SLAC – PUB – 3553
January 1985
(T/AS)

TWO–LOOP CORRECTIONS TO THE COSMOLOGICAL CONSTANT*

BRUCE K. SAWHILL

Stanford Linear Accelerator Center
Stanford University, Stanford, California, 94305

ABSTRACT

The two–loop contributions to the induced cosmological constant are computed in a matterless Einstein theory of gravity expanded about a Minkowski background with no cosmological term present in the classical action. The regularization–independent cancellation of the contribution to one–loop order is found not to occur to two–loop order.

Submitted to *Physics Letters B*

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515.

Introduction

The cosmological constant problem is one of the more tenacious problems of modern quantum field theory. In order to account for the observed vanishingly small value one is required, in a perturbative quantum theory of gravity, to fine tune the classical cosmological constant (Λ) against an infinite set of counterterms. Ideally, one would prefer a theory in which there was a symmetry that insured the vanishing of all orders of corrections to the cosmological constant. Another possibility would be a prescription to identify the ground state or background field about which a perturbative theory would give a zero cosmological constant to all orders. The first calculation of the one–loop counterterms in a theory with no classical cosmological term in the action was performed by 't Hooft and Veltman[1] using dimensional regularization and the background field method. They found no contribution to this order because all of the contributing diagrams are quartically divergent and hence are set equal to zero within the context of dimensional regularization. A similar calculation was done later by M. Mueller,[2] who found that the diagrams that contribute to the cosmological constant vanish even when no regularization method is specified. This occurs because the coefficients of the divergent loop integrals sum to zero; in fact, this cancellation occurs for arbitrary space–time dimensionality. The present letter uses the arbitrary–dimension unregulated quantization procedure of M. Mueller. This procedure differs from the formulation of 't Hooft and Veltman in that it takes into account the dependence of the measure on the fluctuating metric.

Quantization

One begins with the action[3,4]

$$I_G = \frac{1}{2\kappa^2}\int d^n x \left[h^{\rho\sigma}h_{\lambda\mu}h_{\kappa\nu} - 2\delta^\sigma_\kappa \delta^\rho_\lambda h_{\mu\nu} - \frac{1}{(n-2)}\, h^{\rho\sigma}h_{\mu\kappa}h_{\lambda\nu}\right]\left(\partial_\rho h^{\mu\kappa}\partial_\sigma h^{\lambda\nu}\right)$$

where $h^{\mu\nu}$ and $h_{\mu\nu}$ are defined by

$$h^{\mu\nu} = g^{1/2}\, g^{\mu\nu}\ , \qquad g = |\det\, g_{\mu\nu}|$$

$$h_{\mu\nu} = g^{-1/2}\, g_{\mu\nu}\ , \qquad h^{\mu\nu}\, h_{\nu\rho} = \delta^\mu_\rho$$

conventions are $(+---)$ for the metric and $\kappa^2 = 32\pi G$. This prescription for defining $h_{\mu\nu}$, common to this paper and all of its references, is called the background field method because one expands

$$h^{\mu\nu} = \eta^{\mu\nu} + \kappa\phi^{\mu\nu}$$

and

$$g_{\mu\nu} = \eta_{\mu\nu} - \kappa\phi_{\mu\nu} + \kappa^2\phi^2_{\mu\nu} - \kappa^3\phi^3_{\mu\nu} + \ldots..$$

where $\eta_{\mu\nu}$ is the classical Minkowski background field about which quantum fluctuations are considered. Quantization proceeds by choosing a gauge-fixing term,

$$I_{GF} = \frac{1}{2\kappa^2}\int d^n x\ F^\alpha\, \eta_{\alpha\beta}\, F^\beta\ ; \qquad F^\alpha = \partial_\mu\, h^{\mu\alpha}\ .$$

This produces a corresponding Faddeev-Popov ghost term,

$$I_{FP} = \int d^n x\, (-h^{\mu\nu}\, \partial_\mu\, c^*_\alpha\, \partial_\nu\, c^\alpha + \partial_\mu\, c^*_\alpha\, \partial_\nu\, h^{\nu\alpha}\, c^\mu)\ ,$$

where c and c^* are ghost fields. There is an extra term in the quantum action that is not present in dimensionally regularized background field quantum gravity

(RBQG). This term comes from the integral of the product of field configurations (the measure) in the path integral. For any quantum field theory, the measure must be gauge invariant. "Gauge invariance" in quantum gravity means general co–ordinate invariance. In gravity, the measure does not transform properly, and hence it must be multiplied by a compensating term in order to preserve co-ordinate invariance. The new compensated measure is

$$\prod_x [h(x)]^{-(n+1)/2} \prod_{\mu\leq\nu}^{n} dh^{\mu\nu}(x) \ .$$

In RBQG, there is no contribution from this new measure compensating term, as its presence induces a $\delta^n(0)$ term, whose momentum transform produces a contribution that is quartically divergent in four space-time dimensions and is set to zero by the rules of dimensional regularization. In regularization independent gravity it is essential to include this term. The compensating product can be exponentiated, producing a new term in the action,

$$I_M = \frac{1}{2}\int d^n x \ A^i_\mu \ h^{\mu\nu} \ A^i_\nu \ \delta_{ij}$$

where the A are $(n+1)$ new auxiliary fields obeying Bose statistics.

Results

By expanding $h^{\mu\nu}$ as prescribed by the rules above, the Feynman rules can be written down directly (see Table 1). Using them, the one–loop contributions are readily found and are shown in Table 2.[2]

Note that the coefficients of the one–loop integrals sum to zero so it is not necessary to perform the integrals themselves. This compelling result inspired the author to calculate the diagrams to two–loop order. The symbolic manipulation program REDUCE 3 was used to calculate the eight diagrams that contribute.

There are a total of 26 diagrams to two–loop $(\mathcal{O}(\kappa^3))$ order, but 18 vanish by virtue of the one loop result. This involved about 25 hours of computer time on the IBM 3081 at SLAC in order to contract and combine approximately eleven million terms. The results are shown in Table 3.

It is clear that there is no complete cancellation of the kind found in the one–loop result, as the leading terms for $n \rightarrow 0$ and $n \rightarrow \infty$ have non-zero coefficients, and hence the total amplitude for arbitrary space-time dimension is non-zero by analyticity. If the total amplitude is factored by integrands, there is no common root shared by all of the coefficients, so the amplitude does not even pass through zero. Although disappointing, it is not surprising, as there is no underlying reason why they should sum to zero. It is an extremely rigid constraint to require that the contributions to the cosmological constant vanish for every point in the phase space of the loop integrals.

There is an additional subtlety involved in this calculation that is not relevant to the one–loop cosmological constant problem. In a two–loop calculation, it is possible to have sub–divergences, hence requiring a prescription for subtractions. In this particular problem, one–loop corrections occur to internal graviton, ghost, and auxiliary field lines. These all produce non–zero coefficients multiplied by different divergent integrals. This makes it necessary to choose a regularization scheme so as to be able to evaluate these integrals and perform appropriate subtractions. This defeats the original idea of a regularization–independent result. The auxiliary–field quantizaiton method fails to eliminate these one–loop divergences in the same fashion as it eliminated the one–loop cosmological constant divergences. Technically, this occurs because the external momenta of the divergent sub–diagrams are non–zero, whereas in the one–loop cosmological constant calculation the incoming momentum was fixed at zero, producing the necessary cancellations. A further difficulty exists in that it is not consistent to use dimensional regularization to regularize sub–graphs in a process that is divergent by six powers of momentum, because the entire process would automatically be set equal to zero by dimensional regularization.

The computer program was checked by calculating the one–loop corrections to the cosmological constant and to the graviton propagator and comparing them to the literature.[2,3,4] Further calculations being considered by the author include using the computer programs developed here to calculate the four-point graviton scattering amplitude to one-loop order within the context of dimensional regularization and the application of supergravity to results already obtained by the method of this letter.

ACKNOWLEDGEMENTS

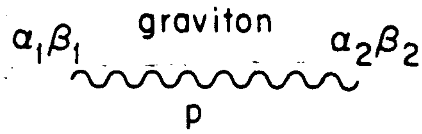
I would like to thank M. Mueller for the original suggestion. Many thanks for enlightening discussions to M. Peskin, M. Namazie, L. Susskind, R.P. Woodard, J. Bagger, D. Nemeschansky, M. Mueller, Ignatios Antoniadis, and S.J. Brodsky.

REFERENCES

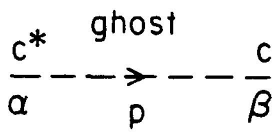
1. G. 't Hooft and M. Veltman, Ann. Inst. Henri Poincare, Vol. XX, No. 1 (1974) 69.
2. M. Mueller, Phys. Lett. **133B** (1983) 385.
3. D. M. Capper and M. A. Namazie, Nucl. Phys. **B142** (1978) 535.
4. D. M. Capper *et al.*, Phys. Rev. **D8** (1973) 4320.

TABLE 1

Feynman Rules

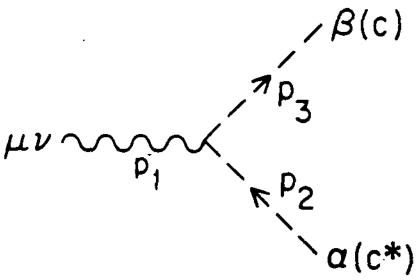


$$D_{\alpha_1\beta_1,\alpha_2\beta_2}(p) = \frac{i}{2p^2}\left(\eta_{\alpha_1\alpha_2}\,\eta_{\beta_1\beta_2}+\eta_{\alpha_1\beta_2}\,\eta_{\beta_1\alpha_2}-\eta_{\alpha_1\beta_1}\,\eta_{\alpha_2\beta_2}\right)$$

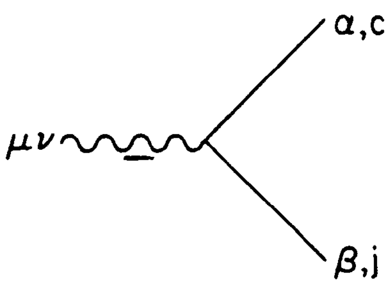


$$G_{\alpha\beta}(p) = -\frac{i\eta_{\alpha\beta}}{p^2}$$

$$\Delta_{ij}^{\alpha\beta}(p) = i\eta_{\alpha\beta}\,\delta_{ij}$$

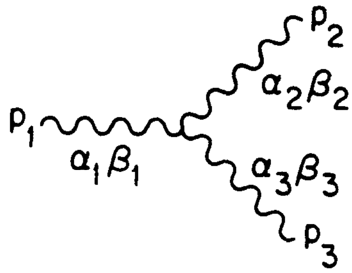


$$V_G^{\mu\nu,\alpha,\beta}(p_1,p_2,p_3) = i\kappa\left[p_2{}^{(\mu}p_3{}^{\nu)}\eta^{\alpha\beta} - \eta^{\alpha(\mu}p_1{}^{\nu)}p_2^{\beta}\right]$$



$$V_a^{\mu\nu,\alpha i,\beta j} = \frac{i}{2}\kappa\left(\eta^{\alpha\mu}\,\eta^{\beta\nu}+\eta^{\alpha\nu}\,\eta^{\beta\mu}\right)\delta^{ij} \quad *$$

3–Graviton Vertex†

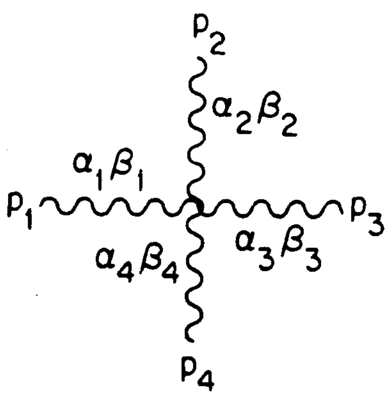


$$V_3^{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3}(p_1,p_2,p_3) = -\frac{i}{2}\kappa\left\{p_2^{\alpha_1}p_3^{\beta_1}\left[\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_2} - \frac{1}{n-2}\eta^{\alpha_2\beta_2}\eta^{\alpha_3\beta_3}\right]\right.$$

$$\left. + 2p_2^{\alpha_3}\eta^{\beta_3\beta_1}\eta^{\alpha_1\beta_2}p_3^{\alpha_2} - 2p_2\cdot p_3\left[\eta^{\alpha_1\beta_2}\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_1} - \frac{1}{n-2}\eta^{\alpha_1\alpha_3}\eta^{\beta_3\beta_1}\eta^{\alpha_2\beta_2}\right]\right\}$$

$$+\,\text{symm. (5)}$$

4–Graviton Vertex†



$$V_4^{\alpha_1\beta_1,\alpha_2\beta_2,\alpha_3\beta_3,\alpha_4\beta_4}(p_1,p_2,p_3,p_4) = \frac{i}{2}\kappa^2\left\{2p_3^{\alpha_1}p_4^{\beta_1}\right.$$

$$\times\left[\eta^{\alpha_2\alpha_3}\eta^{\beta_3\alpha_4}\eta^{\beta_4\beta_2} - \frac{1}{n-2}\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_2}\eta^{\alpha_4\beta_4}\right]$$

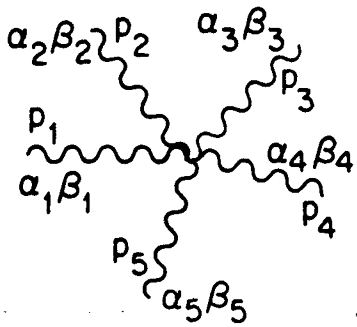
$$+ 2p_3^{\alpha_4}\eta^{\beta_4\beta_2}\eta^{\alpha_2\beta_1}\eta^{\alpha_1\beta_3}p_4^{\alpha_3}$$

$$- p_3\cdot p_4\left[\left(\eta^{\alpha_1\alpha_4}\eta^{\beta_4\beta_2}\eta^{\alpha_2\alpha_3}\eta^{\beta_3\beta_1} + 2\eta^{\alpha_1\beta_4}\eta^{\alpha_4\alpha_3}\eta^{\beta_3\beta_2}\eta^{\alpha_2\beta_1}\right)\right.$$

$$\left.\left. -\frac{1}{n-2}\left(\eta^{\alpha_1\alpha_3}\eta^{\beta_3\beta_1}\eta^{\alpha_2\alpha_4}\eta^{\beta_4\beta_2} + 2\eta^{\alpha_1\alpha_3}\eta^{\beta_3\beta_2}\eta^{\alpha_2\beta_1}\eta^{\alpha_4\beta_4}\right)\right]\right\}$$

$$+\,\text{symm. (23)}$$

5–Graviton Vertex†



$$
\begin{aligned}
V_5^{\alpha_1\beta_1,\ldots,\alpha_5\beta_5}(p_1,\ldots,p_5) &= -\frac{i}{2}\kappa^3 \Big\{ p_4^{\alpha_1} p_5^{\beta_1} \\
&\times \Big[\Big(\eta^{\alpha_2\alpha_4}\eta^{\beta_4\beta_3}\eta^{\alpha_3\alpha_5}\eta^{\beta_5\beta_2} + 2\eta^{\alpha_2\alpha_4}\eta^{\beta_4\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2} \Big) \\
&- \frac{1}{n-2} \Big(\eta^{\alpha_2\alpha_4}\eta^{\beta_4\beta_2}\eta^{\alpha_3\alpha_5}\eta^{\beta_5\beta_3} + 2\eta^{\alpha_2\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2}\eta^{\alpha_4\beta_4} \Big) \Big] \\
&+ 2p_4^{\alpha_5}\eta^{\beta_5\beta_3}\eta^{\beta_3\beta_2}\eta^{\alpha_2\beta_1}\eta^{\alpha_1\beta_4} p_5^{\alpha_4} \\
&- 2p_4\cdot p_5 \Big[\Big(\eta^{\alpha_1\alpha_4}\eta^{\beta_4\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2}\eta^{\alpha_2\beta_1} + \eta^{\alpha_1\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2}\eta^{\alpha_2\alpha_4}\eta^{\beta_4\beta_1} \Big) \\
&- \frac{1}{n-2} \Big(\eta^{\alpha_1\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2}\eta^{\alpha_2\beta_1}\eta^{\alpha_4\beta_4} + \eta^{\alpha_1\alpha_4}\eta^{\beta_4\beta_1}\eta^{\alpha_2\alpha_5}\eta^{\beta_5\beta_3}\eta^{\alpha_3\beta_2} \Big) \Big] \Big\} \\
&+ \text{symm.} \quad (119)
\end{aligned}
$$

---

* In Reference [2], the graviton-auxiliary field vertex has a factor of 4 in the denominator which should be a 2. This does not affect the auxiliary field loop result as that diagram has a symmetry factor of $\frac{1}{2}$ which is not included in the result, hence the two mistakes cancel.

† Symmetrization of the pure graviton vertices is done by permuting the index numbers in all possible ways. For a κ^n vertex, this produces $(n+2)!$ permutations. Also, every α_t, β_t appearing in a given term is symmetrized as follows

$$V_{3,4,5}^{\ldots,\alpha_t\beta_t,\ldots} \equiv \frac{1}{2}\left\{ V_{3,4,5}^{\ldots,\alpha_t\beta_t,\ldots} + V_{3,4,5}^{\ldots,\beta_t\alpha_t,\ldots} \right\}$$

Closed Ghost Loops induce a factor of −1. Greek indices denote space-time, Latin indices denote auxiliary fields. Parentheses about a pair of indices denote symmetrization. All momenta are defined as incoming.

TABLE 2

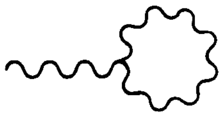
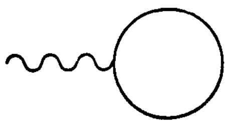
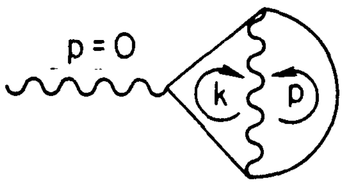
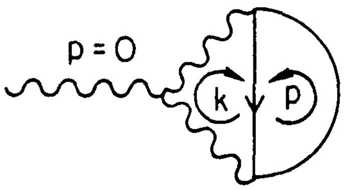
	$\frac{1}{2}(n-1)\,\kappa\eta^{\mu\nu}\int d^n p$
	$\kappa\eta^{\mu\nu}\int d^n p$
	$-\frac{1}{2}(n+1)\,\kappa\eta^{\mu\nu}\int d^n p$

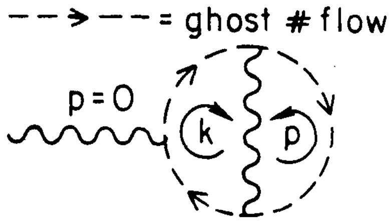
TABLE 3



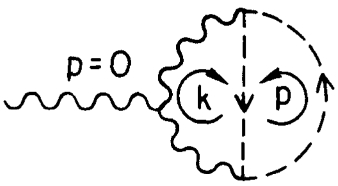
$$-\,i\eta_{\mu\nu}\kappa^3\,\frac{N(N+1)}{2(p+k)^2}$$



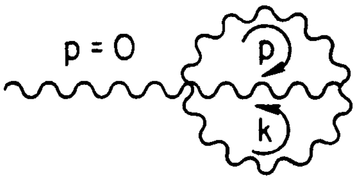
$$\left(\frac{1}{2}\right)\,i\eta_{\mu\nu}\kappa^3\,\frac{(N-2)(N^2-1)}{2Nk^2}$$



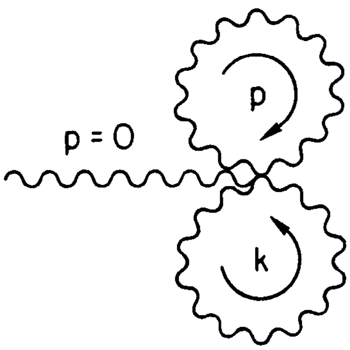
$$(-1)\,i\eta_{\mu\nu}\kappa^3\,\frac{(-Np^2-4k\cdot p-k^2-3p^2)}{2Np^2(p+k)^2}$$



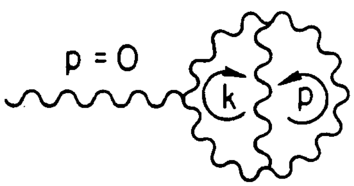
0



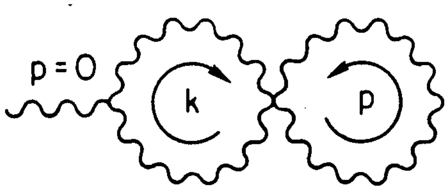
$$\left(\frac{1}{6}\right) i\eta_{\mu\nu}\kappa^3 \{(n-1)(-2n^3-n^2+10n+16)$$
$$\times (k^4+2k^2(k\cdot p)+2p^2(k\cdot p)+p^4)$$
$$+(-4n^4+n^3+21n^2+42n-96)k^2p^2$$
$$+(-2n^4+2n^3+12n^2-24n+48)(k\cdot p)^2\}$$
$$/8p^2k^2\,n(n-2)(p+k)^2$$



$$\left(\frac{1}{8}\right) i\eta_{\mu\nu}\kappa^3 \frac{(p^2+k^2)\,3(n-1)\,(2n^3-n^2-6n-8)}{2p^2k^2\,(n-2)}$$



$$\left(\frac{1}{4}\right) i\eta_{\mu\nu}\kappa^3 \{(p\cdot k)^2(n^3-9n^2+6n+20)$$
$$+(p\cdot k)p^2(2n^3-6n^2+4n-8)$$
$$+2(p\cdot k)k^2n(n-1)^2+p^4(n^3-n^2-8)$$
$$+p^2k^2(2n^3-20)+k^4(n-1)(n^3-n^2-2n-4)$$
$$/(n-2)\}\,/(n-2)^2\,8p^2k^2(p+k)^2$$



$$\left(\frac{1}{6}\right) i\eta_{\mu\nu}\kappa^3 \{(-n^3+2n^2-n+4)p^2k^2$$
$$+2(n-2)(n-1)(k\cdot p)^2-(n-1)k^4\}\,/(p^2k^4n)$$

These results are integrands. The integration $\int d^np\,d^nk$ is assumed. The symmetric integration substitution $\frac{\int d^np\,d^nk\,(p,k)_\mu(p,k)_\nu}{(p,k)=\frac{\eta^{\mu\nu}}{n}((p,k)\cdot(p,k))\int d^np\,d^nk}$ is used for simplifications. Symmetry and ghost factors are enclosed in parentheses at the front of each amplitude. n = number of space-time dimensions.