just this situation that we expect the edge or threshold behavior of the continuum to contribute to the sum rule. It is also easy to see that for integer F the continuum density of states, $\Delta\rho_c(E)$ is identically zero for any fixed $E \neq 0$.

To show that the relative phase is constant in this case and that the imaginary part of $\Delta(z, F)$ vanishes along the continuum cut, compute its energy derivative:

$$\frac{\partial\Delta(z,F)}{\partial x^2} = -\frac{1}{4}\sum_{-\infty}^{\infty}\frac{(n-F)}{[x^2+(n-F)^2]^{3/2}} \; . \tag{2.26}$$

This expression is convergent and unambiguous. Since F is an integer, the sum label can be shifted by F and for non-zero x the derivative vanishes. Since (2.19) fixes the value at infinity, we have

$$\Delta(z,F) = -F \qquad z \neq 0 \; . \tag{2.27}$$

Using our previous formulae for the integer F case, we find

$$\Delta\rho(E) = -F\delta(E) \tag{2.28}$$

and for E near the lowest threshold (which is at zero)

$$J_R(E) \simeq E^{-F}. \tag{2.29}$$

and the contribution from the tip of the continuum is

$$I_\epsilon = F \; . \tag{2.30}$$

It is straightforward to interpret this result. There are F (=integer) states at threshold. They are not isolated because the continuum extends to $E = 0$ (unless a "partial" wave expansion is made in the y eigenmodes in which case all are isolated save one). This result also obtains from the $\Delta\rho_b$ contribution by explicitly evaluating the sum.

Therefore for integer F we find

$$\delta_R(0+) = \delta_R(\infty + i\eta) = -\pi F \ . \tag{2.31}$$

Since there are no isolated bound states, $n_B^- - n_B^+ = 0$, and the low energy form of the index is

$$\text{Index}(i\not{D}_0) = -\frac{1}{2\pi i} \int\limits_{\epsilon} \frac{d}{dE} \ell n J_R(E)\, dE = -F \ . \tag{2.32}$$

The sum rule is satisfied since

$$\frac{1}{2\pi i} \oint\limits_{\Gamma\to\infty} \frac{d}{dE} \ell n J_R(E) = -F \ , \tag{2.33}$$

and the high energy form for the index becomes

$$\text{Index}(i\not{D}_0) = \frac{1}{\pi} [\delta_R(0) - \delta_R(\infty)] - F = -F \ . \tag{2.34}$$

From this equation we clearly see the interplay between the physics operating at high and low energies. The "anomaly" removes F states (in the relative spectrum) at infinite energies; these states do not disappear- there is an equal surplus of states at $E = 0$. These are the zero modes of definite chirality.

Thus when F is an integer there is no structure in the continuum; the spectrum moves rigidly. This is the magic of the compactification that occurs when the flux F is an integer. Recall that we still have to examine both F^+ and F^- and subtract to obtain the final sum rule. Before doing this let us first discuss the case of non-integer F.

$F \neq$ Integer:

For these values of F, we write

$$F = [F] + \langle F \rangle \ , \tag{2.35}$$

where $[F]$ is the integer part of F and $\langle F \rangle$ is its positive fractional part. For a non-zero value of $\langle F \rangle$, all the thresholds are also non-zero. The edge of the continuum does not contribute, i.e. $I_\epsilon = 0$. We also expect that the density of states will not be trivial and indeed

$$\Delta\rho_c = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \theta\left(E(L/2\pi)^2 - (n-F)^2\right) \frac{d}{dE}\,\delta_R^n(E) \tag{2.36}$$

where

$$\delta_R^n(E) = \tan^{-1}\left(\frac{n-F}{\sqrt{E(L/2\pi)^2 - (n-F)^2}}\right) \ . \tag{2.37}$$

and

$$\delta_R(E) = \sum_{n=-\infty}^{\infty} \theta(E - E_T^n)\ \delta_R^n(E) \tag{2.38}$$

where $E_T = (2\pi(n-F)/L)^2$. The quantity $\delta_R(0+)$ that arises from the lower limit of the phase space integral has a precise definition, but from the above, it is formally

$$\delta_R(0+) = \sum_{n=-\infty}^{\infty} \delta_R^n(E_T^n) \ . \tag{2.39}$$

From (2.29) we can evaluate the integral of the density of states up to a large energy $E = \Gamma = M^2$ ($M \gg (F/L)$). In the notation of (2.24)

$$\int_{E_T}^{M^2} \Delta\rho_c(E)\, dE = -\frac{1}{2\pi} \sum_{-\infty}^{\infty} \int_{E_T}^{M^2} \{n-F\}\, dE \ . \tag{2.40}$$

The interchange of sum and integral is legitimate since M^2 is finite and the sum is cutoff. In Appendix B we compute the above in two ways: fixing M large and taking the limit and by ς-function regularization. Both yield the same result:

$$\begin{aligned}\lim_{M^2\to\infty}\int_{E_T}^{M^2}\Delta\rho_c(E)\,dE &= -\langle F\rangle+\frac{1}{2} &&\langle F\rangle\neq 0\\ &= 0 &&\langle F\rangle = 0\;.\end{aligned}\tag{2.41}$$

From this result and (1.30) we find

$$\frac{\delta_R(0+)}{\pi}=-[F]-\frac{1}{2}\tag{2.42}$$

and hence

$$\mathrm{Index}(i\not{D}_0)=n_B^- - n_B^+ = -[F]-\frac{1}{2}\;.\tag{2.43}$$

The reader should not yet worry about the one-half, we still have to compute the difference in (2.17) . This result could also be obtained "naively" from the bound state contribution

$$\begin{cases}\Delta\rho_b=\dfrac{1}{2}\left\{(-1)+\displaystyle\sum_1^\infty[\epsilon(n-F)-\epsilon(n+F)]\right\}\delta(E)\\[2ex] \quad\;\;=\dfrac{1}{2}\left(-1-2[F]\right)\delta(E)\end{cases}\tag{2.44}$$

but the reader should feel uneasy about such manipulations.

Now we can collect results and write

$$\mathrm{Index}(i\not{D}_0)=\frac{1}{\pi}\left[\delta_R(0)-\delta_R(\infty)\right]-\text{“Anomaly''}\;.\tag{2.45}$$

This is the expression given in the literature for the index of an operator defined on a space with boundary. The Atiyah-Patodi-Singer[31,9] invariant or spectral asymmetry for $(i\not{D}_0)$ is now recognized as the index plus the anomaly:

$$\eta_0 = \frac{1}{\pi}\left[\delta_R(0) - \delta_R(\infty)\right] = \langle F\rangle - \frac{1}{2} \tag{2.46}$$

This is periodic and jumps discontinuously for integer F.

There is an important feature of (2.41) that should be made at this point. The integral of the density of states as M goes to infinity is independent of L, the width of the strip. Even though the density of states depends on this scale, the total integral, i.e. the total number of (relative) states in the continuum is scale independent. This remarkable feature has far reaching consequences which we postpone discussing until the treatment of the vortex problem.

Now we can complete the evaluation of the sum rule for the strip problem by combining the results for $F = F^{\pm}$. One must distinguish three cases: i) F^+ and F^- non-integer, ii) one is integral, iii) both are integral. In all of these cases the anomaly and the phase shift at infinite energies are

$$A = (F^+ - F^-) = \text{total flux} \tag{2.47}$$

and

$$\frac{1}{\pi}\,\delta_R(\infty + i\eta) = -A\ . \tag{2.48}$$

<u>Case i:</u>

From eqn (2.20) and (2.41) one gets

$$\begin{gathered}\frac{1}{\pi}\,\left[\delta_R(0+) - \delta_R(\infty)\right] = \left\langle F^+\right\rangle - \left\langle F^-\right\rangle \\ \text{Index}(i\not{D}) = n_B^- - n_B^+ = [F^-] - [F^+]\ .\end{gathered} \tag{2.49}$$

<u>Case ii:</u>

Suppose F^+ is an integer and F^- is not. The edge of the continuum contributes and using (2.20), (2.31), (2.32), and (2.41) we find

$$\frac{1}{\pi}\left[\delta_R(0+) - \delta_R(\infty)\right] = \langle F^- \rangle - \frac{1}{2}$$
$$\text{Index}(i\not{D}) = [F^-] + \frac{1}{2} - F^+ \ . \tag{2.50}$$

The one-half arises because one of the "zero modes", the one corresponding to $n = F^+$ is <u>not</u> a bound state. It is an unbound resonance whose wave function is continuum normalizable and approaches a constant at plus infinity. This corresponds to the anomalous Levinson's theorem that has been studied in Ref. 7 . In the mathematical expression for the index there is a term $h(0)/2$, where $h(0)$ is the number of harmonic spinors on the boundary.[9,31] This term is added to compensate for the fraction 1/2 arising from the anomalous Levinson's theorem.

Case iii:

From eqn (2.31) and (2.32) one gets

$$\frac{1}{\pi}\left[\delta_R(0+) - \delta_R(\infty)\right] = 0$$
$$\text{Index}(i\not{D}) = (F^- - F^+) = -A \ . \tag{2.51}$$

Let us now turn to the second example.

B) Joe's Bar- The Vortex:

In this section we will study the problem of fermions interacting with a gauge vortex.[32] This problem has been investigated by Kiskis, ref. 25, with special emphasis on the low energy aspects, namely the zero modes. In his paper, Kiskis took the anomaly structure and the high energy behavior from the well known diagrammatic results. More recently, this problem has been studied in relation to the index theorems (Ref. 9).

Here we propose to study this problem using the sum rule exposing the physics of both the high and low energy aspects of the problem. Again we are trying to understand the behavior of the relative spectrum at the intermediate energy scales as well. This problem offers an unusual mathematical setting because of the long range nature of the gauge fields.

We take the background gauge field to be (in polar coordinates)

$$A_\mu = \epsilon_{\mu\nu} \frac{x_\nu}{r} A(r) \ . \tag{2.52}$$

In (1.12) the operator L reads

$$L = i\, e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} + A(r) \right) \ . \tag{2.53}$$

The restriction of $A(r)$ to be spherically symmetric can be relaxed without modifying our conclusions.[9] Since the total angular momentum operator ($\hat{J} = -i(\partial/\partial\theta) + (\sigma_3/2)$) commutes with the Dirac operator, the wave functions can be written as

$$\psi = \frac{e^{iJ\theta}}{\sqrt{r}} \begin{pmatrix} \psi_+ e^{-i\theta/2} \\ \psi_- e^{i\theta/2} \end{pmatrix} \tag{2.54}$$

where

$$J = \ell + \frac{1}{2} \qquad \ell = 0\,, \pm 1\,, \pm 2\,, \ldots \tag{2.55}$$

In this basis we can write for every J

$$i\not{D}_J = \begin{pmatrix} 0 & L_J \\ L_J^\dagger & 0 \end{pmatrix} \qquad \begin{aligned} L_J &= i(\partial_r + \phi_J) \\ L_J^\dagger &= i(\partial_r - \phi_J) \end{aligned} \tag{2.56}$$

where

$$\phi_J = \frac{J}{r} + A\,. \tag{2.57}$$

For a regular vortex configuration, the potential satisfies

$$A(r) \xrightarrow[r\to 0]{} 0 \qquad\qquad A(r) \xrightarrow[r\to\infty]{} -\frac{F}{r} \tag{2.58}$$

and

$$F_{\mu\nu} = \epsilon_{\mu\nu}\, B(r) \qquad\qquad \int B(r)\, r dr = F\,. \tag{2.59}$$

From (2.56) and (2.58) we see that at short distances the fermions interact with the magnetic field, whereas at long distances (we assume that the vortex has a radius of order μ) the interaction is with the long range gauge field. This long range part causes a shift in the effective angular momentum ($J \to J - F$).

The zero modes can be found easily; they satisfy

$$\left[\frac{\partial}{\partial r} \pm \phi_J(r)\right] \psi_{\mp}(r) = 0 \tag{2.60}$$

whose solutions are

$$\psi_- \sim \exp\left(-\int^r \phi_J(r')\, dr'\right) \qquad \begin{array}{l} \psi_-(r) \xrightarrow[r\to 0]{} r^{-J} \\ \psi_-(r) \xrightarrow[r\to\infty]{} r^{-(J-F)} \end{array} \tag{2.61}$$

or

$$\psi_+ \sim \exp\left(+\int^r \phi_J(r')\, dr'\right) \qquad \begin{array}{l} \psi_+(r) \xrightarrow[r\to 0]{} r^{J} \\ \psi_+(r) \xrightarrow[r\to\infty]{} r^{(J-F)} \end{array} \tag{2.62}$$

Regularity at the origin requires $J < 0$ for ψ_- and $J > 0$ for ψ_+ . For negative J the solutions for ψ_- are normalizable in the measure dr for $|J| + F < -\frac{1}{2}$ while for positive J ψ_+ is normalizable if $J - F < -\frac{1}{2}$.

The long range nature of the gauge field introduces very strong infrared problems. All the usual theorems of scattering theory are highly suspect in the presence of such long range potentials.[33] Therefore we shall proceed in a rather unconventional manner. We shall "imbed" the radial half-line problem onto the full line and recover the problem of interest by a simple and tame limiting procedure. This device may be new (but we doubt it) and of interest in other applications. We will regulate the long and short distance behavior of the differential equation by modifying the effective potential for distances smaller than a, where $a \ll \mu$ and for distances larger than R, where $R \gg \mu$.

The imbedding potential for this problem is chosen to be ($a > 0$)

$$\phi_J(r) = \begin{cases} J/a & -\infty < r < a \qquad a \ll \mu \\ (J + rA(r))/r & a < r < R \qquad (R \gg \mu) \\ (J - F)/R & R < r < \infty \end{cases} \tag{2.63}$$

That is, each partial wave is imbedded into a full one-dimensional problem on the line. We shall demonstrate that the vortex case is recovered smoothly in the limit that (1) a goes to zero (recall $A(0) = 0$), and (2) R goes to infinity. In fact, by studying the limiting process in some detail, physical insight can be extracted that is relevant to the physics of the vortex. The imbedding will introduce a length scale in the problem. This will allow us to track the behavior of the density of states as the cut-offs reach their physical values.

Another advantage of this formulation is that it introduces a "gap" in the spectrum, and the zero modes have an exponential fall-off (for $F > J$) for $r > R$, and are continuous across a and R. This procedures binds zero modes (for ψ_+

for example) for every $J > 0$ satisfying $0 < J \leq F$. In the original problem some of these states are resonances in the continuum, however we must and will recover all the known results in the limits. As a payoff of this procedure we will be able to expose clearly the physics of the anomaly. We can now carry-over certain results from the strip problem. In particular, we find using (2.13) for each one dimensional problem for the partial wave J:

$$\Delta_J(z) = \frac{1}{2}\left[\frac{(J-F)/R}{\sqrt{z+(J-F)^2/R^2}} - \frac{J/a}{\sqrt{z+J^2/a^2}}\right] \tag{2.64}$$

$$\Delta(x,F) = \sum_J \Delta_J(x) \qquad J = \ell + \frac{1}{2} \qquad \ell = 0, \pm 1\ , \pm 2\ , \ldots$$

Again we see that the sum over J is conditionally convergent, but we can calculate unambiguously the quantity

$$\frac{\partial \Delta(x,F)}{\partial F} = -\frac{1}{2}\, x^2 \sum_{\ell=-\infty}^{\infty} \frac{1}{[x^2+(\ell-f)^2)]^{3/2}}\ , \tag{2.65}$$

where

$$x^2 = zR^2 \qquad f = F - \frac{1}{2}\ .$$

Following our previous arguments (see also Appendix A) we find

$$\lim_{z\to\infty} \frac{\partial \Delta(x,F)}{\partial F} = -1 \qquad \Delta(x,F) \xrightarrow[z\to\infty]{} -F \tag{2.66}$$

and therefore

$$\frac{1}{2\pi i} \oint\limits_{\Gamma\to\infty} \frac{d}{dE}\, \ell n J_R(E)\, dE = -F \qquad \delta_R(\infty + i\eta) = -\pi F\ . \tag{2.67}$$

Indeed it can be seen that $\Delta(z, F = 0)$ vanishes identically since there is a cancellation among all the partial waves. For $F = 0$ there is no anomaly. We also compute the quantity (again see Appendix A)

$$\frac{\partial\Delta(x,F)}{\partial x^2} = -\frac{1}{4}\sum_{-\infty}^{\infty}\left[\frac{(\ell-f)}{[x^2+(\ell-f)^2]^{3/2}} - \frac{(\ell+\frac{1}{2})}{[x^2+(\ell+\frac{1}{2})^2]^{3/2}}\right] \tag{2.68}$$

It is easy to demonstrate the following properties of the above expression:

$$\frac{\partial\Delta(x,F)}{\partial x^2} \xrightarrow[x^2\to\infty]{} 0\ ;$$

and for any $x^2 \neq 0$,

$$\frac{\partial\Delta(x,F)}{\partial x^2} = 0 \qquad \left\{\begin{array}{c} F = \text{half integer} \\ \text{or } F = \text{integer} \end{array}\right\} \tag{2.69}$$

The relative density of states is evaluated as before. It reads

$$\begin{aligned} \Delta\rho(E) &\equiv \Delta\rho_b(E) + \Delta\rho_c(E) \\ \Delta\rho_b(E) &= \frac{1}{2}\left\{\sum_{\ell=-\infty}^{\infty}[\epsilon(\ell-f)-\epsilon(\ell+\frac{1}{2})]\right\}\delta(E) \\ \Delta\rho_c(E) &= -\frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}\left[\{\ell-f\}-\{\ell+\frac{1}{2}\}\right] \end{aligned} \tag{2.70}$$

and where the curly bracket is now defined as

$$\{j\} = \frac{j\theta\left(E-j^2/R^2\right)}{ER\sqrt{E-j^2/R^2}}\ . \tag{2.71}$$

For every fixed E it is easily seen that $\sum\{\ell+\frac{1}{2}\}$ vanishes identically (the θ functions cut off the sum, the sum index can be shifted and the terms cancel pairwise. *Therefore there is automatically no dependence on the cut-off a.* The passage to the limit $a \to 0$ is trivial. This is because the gauge field is regular at the origin, and very near $r \sim 0$ the fermions only feel the centrifugal barrier as in the free problem.

The same can be seen from $\Delta\rho_c(E)$ for the case $f =$ integer. For such values of f the continuum extends to zero, and we expect a contribution from the edge of the continuum. There are no isolated bound states at $E = 0$. From Eq. (2.68) and (2.69) for integer f

$$\Delta(x, F) = \Delta(\infty, F) = -F \qquad x \neq 0 \, . \tag{2.72}$$

Also (similar to Eq. (2.28) and (2.29))

$$J_R(E) \cong E^{-F} \qquad (E \sim 0) \tag{2.73}$$

and

$$I_\epsilon = F \qquad\qquad \text{Index}(i\not{D}) = -F \, . \tag{2.74}$$

But when f is integer F $(= f + \frac{1}{2})$ is a half integer. This half integer in the index $(i\not{D})$ is again a realization of the anomalous Levinson's theorem. The wavefunction for $\ell = f$ approaches a constant for $r > R$ and corresponds to a resonant continuum state. In the physical vortex problem $(R \to \infty)$ the norm of this state diverges linearly with the length of the system.

For $f \neq$ integer we write $f = [f] + \langle f \rangle$ $(0 < \langle f \rangle < 1)$. Integrating the continuum density of states from threshold up to a large energy M, and passing to the limit $M^2 \to \infty$ we find (using the results of Appendix B)

$$\int_0^{M^2 \to \infty} \Delta\rho_c(E)\, dE = \frac{1}{\pi}\left[\delta_R(\infty + i\eta) - \delta_R(0+)\right] = -\langle f \rangle + \frac{1}{2} \, . \tag{2.75}$$

Collecting results and using the sum rule (1.24) or (1.25) , one finally achieves

$$\text{Index}(i\not{D}) = n_B^- - n_B^+ = -[f] - 1 \, . \tag{2.76}$$

This is seen to be the correct relative number of bound states from the analysis following (2.62) with the "potential" (2.63). This result can again be obtained "naively" by carrying out the sum in $\Delta\rho_b(E)$ in (2.70) for R finite and f not an integer.

The interpretation of the above analysis is the following. The gauge field removes F states at infinity (of one chirality relative to the other), and $(n_B^- - n_B^+)$ states "spill over" the threshold and become bound states; however $\langle f \rangle - \frac{1}{2}$ states *remain* in the continuum.

Our next step is to study the limit $R \to \infty$ to recover the physical problem. In (2.70) the continuum contribution to the density of states vanishes as $R \to \infty$, however the integral from threshold to infinity is *independent* of R and given by (2.73).

From (2.70) for $\Delta\rho_c(E)$ we see that as $R \to \infty$ *every* threshold approaches $E = 0$, and the origin becomes an accumulation point. The reader may wish to repeat the evaluations using the contour e of Figure 1 which has a smoother behavior in this limit.

The fact that for $E \neq 0$ the density of continuum states vanishes can also be seen from the fact that for $z \neq 0$,

$$\frac{\partial\Delta(x,F)}{\partial x^2} \xrightarrow[R\to\infty]{} 0 \ , \qquad \Delta(z,F) \xrightarrow[R\to\infty]{} -F \ . \tag{2.77}$$

Therefore in the limit $R \to \infty$ we find

$$\Delta\rho(E) = -F\,\delta(E) \ . \tag{2.78}$$

There are F states at $E = 0$ (threshold). This is the result found by Kiskis.[25] . The interpretation of this effect is simple, and yet it *is* the magic of the anomaly.

As R becomes large, the density of continuum states becomes peaked near the lowest threshold. There are $(-\langle f\rangle + \frac{1}{2})$ states in a region of order $(1/R^2)$ near the lowest threshold $(E_T \sim \langle f\rangle^2/R^2)$; as $R^2 \to \infty$ the *continuum* density of states approaches a δ-function at $E = 0$ with total integral $(-\langle f\rangle + \frac{1}{2})$. In addition *it is vanishingly small* for finite $E \neq 0$ in this limit.

Suppose we consider the contour of Fig. 1 in the case $\epsilon \gg 1/R^2$. We can write the total contribution from the continuum states as

$$\int_{E_T}^{\infty} \Delta\rho_c(E)\, dE = \frac{1}{\pi}\left[\delta_R(\infty) - \delta_R(\epsilon)\right] + \frac{1}{\pi}\left[\delta_R(\epsilon) - \delta_R(0)\right] = -\langle f\rangle + \frac{1}{2}\,. \tag{2.79}$$

and from (2.77) in the same limit

$$\delta_R(\infty) - \delta_R(\epsilon) \simeq 0\,. \tag{2.80}$$

Hence

$$\frac{1}{\pi}\left[\delta_R(\epsilon) - \delta_R(0)\right] \sim -\langle f\rangle + \frac{1}{2}\,. \tag{2.81}$$

Clearly the total contribution from the continuum then comes from a small region $O(1/R^2)$ near threshold. As $R \to \infty$ these continuum states all move to the *bound* state poles at $E = 0$ (since in this limit all the thresholds collapse to zero). Therefore the total number of states at $E = 0$ in the limit is

$$N = -[f] - 1 - \langle f\rangle + \frac{1}{2} = -F\,. \tag{2.82}$$

This is precisely the total number of states removed at infinite energy (via the phase shifts). When the limit $R \to \infty$ is taken, there are no energy scales left in the problem; therefore any deficit of states at infinite energy has to be compensated by an excess of states at zero energies. These are the "zero modes" induced by the anomaly. In this manner the high and low energy aspects of the anomaly are reconciled.

In the absence of any mass scale (chiral symmetry) the relative density of states behaves as an incompressible fluid. The anomaly "pushes on" states at infinite energy and these states "spill out" at zero energy. After the theory is regulated, the states above the regulator scale are missed in the counting even after this scale is taken to infinity. Therefore there is an imbalance of chiral states at low energies.

3. Relation to Other Treatments

Before proceeding to a brief discussion of perturbation theory, we would like to compare our results with other approaches to this problem.

Fujikawa[12] was the first to point out that anomalies arise from a non-trivial change in the fermionic measure under a chiral transformation. This change is reflected in the existence of a Jacobian of the transformation. Fujikawa has shown that the regulated Jacobian is

$$J = \lim_{M\to\infty} \operatorname{Tr}\left[\gamma_5 e^{-(i\not{D}/M)^2}\right] \tag{3.1}$$

which can be seen to be the heat kernel regularization of our eq. (1.9). We can write the above as

$$J = \lim_{M\to\infty} \int_0^\infty dE\,\Delta\rho(E)\,e^{-E/M^2}\ . \tag{3.2}$$

Using the sum rule, we find the result

$$J = F \tag{3.3}$$

given by the circle at infinity. The Jacobian is counting only states at zero energy, and in the continuum up to M^2. Hence it counts only those states pushed from infinity in the limit of M^2 going to infinity. Note also that Fujikawa's heat kernel regularization gives the same result as the two methods discussed in Appendix B, as must be the case. In the next section we also establish contact with the beautiful work of Dolgov and Zakharov[14] , Coleman and Grossman[16] and Frishman et al.[15] These authors used unitarity (Coleman, Grossman) and dispersion relations (Dolgov et al.; Frishman et al.) to show that the anomaly implies the existence of singularities at zero four momentum. The present examples and our formalism reproduce these results in a non-perturbative fashion, thereby establishing a simple bridge between high and low energy.

4. (3 + 1) Dimensions

A) Perturbation Theory:

At this point the learned reader may ask what is the relation between our results and the well-known results from perturbation theory, namely the "triangle diagram"[1] in $3+1$ dimensions. To answer this question, we will choose to keep gauge invariance and introduce a Pauli-Villars regulator of mass M and compute the usual matrix elements of J^5_μ and J^5 between the vacuum and a state with two real photons of momentum k^1 and k^2 ($q = k^1 + k^2$):

$$\begin{aligned}\langle 0|J^5_\mu|k^1_\alpha, k^2_\beta\rangle &= T^{\mu\alpha\beta}(k^1,k^2) \cong \epsilon_{\alpha\beta\gamma\delta}\, k^1_\gamma k^2_\delta\, F(q^2) + \ldots \\ \langle 0|J^5|k^1_\alpha, k^2_\beta\rangle &= T^{\alpha\beta}(k^1,k^2) \cong \epsilon_{\alpha\beta\gamma\delta}\, k^1_\gamma k^2_\delta\, m\,\Delta(q^2)\ ,\end{aligned} \tag{4.1}$$

where m is the mass of the fundamental fermions, and the dots refer to terms that vanish when contracted with q_μ. Therefore we write the analog of Eq. (1.6) as

$$q_\mu T^{\mu\alpha\beta} = 2m\, T^{\alpha\beta}(q,m^2) - \lim_{M\to\infty} 2M\, T^{\alpha\beta}(q,M^2) \tag{4.2}$$

and in terms of the invariant amplitudes $F(q^2)$ and $\Delta(q^2)$ we have

$$q^2 F(q^2) = 2m^2\, \Delta(q^2,m^2) - 2M^2\, \Delta(q^2,M^2)\ . \tag{4.3}$$

It is easy to calculate $\Delta(q^2,m^2)$ for on-shell photons (see Ref. 1):

$$\Delta(q^2,m^2) = A \int_0^1 dx \int_0^{1-x} dy\, \frac{1}{[m^2 - q^2 xy]}\ . \tag{4.4}$$

where $A =$ numerical constant. The statement of the anomaly is that

$$\lim_{M\to\infty} M^2\, \Delta(q^2,M^2) \equiv A\ . \tag{4.5}$$

However it is interesting to delay taking the limit $M \to \infty$. Let us first analyze the imaginary part of $F(q^2)$ in (4.3), following the spirit of Dolgov et al.,[14] Coleman et al.[16] and Frishman et al.[15] :

$$\mathrm{Im}\ F(q^2) = 2\frac{m^2}{q^2}\mathrm{Im}\ \Delta(q^2,m^2) - \frac{2M^2}{q^2}\mathrm{Im}\ \Delta(q^2,M^2)\ . \tag{4.6}$$

The imaginary part (absorptive) of $F(q^2)$ gives the discontinuity across the two-particle cut. It corresponds to the amplitude for creating a fermion-antifermion state (of opposite chirality in the $m \to 0$ limit). The contribution from $\delta(q^2)$ cancels between the two terms on the r.h.s. of (4.6). A short computation yields

$$\frac{m^2}{q^2}\mathrm{Im}\,\Delta(q^2,m^2) = I(q^2,m^2) \cong \frac{m^2}{q^4}\,\theta(q^2-4m^2)\,\ell n\left[\frac{1-\sqrt{1-4m^2/q^2}}{1+\sqrt{1-4m^2/q^2}}\right] . \quad (4.7)$$

The remarkable property of Eq. (4.7) is that

$$\int_{4m^2}^{\infty} I(q^2,m^2)\,dq^2 = c = \text{constant} , \quad (4.8)$$

where the constant c is *independent* of the mass m. Therefore as $M \to \infty$ $I(q^2,M^2)$ becomes a distribution peaked at infinite q^2, while in the $m \to 0$ limit, the chiral limit, $I(q^2,m^2)$ becomes $c\delta(q^2)$. This is the same behavior of the densities of states found in our examples. In our examples and formalism the variable E corresponds to q^2. The fact that the limit of $M \to \infty$ in (4.5) corresponds to states being removed from infinite q^2; in the chiral limit, these states appear at $q^2 = 0$. This is the "anomaly pole", again a low energy manifestation of physics taking place at infinite energies.

B) Generalized Discussion:

The results of the first section are not restricted to a particular dimension. Since the result was based on analyticity, the relative chirality phase is still given by

$$\frac{1}{\pi}\delta_R(\infty + i\eta) = \lim_{z\to\infty} \Delta(z) . \quad (4.9)$$

Now the results of perturbation theory can be reconciled with the discussion of the earlier sections. In order to see this explicitly we borrow the result of Brown et.al.(Ref. 27) for the Abelian theory in $(3+1)$ dimensions:

$$\lim_{m\to\infty} \Delta(m) = -\frac{1}{32\pi^2}\int d^4x F\widetilde{F}\ . \tag{4.10}$$

This result is obtained from the expansion of the fermion propagator in the background field up to the appropriate order in the field strengths. After the trace is performed over Dirac matrices (with γ_5) and the limit of large m taken, only the triangle diagram survives. Finally

$$\delta_R(\infty + i\eta) = -\frac{1}{32\pi}\int d^4x F\widetilde{F}. \tag{4.11}$$

The interpretation of this result is again based on the removal of states at infinite energy by the background field. The threshold singularity indicates that these states are compensated by an enhancement of states at threshold.

We *conjecture* that this behavior will not be changed in perturbation theory since the states have been pushed to the extremes of the energy scale. For finite mass a "reasonable" perturbation can only distort the relative density of states locally without changing the number of states regionally. In the limit they all are pinned to the energy boundary anyway. Then the sum could not have changed. Perhaps this picture provides a physical basis for and helps to clarify the remarkable non-renormalization theorem of Adler and Bardeen.[34]

5. Charge in (2 + 1) Dimensions

A) Vacuum Charge and Parity Anomaly:

Recently a very interesting new anomaly has been discovered in 2-space–1-time dimensions. The presence of this anomaly was pointed out by the authors of Ref. 18,19,20. There the vacuum charge and currents induced by b.f. were computed.[19] The induced current turns out to be of abnormal *parity*. As in the axial anomaly case, it was also realized that there is a conflict between gauge invariance and parity.[20] Insisting on gauge invariance leads to parity breaking.

The effective action for the gauge fields is seen to have a *topological mass term*[19,20,35] which breaks parity.

It was also suggested that the vacuum charge arises from the presence of "zero modes" in the spectrum of the Dirac Hamiltonian.[22] However in Ref. 19 it was obtained from the $(1+1)$-dimensional anomaly. Again we find the same puzzle, high or low energy?

We propose to study the same theory as looked at in Ref. 19 but only the Abelian case. We will study *time-independent* background fields in the Weyl $A_0 = 0$ gauge. The Dirac equation is

$$\left[i\gamma_0\partial_0 + i\overrightarrow{\gamma}.(\overrightarrow{\partial} - i\overrightarrow{A}) + m\right]\psi(x,t) = 0 \; . \tag{5.1}$$

Separating the time variable and choosing a representation such as:

$$\alpha_1 = \sigma_1 \qquad \alpha_2 = \sigma_2 \qquad \beta = \gamma_0 = \sigma_3 \tag{5.2}$$

it follows that

$$\left[i\overrightarrow{\alpha}.(\overrightarrow{\partial} - i\overrightarrow{A}) + \beta m\right]\psi = H\psi = E\phi \; . \tag{5.3}$$

The necessity of the mass in (5.1), (5.3)will become apparent shortly. However it may also be argued that it is needed to define the "zero" of energy. In the presence of long range fields, there are severe infrared problems. A mass is needed to tame these ambiguities and to separate the positive and negative parts of the spectrum. When this mass term vanishes β anticommutes with the Hamiltonian (4.3) and the spectrum is symmetric about zero energy. The mass breaks this symmetry.

Equation (5.3)with (5.2)makes the physics of the vacuum charge very clear, indeed now the piece $i\vec{\alpha}(\vec{\partial} - i\vec{A})$ in H is the 1+1 dimensional Dirac operator $i\not{D}$ studied in Sections 1 and 2. Therefore the "zero modes" of $i\not{D}$ became the bound states of H (some of them may be continuum resonant states). Take the example of the vortex case studied in Section 2. For positive flux the "'zero modes" only have upper component, are eigenstates of parity γ_0 with eigenvalue $+1$ therefore are solutions of (5.3)with energy $+m$. For negative flux the situation is the opposite. Only these threshold states can have definite "parity" (eigenstates of γ_0). Therefore an excess of "parity" (+) over "parity" (−) states induces breaking of parity in the same sense as an excess of chirality (+) over (−) states (or vice versa) breaks chiral symmetry. These "zero modes" induce a local charge density that is "localized" (the states fall off algebraically) near the vortex, see (2.61) and (2.62) . The background field treats positive and negative energy eigenstates differently. For positive flux, positive energy states are attracted to the vortex (some of them bind to it) and negative energy states are repelled (the opposite for negative flux). A polarization cloud is formed, and this is the origin of the charge density.

The vacuum charge is related to this asymmetry in the Dirac spectrum,[30] and the quantity η - the Atiyah-Patodi-Singer invariant.[31] We will closely follow our previous work, Ref. 7, in which the fundamental gauge invariant quantity B(E) was introduced. This together with the derived quantity $G_e(E)$, have a very natural and intuitive meaning. In BB we found that the ground state charge was given in terms of the relative phase shifts at infinite energies[36] when spectral flow cannot occur. We write

$$Q = -\frac{1}{2}\int_0^\infty [\rho(E) - \rho(-E)]\, dE = -\int_0^\infty \rho_{\rm odd}(E)\, dE \tag{5.4}$$

and

$$\eta = 2\int_0^\infty \rho_{\rm odd}(E)\, dE$$

where

$$\begin{aligned} \rho_{\rm odd}(E) &= \frac{1}{2\pi}\, \mathrm{Im}\; G_e(E + i\eta) \\ Ge &= \frac{1}{2}\, \mathrm{Tr}\left[\frac{1}{H+E} + \frac{1}{H-E}\right] = \frac{1}{2}\,\frac{d}{dE}\,\ell n\, B(E) \\ B(E) &= \det\left[\frac{H+E}{H-E}\right] \end{aligned} \tag{5.5}$$

The fundamental quantity $B(E)$ measures the asymmetry in the spectrum. The quantity $\rho_{\rm odd}$ is the odd part of the density of states of the *Dirac* Hamiltonian (5.3).

Regulators are not necessary in (5.4) since the background fields only move a *finite* number of states. Indeed we can relate the quantity G_e in (5.5) to $J_R(z)$ and $\Delta(z)$ defined in the previous sections, after recognizing that Eq. (5.3) can be written as (this corresponds to the κ constant case on BB)

$$H = i\not{D} + m\,\sigma_3 \tag{5.6}$$

where $\not{D}$ is the Dirac operator studied in the first sections. Therefore

$$G_e = m \text{ Tr } \frac{\sigma_3}{H^2 - E^2} = \frac{m}{m^2 - E^2}\,\Delta(m^2 - E^2) \tag{5.7}$$

where $\Delta(z)$ was introduced in (1.16). We can now use all the results from the previous section, and will return to a reexamination of the vortex problem of Section 2 when $f \neq$ integer,while keeping the cut-off R finite, and then taking the physical limit at the end of the computation. In this way we will clearly expose the high and low energy physics.

For this problem $\Delta(z, F)$ is given by Eq. (2.61). Writing $\rho_{\text{odd}}(E)$ in terms of ρ_b^{odd} and ρ_c^{odd} the bound states and continuum contribution respectively and writing above thresholds

$$B(E) = |B(E)|\,e^{i\delta(E)}\ . \tag{5.8}$$

We find for positive energy

$$\rho_c^{\text{odd}}(E) = \frac{1}{2\pi}\,\frac{d}{dE}\,\delta(E)\ . \tag{5.9}$$

Below thresholds we find using (5.5) and (5.7)

$$\begin{aligned}\rho_b^{\text{odd}}(E > 0) &= \frac{1}{2}\,\text{sign}(m)\,\Delta(0)\,\delta(E - |m|)\\ &= \frac{1}{2}\,\text{sign}(m)\,\text{Index}(i\not{D})\,\delta(E - |m|)\end{aligned} \tag{5.10}$$

where Index($i\not{D}$) is given by (2.73). From the equations (5.5), (5.7) and using

(2.67) we find

$$\rho_c^{\mathrm{odd}}(E>0) = m\,\Delta\rho_c(E^2-m^2)\ . \tag{5.11}$$

The dimensions are reconciled by noting that the left side is the density per E whereas the ρ on the right side is the density per E^2.

The ground state charge is given by (5.2), (5.9) and (5.10)

$$Q = -\frac{1}{2}\,\mathrm{sign}(m)\,\mathrm{Index}(i\not{D}) - \frac{1}{2\pi}\left[\delta(\infty+i\eta) - \delta(0+i\eta)\right]\ . \tag{5.12}$$

Clearly the threshold phase shift $\delta(0+i\eta)$ must cancel the bound state contribution since it is precisely the number of states that "spilled over" the thresholds. To be convinced of this fact, notice that Eq. (5.12) can be written in the form (see Ref. 3)

$$\rho_c^{\mathrm{odd}}(E>0) = -\frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}\left[\varrho(\ell-f,E) - \varrho(\ell+\frac{1}{2},E)\right] \tag{5.13}$$

with the definition

$$\varrho(j,E) = \theta\left(E^2-m^2-j^2/R^2\right)\frac{d}{dE}\tan^{-1}\left((\frac{mR}{E\,j})[E^2-m^2-j^2/R^2]^{1/2}\right)\ . \tag{5.14}$$

For $E>0$ there are thresholds at $[m^2+j^2/R^2]^{1/2}$. The $\tan^{-1}(x)$ terms are the phase shifts above these thresholds. From this result it is straightforward to check that $\delta(0+i\eta)$ is the same as the term in brackets in Eq. (2.67), namely the number of bound states.

Therefore we are left with only the $\delta(\infty)$ term in (5.12) . This phase shift is related to the $1+1$ dimensional anomaly as follows:

Continue E to the imaginary axis $E = +i\omega$, and by using previous results we find

$$G_e = m \left. \frac{\Delta(z)}{z} \right|_{z=m^2+\omega^2} \tag{5.15}$$

and since

$$\frac{\Delta(z)}{z} \xrightarrow[z\to\infty]{} -\frac{F}{z}$$

then

$$G_e(i\omega) \xrightarrow[\omega\to\infty]{} -\frac{mF}{m^2+\omega^2} \ . \tag{5.16}$$

And from the above and their relation to B(E), is easy to see that

$$B(i\omega) = \det\left[\frac{H+i\omega}{H-i\omega}\right] \xrightarrow[\omega\to\infty]{} B_0 \left[\frac{|m|+i\omega}{|m|-i\omega}\right]^{-F\,\mathrm{sign}(m)} \tag{5.17}$$

with B_0 a constant. Continuing to $E + i\eta$, the positive energy cut is approached from above and the negative cut from below. Hence

$$B(E+i\eta) \xrightarrow[E\to\infty]{} B_0 \; e^{-i\pi F \mathrm{sign}(m)} \ . \tag{5.18}$$

Therefore from (5.8)

$$\delta(\infty + i\eta) = -\pi F \,\mathrm{sign}(m) \ . \tag{5.19}$$

in agreement with (1.37) . Finally the high energy form for the charge is

$$Q = \frac{F}{2}\,\mathrm{sign}(m) \ . \tag{5.20}$$

This is the result of Ref. 19. However we see here that it arises from the phase shifts at infinite energy. The number of bound states at $E = |m|$ is given by

$N = [f] + 1$ (recall the Index of $(i\not{D})$ and equations (2.76) and (2.82)). And there are $\langle f \rangle - \frac{1}{2}$ states in the continuum (positive and negative). As the cut-off R is taken to infinity, all these continuum states accumulate at $E = |m|$ and as in the physical vortex problem, there are F states at $E = \pm|m|$. Some of these are bound states, others are resonant states, hence the number need not be an integer.

In the same spirit as (2.81), Eq. (5.4) becomes for large R ($R^2 \gg 1/\epsilon$).

$$Q = \frac{1}{2}\text{sign(m)} \; ([\text{f}] + 1) + \frac{1}{2\pi} \left[\delta(0) - \delta(\infty)\right]. \tag{5.21}$$

Now for very large R, we have shown that the phase is constant for all energies larger than ϵ. Hence in this limit we can write

$$\begin{aligned} \frac{1}{2\pi} \left[\delta(0) - \delta(\infty)\right] &= \frac{1}{2\pi} \left[\delta(0) - \delta(\epsilon)\right] \\ &= \frac{1}{2} sign(m) \left(\langle f \rangle - \frac{1}{2}\right) . \end{aligned} \tag{5.22}$$

The charge evaluated from the low energy part of the spectrum then becomes

$$Q = \frac{1}{2} \, \text{sign}(m) \; \{[f] + \langle f \rangle + \frac{1}{2}\} \tag{5.23}$$

in agreement with the high energy result (5.20) .

These extra states appear because the anomaly removes states at infinity,and as a consequence 'enhances' one value of parity relative to the other at threshold. Parity is broken when the theory is regulated. From the expression for Q and, as usual, invoking Lorentz covariance,[19]

$$\langle J^{\mu} \rangle \sim \mathrm{sign}(m)\, \epsilon^{\mu\nu\rho} F_{\nu\rho} + \ldots \tag{5.24}$$

The dots stand for higher derivative terms. Expression (5.24) shows the abnormal parity (Hall current) and gives rise to the mass term in the effective action for the gauge fields.[35] The sign(m) factor determines whether these states are in the positive or negative continuum, i.e. whether they are empty or filled. This factor plays the same role as the $\pm\frac{1}{2}$ of the soliton case in $1+1$ dimensions.

B) Spectral Flow:

Now we can generalize the problem to the situation in which the mass term has a compact (localized) variation in space $m \sim m(r)$. Following BB we construct the spectral asymmetry for this problem as

$$\eta = \eta_m + (\eta - \eta_m) \ , \tag{5.25}$$

where η_m is the index for the problem with a constant m. This amounts to isolating the phase at infinite energy completely. The index η_m contains <u>all</u> the information about the high energy aspects of the problem. The quantity $(\eta - \eta_m)$ is an even integer (or zero). It depends on the local variation of $m(r)$ and contains the information about spectral flow- i.e. the energy levels that cross zero in the process of deforming the mass from the constant m to the final $m(r)$.

6. Discussion and Conclusions

We have studied the physics of axial and parity anomalies in Abelian theories from a simple perspective. Our approach consists in studying the relative density of states of different chirality—(or parity in $(2+1)$ dimensions).

We find that the gauge field induces a phase shift in states of one chirality with respect to the opposite chirality at infinite energy (or alternatively read intrinsic parity respectively). The *relative* phase shift is given by the anomaly. The number of states is conserved. A deficit of states at infinite energy is compensated for by an excess of states in the finite part of the spectrum. When the theory is regulated only the excess of states remain, and chiral symmetry is broken.

For long-range fields, when there are no other energy scales in the fermion spectrum, these states appear at zero energy (energy here stands for the eigenvalues of the Dirac operator). In this case the relative density of states behaves like an incompressible fluid.

These states at zero energy ("zero modes") have definite chirality. When regulators are introduced, they suppress the counting of states above the regulator scale. Even as the regulators are taken to infinity, the states removed at infinite energy (by the gauge fields) are missed in the counting. A net number of states of one chirality with respect to the other is found. Chiral symmetry is broken.

The same physics is seen to happen in $(2+1)$ dimensions with static background fields. This time the symmetry that is broken is parity. The ground state charge arises from states that are phase-shifted at infinite energy. This charge can be expressed as this phase shift which is in turn determined completely by the chiral anomaly in $(1+1)$ dimensions.

Again the relative number of states (of opposite parity) lost at infinity are found at threshold (for long-range fields) which is now at an energy of $E = \pm|m|$ (m = mass of the fermions). These states are localized near the spatial region in which the magnetic field is concentrated, thus creating a polarization cloud and giving rise to the vacuum charge. Each of these states is an eigenstate of (intrinsic) parity. A formulation of spectral flow was briefly discussed.

As a payoff, our simple formulation can offer a physical re-interpretation of some well-known mathematical indices. We find that the expression for the index of the Dirac operator is related to an extension of Levinson's theorem (or to the conservation of states). This index was shown to be given by relative phase shifts (of opposite chirality states) at threshold.

We have also made contact with the perturbative treatment of the axial anomaly in $(3 + 1)$ dimensions, where an analysis of the absorptive parts of amplitudes yielded the same qualitative physics as in the $(1 + 1)$- dimensional examples. We also showed that the relative phase-shift at infinite energy is completely determined by the anomaly expression.

APPENDIX A

We want to compute the sum in Eq. (2.18) in the text. For this we use an equation given in Bateman[37] and by taking an appropriate derivative we find

$$\frac{\partial\Delta(x,F)}{\partial F} = -1 - 4\pi x \sum_{k=1}^{\infty} k\cos(2\pi kF)K_1(2\pi kx) \ . \tag{A.1}$$

This equation can also be obtained directly by taking a suitable Bessel transform of each term and summing. Integrating (A.1) in F from zero to F we find $\Delta(z,F)$ as given in (2.15) :

$$\Delta(x,F) = -F - 2x \sum_{k=1}^{\infty} \sin(2\pi Fk)K_1(2\pi kx) \ . \tag{A.2}$$

From this it follows that

$$\frac{\partial\Delta}{\partial x^2} = 2\pi \sum_{k-1}^{\infty} k\sin(2\pi kF)K_0(2\pi kx) \ . \tag{A.3}$$

In the above expressions, $K_0(\omega)$ and $K_1(\omega)$ are the modified Bessel functions. The expression for $d\delta_R(E)/dE$ can be obtained by analytically continuing $x^2 \rightarrow -E - i\eta$ in the above expressions, and computing the imaginary part according to eqns. (1.20) and (1.29) in the text.

From the asymptotic expansions for the modified Bessel functions we can bound the sum

$$|\Delta(z,F) + F| \ \leq c\sqrt{x}e^{-2\pi x} \tag{A.4}$$

with c being a numerical constant.

APPENDIX B

In this appendix we give the details leading to Eq. (2.38) in the text using two alternative methods.

1. Computing the integral in the limit $M^2 \to \infty$ using the expression (2.41) in terms of the phase shifts (2.37) we can write

$$\int_{E_T}^{M^2\to\infty} \Delta\rho_c(R)\, dE = \frac{1}{\pi} \lim_{\mathcal{M}\to\infty} \times \left\{ \sum_{n=0}^{U^+} \tan^{-1}\left(\frac{n-F}{k(E)}\right)\Bigg|_{k=0}^{k=K^-} - \sum_{n=1}^{U^-} \tan^{-1}\left(\frac{n+F}{k(E)}\right)\Bigg|_{k=0}^{k=K^+} \right\}, \tag{B.1}$$

where

$$\mathcal{M} = M\left(\frac{L}{2\pi}\right) \tag{B.2}$$

$$U^{\pm} = [\mathcal{M} \pm F] \tag{B.3}$$

$$K^{\pm} = \sqrt{\mathcal{M}^2 - (n \pm F)^2} \tag{B.4}$$

and $[P]$ is the integer part of P. Evaluating the expression for (B.1) at the upper limits $K^{\pm}$, we achieve

$$U.L. = \frac{1}{\pi}\left[\sum_{n=0}^{U^+} \sin\left(\frac{n-F}{\mathcal{M}}\right) - \sum_{1}^{U^-} \sin^{-1}\left(\frac{n+F}{\mathcal{M}}\right)\right] . \tag{B.5}$$

For $\mathcal{M} \to \infty$ only the large values of n contribute to (B.5). Therefore we can replace the sums in the above expression by integrals.

Hence

$$U.L. \xrightarrow[\mathcal{M}\to\infty]{} \frac{\mathcal{M}}{\pi}\left\{\int\limits_{-F/\mathcal{M}}^{A^+} \sin^{-1}(x)\,dx - \int\limits_{(1-F)/\mathcal{M}}^{A^-} \sin^{-1}(x)\,dx\right\} . \tag{B.6}$$

where $A^{\pm} = (U^{\pm} - F)/\mathcal{M}$. The lower limits in (B.6) go to zero at $\mathcal{M} \to \infty$. Now (B.6) becomes

$$= \frac{\mathcal{M}}{\pi}\int\limits_{A^-}^{A^+} \sin^{-1}(x)\,dx . \tag{B.7}$$

In the limit, the integrand is approximately a constant in (A^+, A^-) $(\sin^{-1}(x) \simeq \pi/2)$ therefore the expression in (B.5) yields

$$U.L. = \frac{1}{2}\ \{[\mathcal{M} + F] - [\mathcal{M} - F] - 2F\} . \tag{B.8}$$

The evaluaton of (B.1) in the lower limits $k = 0$ is straightforward. It gives

$$\begin{aligned} L.L. &= -\frac{1}{\pi}\,\frac{\pi}{2}\left\{\sum_{0}^{U^+}\epsilon(n-F) - \sum_{1}^{U^-}\epsilon(n+F)\right\} \\ &= -\frac{1}{2}\ \{-2[F] - 1 + [\mathcal{M} + F] - [\mathcal{M} - F]\} . \end{aligned} \tag{B.9}$$

Combining (B.8) and (B.9) we find Eq. (2.41) . Notice that the dependence of (B.8) and (B.9) on $U^{\pm}$ is because of the sharp cut-off. However this dependence cancels in the final answer.

2. Using ς-function regularization. Let us define the regularized continuum

contribution $\Delta\rho(E,s)$ by

$$\Delta\rho_c(E,s) = -\frac{1}{2\pi}\sum_{-\infty}^{\infty}\frac{(n-F)k^{-s}(E)\theta\left(k^2(E)\right)}{E\,k(E)} \tag{B.10}$$

with

$$k^2(E) = E\left(\frac{L}{2\pi}\right)^2 - (n-F)^2 . \tag{B.11}$$

Then we are interested in $\lim\limits_{s\to 0^+}\int_0^\infty \Delta\rho(E,s)\,dE$. With this regularization we can safely interchange the integral over E and the sum over n. For every n the integral over E is

$$\int_0^\infty \Delta\rho_c^n(E,s)\,dE = -\frac{(n-F)}{2\pi}\int_{-\infty}^{\infty}\frac{k^{-s}dk}{k^2+(n-F)^2} . \tag{B.12}$$

Therefore (for non-integer F)

$$\lim_{s\to 0^+}\int_0^\infty \Delta\rho_c(E,s)\,dE = -\frac{1}{2}\lim_{s\to 0^+}\sum_{n=-\infty}^{\infty}\mathrm{sign}(n-F)|n-F|^{-s} . \tag{B.13}$$

Now the series is convergent and we can shift the integer part of F. Separating the positive, negative and $n=0$ terms in (B.13) the sum in (B.13) can be written as

$$\sum_0^\infty |n+1-\langle F\rangle|^{-s} - \sum_0^\infty |n+\langle F\rangle|^{-s} = \varsigma(s,1-\langle F\rangle) - \varsigma(s,\langle F\rangle) \tag{B.14}$$

where ς is Riemann's ς function. Using $\lim\limits_{s\to 0^+}\varsigma(s,a) = \left(\frac{1}{2}-a\right)$ we find

$$\lim_{s\to 0^+}\int_0^\infty \Delta\rho_c(E,s)\,dE = -\langle F\rangle + \frac{1}{2} . \tag{B.15}$$

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FIGURE CAPTIONS

1. The contours of integration in the E-plane for computing the index. When there is a gap, use the solid curve ϵ. When the gap is small or zero, use the dashed curve e.

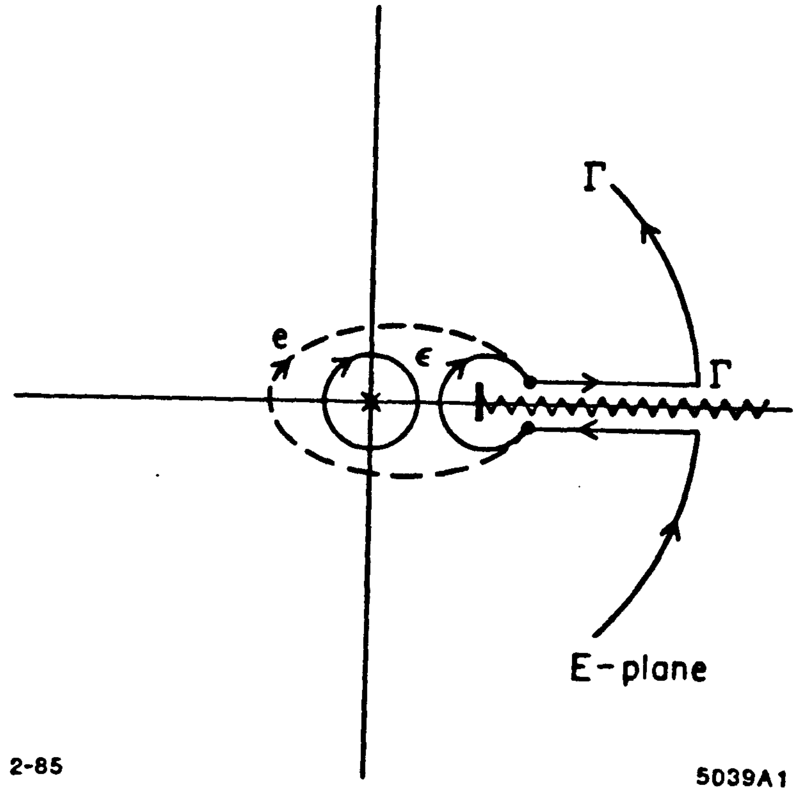

Fig. 1