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ACCELERATION WITHOUT RADIATION*

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ABSTRACT

We study the radiation generated by electric currents in (1) infinite cylinders with longitudinal flow, (2) infinite cylinders with solenoidal flow, and (3) infinite planes. In each case we work out four specific examples, for which the retarded fields can be calculated exactly, and we derive a "Larmor-like" formula for the power radiated, in the limit of infinitesimal cross-section. We then consider sinusoidal currents with finite cross-section, and discover that for certain special frequencies the external fields are zero and there is no radiation. We relate our results to the work of Goedecke and others, and conclude with some remarks on the radiation reaction in these configurations.

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1. INTRODUCTION

When a *point* charge accelerates, it radiates. The power radiated is given by the Larmor formula[1]

$$P = \frac{1}{4\pi\epsilon_0}\,\frac{2e^2}{3c^3}\,a^2 \;, \tag{1}$$

where e is the charge, a is its acceleration, and c is the speed of light. Surprisingly, however, it is possible for an *extended* charge to accelerate without radiating. For example, a nonrotating uniformly charged spherical shell (radius R) will not radiate if its center oscillates sinusoidally at a frequency[2] such that

$$\sin(\omega R/c) = 0 \;,$$

which is to say

$$\omega_j = j\pi(c/R) \;, \quad j = 0, 1, 2, \ldots \;. \tag{2}$$

In fact, any superposition of such oscillations is radiationless;[3] if the position of the center of the sphere is given by

$$\vec{r}(t) = \sum_{j=0}^{\infty}\left[\vec{a}_j \cos(\omega_j t) + \vec{b}_j \sin(\omega_j t)\right] \;, \tag{3}$$

for constant vectors $\vec{a}_j$ and $\vec{b}_j$, the sphere will not radiate. Since any periodic function can be expanded in such a Fourier series, it follows that the sphere does not radiate as long as its motion has period $2R/c$ (the time it takes light to cross a diameter). Similarly, a spherical shell which rotates sinusoidally about a diameter will not radiate if its frequency is such that[4]

$$j_1(\omega R/c) = 0 \tag{4}$$

where $j_1(z)$ is the first-order spherical Bessel function.

We would like to study the phenomenon of radiationless motion in greater detail. Unfortunately, there are very few nontrivial localized configurations for which one can calculate the electromagnetic fields exactly. In this paper, therefore, we examine three special classes of nonlocalized currents: infinite cylindrical "pipes" with longitudinal flow (Section 2); infinite solenoids (Section 3); and infinite planes which carry uniform surface currents (Section 4). For each geometry we first analyse the case of infinitesimal cross-section, working out four specific examples and deriving a "Larmor" formula for the power radiated. We then consider sinusoidal currents with *finite* cross-section, obtaining an infinite set of frequencies which do not radiate. In Section 5 we show that our results are consistent with Goedecke's general criterion for radiationless motion,[5] and in Section 6 we conclude with some remarks about the radiation reaction force in these configurations.

2. THE INFINITE PIPE

Suppose an infinite straight wire, lying along the z axis, carries a time-dependent current $I(t)$. We assume that the wire is electrically neutral,[6] so the scalar potential is zero, while the retarded vector potential is given by[7]

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\,\hat{z}\int_{-\infty}^{\infty}\frac{I(t-\rho/c)}{\rho}\,dz = \frac{\mu_0}{2\pi}\,\hat{z}\int_{r}^{\infty}\frac{I(t-\rho/c)}{\sqrt{\rho^2-r^2}}\,d\rho\ , \qquad (5)$$

where (see Fig. 1)

$$\rho = \sqrt{r^2+z^2}\ . \qquad (6)$$

Since $\vec{A}$ has only one component, we'll write $\vec{A} = A\hat{z}$. The fields, then, are

$$\vec{E}(r,t) = -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial A}{\partial t}\,\hat{z}\ ; \quad \vec{B}(r,t) = \vec{\nabla}\times\vec{A} = -\frac{\partial A}{\partial r}\,\hat{\phi}\ . \qquad (7)$$

The Poynting vector is

$$\vec{S}(r,t) = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = -\frac{1}{\mu_0}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial r}\right)\hat{r} \ , \tag{8}$$

and hence the energy per unit time passing out through a cylinder of radius r and length L is

$$P_r(t) = \int \vec{S}\cdot d\vec{a} = -\frac{1}{\mu_0}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial r}\right) 2\pi r L \ . \tag{9}$$

To get the total power radiated we take the limit of P_r as $r \to \infty$. However, there is a delicate point here: because of the retardation, as we go farther and farther from the wire (at a given time t), we are sampling fields that *left the wire* at earlier and earlier times. If we want the energy that left the wire at a fixed time t_0, we must "follow the fields out" to infinity — that is, we want the limit of P_r with $t_0 = t - r/c$ (rather than t itself) held constant. Thus the power per unit length radiated from the wire at time t_0 is given by

$$\mathcal{P} = \frac{1}{L}\lim_{r\to\infty} P_r\left(t_0 + \frac{r}{c}\right) \ , \quad \text{with } t_0 \text{ held constant} \ . \tag{10}$$

Example 1.[8] Suppose a constant current I_0 is turned on abruptly at time $t = 0$:

$$I(t) = \begin{cases} 0, & t \le 0 \\ I_0, & t > 0 \end{cases} \ .$$

For $t > r/c$ (i.e. $t_0 > 0$) only points on the wire out to

$$z_0(r,t) = \sqrt{(ct)^2 - r^2} \tag{11}$$

contribute, and we have

$$A(r,t) = \frac{\mu_0 I_0}{2\pi} \int_0^{z_0} \frac{1}{\sqrt{z^2+r^2}}\, dz = \frac{\mu_0 I_0}{2\pi}\, \ln\left(\frac{z_0+ct}{r}\right) \;;$$

$$\vec{E}(r,t) = -\frac{\mu_0 I_0}{2\pi}\,\frac{c}{z_0}\,\hat{z}\;;\quad \vec{B}(r,t) = \frac{\mu_0 I_0}{2\pi}\,\frac{ct}{r z_0}\,\hat{\phi}\;;\quad \mathcal{P} = \frac{\mu_0}{4\pi}\,\frac{I_0^2}{t_0}\;.$$

Example 2. If the current increases linearly,

$$I(t) = \begin{cases} 0, & t \le 0 \\ \alpha t, & t \ge 0 \end{cases}\;.$$

We find (for $t > r/c$):

$$A(r,t) = \frac{\mu_0 \alpha}{2\pi} \int_0^{z_0} \frac{(t - \sqrt{r^2+z^2}/c)}{\sqrt{r^2+z^2}}\, dz = \frac{\mu_0\alpha}{2\pi}\left[t \ln\left(\frac{ct+z_0}{r}\right) - \frac{z_0}{c}\right]\;;$$

$$\vec{E}(r,t) = -\frac{\mu_0 \alpha}{2\pi}\,\ln\left(\frac{ct+z_0}{r}\right)\hat{z}\;;\quad \vec{B}(r,t) = \frac{\mu_0\alpha}{2\pi}\,\frac{z_0}{cr}\,\hat{\phi}\;;\quad \mathcal{P} = \frac{\mu_0\alpha^2}{\pi}\,t_0\;.$$

Example 3. If the current is a sudden burst at $t = 0$,

$$I(t) = q_0\,\delta(t)\;,$$

then (for $t > r/c$):

$$A(r,t) = \frac{\mu_0 q_0}{2\pi} \int_r^{\infty} \frac{\delta(t-\rho/c)}{\sqrt{\rho^2 - r^2}}\, d\rho = \frac{\mu_0 q_0 c}{2\pi z_0}\;;$$

$$\vec{E}(r,t) = \frac{\mu_0 q_0}{2\pi}\,\frac{c^3 t}{z_0^3}\,\hat{z}\;;\quad \vec{B}(r,t) = -\frac{\mu_0 q_0}{2\pi}\,\frac{cr}{z_0^3}\,\hat{\phi}\;;\quad \mathcal{P} = \frac{\mu_0}{\pi}\left(\frac{q_0}{4}\right)^2\frac{1}{t_0^3}\;.$$

Example 4. For a sinusoidal current,

$$I(t) = I_0 \ \sin(\omega t) \ , \tag{12}$$

the results are as follows:[9]

$$\begin{aligned} A(r,t) &= \frac{\mu_0 I_0}{2\pi} \int_r^\infty \frac{\sin\omega(t-\rho/c)}{\sqrt{\rho^2 - r^2}} \, d\rho \\ &= -\frac{\mu_0 I_0}{4} \left[\sin(\omega t) N_0 \left(\frac{\omega r}{c}\right) + \cos(\omega t) J_0 \left(\frac{\omega r}{c}\right) \right] \ , \end{aligned} \tag{13}$$

where J_0 and N_0 are the Bessel and Neumann functions of order zero;

$$\vec{E}(r,t) = \frac{\mu_0 I_0 \omega}{4} \left[\cos(\omega t) N_0 \left(\frac{\omega r}{c}\right) - \sin(\omega t) J_0 \left(\frac{\omega r}{c}\right) \right] \ \hat{z} \ ; \tag{14}$$

$$\vec{B}(r,t) = -\frac{\mu_0 I_0 \omega}{4c} \left[\sin(\omega t) N_1 \left(\frac{\omega r}{c}\right) + \cos(\omega t) J_1 \left(\frac{\omega r}{c}\right) \right] \ \hat{\phi} \ ; \tag{15}$$

and[10]

$$\mathcal{P} = \frac{\mu_0 \omega I_0^2}{8} \ [1 + \sin(2\omega t_0)] \ . \tag{16}$$

If all we are interested in is the total power radiated, there is no need to calculate the fields exactly; what we require are the "radiation fields" — the terms in $\vec{E}$ and $\vec{B}$ which go like $1/\sqrt{r}$ at large distances from the wire.[11] To pick out the radiation term in the vector potential, we make the substitution

$$u = (\rho - r)/c \tag{17}$$

in Eq. (5):

$$A(r,t) = \frac{\mu_0}{2\pi} \int_0^\infty \frac{I(t_0 - u)}{\sqrt{u}\,\sqrt{u + 2r/c}} \, du \ . \tag{18}$$

Recall that

$$t_0 = t - r/c \tag{19}$$

is to be held constant as we send $r \to \infty$. Expanding the radical,[12] we obtain a series of the form

$$\frac{1}{r^{1/2}}(\ ) + \frac{1}{r^{3/2}}(\ ) + \frac{1}{r^{5/2}}(\ ) + \dots \ . \tag{20}$$

The first term is the one we want:

$$A_{\text{rad}}(r,t) = \frac{\mu_0}{2\pi}\sqrt{\frac{c}{2r}} \int_0^\infty \frac{I(t_0 - u)}{\sqrt{u}}\, du \ . \tag{21}$$

Since the only time-dependence in A_{rad} is carried by t_0, it follows that

$$\vec{E}_{\text{rad}}(r,t) = -\frac{\mu_0}{2\pi}\sqrt{\frac{c}{2r}}\, \hat{z} \int_0^\infty \frac{\dot{I}(t_0 - u)}{\sqrt{u}}\, du \ , \tag{22}$$

where the dot denotes differentiation. A_{rad} depends on r both explicitly (through the $1/\sqrt{r}$ in front of the integral) and implicitly (through t_0), so there are two terms in $\partial A_{\text{rad}}/\partial r$. However, the first goes like $r^{-3/2}$, so

$$\vec{B}_{\text{rad}}(r,t) = \frac{\mu_0}{2\pi c}\sqrt{\frac{c}{2r}}\, \hat{\phi} \int_0^\infty \frac{\dot{I}(t_0 - u)}{\sqrt{u}}\, du = \frac{1}{c}(\hat{r} \times \vec{E}_{\text{rad}}) \ , \tag{23}$$

and the power radiated, per unit length, is

$$\mathcal{P} = \frac{\mu_0}{4\pi}\ [Q(t_0)]^2 \ , \tag{24}$$

where

$$Q(t) = \int_0^\infty \frac{\dot{I}(t - u)}{\sqrt{u}}\, du \ . \tag{25}$$

Equation (24) is the analog, in this geometry, to the Larmor formula — or rather, to the Liénard formula, since no nonrelativistic approximation has been invoked. The reader is invited to check Eq. (24) for each of the examples considered earlier. Notice that a steady current ($\dot{I} = 0$) does not radiate (of course); evidently this is the only radiationless case, just as constant velocity is the only radiationless motion for a point charge.

Suppose now that our infinite wire has a nonzero radius R, and the current $I(t)$ is uniformly distributed over its surface, so that the surface current density is

$$K(t) = \frac{I(t)}{2\pi R} \ . \tag{26}$$

The vector potential is

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\,\hat{z} \int_0^{2\pi} \int_{-\infty}^{\infty} \frac{K(t-\rho/c)}{\rho}\, dz\, R\, d\phi \ , \tag{27}$$

where (see Fig. 2)

$$\rho = \sqrt{\xi^2 + z^2} \ , \tag{28}$$

and

$$\xi = \sqrt{R^2 + r^2 - 2Rr\cos\phi} \ . \tag{29}$$

Consider a sinusoidal current,

$$I(t) = I_0 \sin(\omega t) \ . \tag{30}$$

The z-integral is performed as before:[9]

$$A(r,t) = \frac{\mu_0 I_0}{4\pi^2} \int_0^{2\pi} \int_\xi^\infty \frac{\sin\omega(t-\rho/c)}{\sqrt{\rho^2-\xi^2}}\, d\rho\, d\phi$$
$$= -\frac{\mu_0 I_0}{8\pi} \int_0^{2\pi} \left[\sin(\omega t) N_0\left(\frac{\omega\xi}{c}\right) + \cos(\omega t) J_0\left(\frac{\omega\xi}{c}\right)\right] d\phi\ . \tag{31}$$

The ϕ-integral can also be evaluated exactly:[13]

$$A(r,t) = -\frac{\mu_0 I_0}{4} J_0\left(\frac{\omega R}{c}\right)\left[\sin(\omega t) N_0\left(\frac{\omega r}{c}\right) + \cos(\omega t) J_0\left(\frac{\omega r}{c}\right)\right]\ . \tag{32}$$

Apart from an overall constant factor of $J_0(\omega R/c)$ — which reduces to 1 in the limit $R \to 0$ — this is exactly what we found in Example 4 for the wire of zero radius! The fields will be multiplied by the same factor, and the power radiated at time t_0 can be read from Eq. (16):

$$\mathcal{P} = \frac{\mu_0 \omega I_0^2}{8}\left[J_0\left(\frac{\omega R}{c}\right)\right]^2 [1+\sin(2\omega t_0)]\ . \tag{33}$$

(The same result holds for a current $I(t) = I_0\cos(\omega t)$, except that the plus sign becomes a minus sign.) Evidently the wire will not radiate if the current is sinusoidal with a frequency ω such that $(\omega R/c)$ is a zero of $J_0(z)$:

$$\omega_j = \lambda_j(c/R)\ , \quad j = 1,2,3,\dots\ , \quad \text{where } J_0(\lambda_j) = 0\ . \tag{34}$$

Notice that for these special frequencies the *fields are precisely zero everywhere outside the wire.* There is, as it were, perfect destructive interference in all directions. Because the fields obey the Superposition Principle, it follows that any current of the form

$$I(t) = \sum_{j=1}^{\infty} \left[a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\right] \; , \tag{35}$$

with ω_j given by Eq. (34), will generate no external fields, and produce no radiation.

This raises an intriguing mathematical question: what is the most general function $I(t)$ that can be expanded in such a "Fourier-like" series? It is not a *true* Fourier series, of course, because the frequencies ω_j are not integer multiples of a fundamental, but rather are proportional to the zeros of $J_0(z)$. Nevertheless, it turns out that the functions $\{\cos\omega_j t,\ \sin\omega_j t\}$ are complete[14] on the interval $(-R/c,\ R/c)$: *any* well-behaved function on this interval can be written in the form (35). (In a related paper[15] we show how to determine the coefficients a_j and b_j, for a given function.) Unlike a Fourier series, however, the extension *outside* this interval does not generate a periodic function. For example, the step function

$$I(t) = \begin{cases} -1, & \text{for } -R/c < t < 0 \\ +1, & \text{for } 0 < t < R/c \end{cases} \; , \tag{36}$$

expanded according to (35), extrapolates as shown in Fig. 3. What this means is that a nonradiating current can be anything whatever, on an interval of length $T = 2R/c$ (the time it takes light to cross a diameter), provided $I(t)$ has just the right matching form [given by (35)] for all earlier and later times — in fact, the electric and magnetic fields outside the pipe will vanish identically.[16]

3. THE INFINITE SOLENOID

Suppose now that the current flows *around* the pipe, rather than *along* it. The surface current density is

$$\vec{K}(t) = K(t)\,\hat{\phi}\ . \tag{37}$$

In this case the vector potential is given by

$$\vec{A}(r,t) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(t-\rho/c)}{\rho}\,dz\,R\,d\phi = \frac{\mu_0 R}{4\pi}\hat{\phi}\int_0^{2\pi}\int_{-\infty}^{\infty}\frac{K(t-\rho/c)}{\rho}\cos\phi\,dz\,d\phi\ , \tag{38}$$

where (see Fig. 2)

$$\rho = \sqrt{\xi^2+z^2} \qquad \text{and} \qquad \xi = \sqrt{R^2+r^2-2Rr\cos\phi}\ . \tag{39}$$

This time [writing $\vec{A}(r,t) = A(r,t)\hat{\phi}$], we have

$$\vec{E}(r,t) = -\frac{\partial A}{\partial t}\,\hat{\phi}\ ; \quad \vec{B}(r,t) = \frac{1}{r}\,\frac{\partial}{\partial r}\,(rA)\hat{z}\ ; \tag{40}$$

$$\vec{S}(r,t) = -\frac{1}{\mu_0 r}\left(\frac{\partial A}{\partial t}\right)\frac{\partial}{\partial r}(rA)\hat{r}\ ; \quad P_r(t) = -\frac{2\pi}{\mu_0}\left(\frac{\partial A}{\partial t}\right)\frac{\partial}{\partial r}(rA)L\ . \tag{41}$$

As before, the power radiated per unit length is

$$\mathcal{P} = \frac{1}{L}\lim_{r\to\infty} P_r\left(t_0+\frac{r}{c}\right)\ , \quad \text{with } t_0 \text{ held constant}\ . \tag{42}$$

Once again, we begin with the limiting case $R \to 0$ — physically, this represents a string of infinitesimal magnetic dipoles. Expanding the integrand to first order in R, and performing the ϕ integral, we find

$$A(r,t) = \frac{\mu_0 r}{4\pi}(\pi R^2) \int_{-\infty}^{\infty} \left\{ \frac{\dot{K}(t - \sqrt{r^2+z^2}/c)}{c(r^2+z^2)} + \frac{K(t-\sqrt{r^2+z^2}/c)}{(r^2+z^2)^{3/2}} \right\} dz \ . \quad (43)$$

In terms of the magnetic dipole moment per unit length,

$$M(t) = \pi R^2 \, K(t) \ , \quad (44)$$

then,

$$\begin{aligned} A(r,t) &= \frac{\mu_0 r}{2\pi} \int_r^{\infty} \left[\frac{\dot{M}(t-\rho/c)}{c\rho} + \frac{M(t-\rho/c)}{\rho^2} \right] \frac{d\rho}{\sqrt{\rho^2 - r^2}} \\ &= -\frac{\mu_0 r}{2\pi} \int_r^{\infty} \frac{d}{d\rho} \left[\frac{M(t-\rho/c)}{\rho} \right] \frac{d\rho}{\sqrt{\rho^2-r^2}} \ . \end{aligned} \quad (45)$$

Example 1. Suppose the current is turned on abruptly at time $t = 0$:

$$M(t) = \begin{cases} 0, & t \le 0 \\ M_0, & t > 0 \end{cases} \ .$$

Then

$$\dot{M}(t) = M_0 \, \delta(t) \ ,$$

and (for $t > r/c$)

$$A(r,t) = \frac{\mu_0 r}{2\pi} M_0 \left\{ \int_r^{\infty} \frac{\delta(t-\rho/c)}{c\rho\sqrt{\rho^2-r^2}} \, d\rho + \int_r^{ct} \frac{1}{\rho^2\sqrt{\rho^2-r^2}} \, d\rho \right\} = \frac{\mu_0 M_0}{2\pi} \, \frac{ct}{r z_0} \ ,$$

where, as before,

$$z_0 = \sqrt{(ct)^2 - r^2} \ .$$

It follows that

$$\vec{E}(r,t) = \frac{\mu_0 M_0}{2\pi}\,\frac{cr}{z_0^3}\,\hat{\phi}\ ; \quad \vec{B}(r,t) = \frac{\mu_0 M_0}{2\pi}\,\frac{ct}{z_0^3}\,\hat{z}\ ; \quad \mathcal{P} = \frac{\mu_0}{\pi}\left(\frac{M_0}{4c}\right)^2 \frac{1}{t_0^3}\ .$$

Example 2. If the current increases linearly with time:

$$M(t) = \begin{cases} 0, & t \leq 0 \\ \alpha t, & t \geq 0 \end{cases}\ ,$$

we find (for $t > r/c$)

$$A(r,t) = \left(\frac{\mu_0\alpha}{2\pi}\right)\frac{z_0}{rc}\ ; \quad \vec{E}(r,t) = -\left(\frac{\mu_0\alpha}{2\pi}\right)\frac{ct}{z_0 r}\,\hat{\phi}\ ;$$

$$\vec{B}(r,t) = -\left(\frac{\mu_0\alpha}{2\pi}\right)\frac{1}{z_0 c}\,\hat{z}\ ; \quad \mathcal{P} = \frac{\mu_0\alpha^2}{4\pi c^2}\,\frac{1}{t_0}\ .$$

Example 3. If the current consists of a sudden burst at $t = 0$:

$$M(t) = \beta\delta(t)\ ,$$

then (for $t > r/c$)

$$A(r,t) = -\left(\frac{\mu_0\beta}{2\pi}\right)\frac{rc}{z_0^3}\ ; \quad \vec{E}(r,t) = -\left(\frac{\mu_0\beta}{2\pi}\right)\frac{3c^3 rt}{z_0^5}\,\hat{\phi}\ ;$$

$$\vec{B}(r,t) = -\left(\frac{\mu_0\beta c}{2\pi}\right)\frac{[r^2 + 2(ct)^2]}{z_0^5}\,\hat{z}\ ; \quad \mathcal{P} = \frac{9}{64}\,\frac{\mu_0\beta^2}{\pi c^2}\,\frac{1}{t_0^5}\ .$$

Example 4. For a sinusoidal current,

$$M(t) = M_0\ \sin(\omega t)\ , \tag{46}$$

the results are as follows:[17]

$$A(r,t) = -\frac{\mu_0 M_0\omega}{4c}\left[\sin(\omega t)N_1\left(\frac{\omega r}{c}\right) + \cos(\omega t)J_1\left(\frac{\omega r}{c}\right)\right]\ ; \tag{47}$$

$$\vec{E}(r,t) = \frac{\mu_0 M_0 \omega^2}{4c}\left[\cos(\omega t) N_1\left(\frac{\omega r}{c}\right) - \sin(\omega t) J_1\left(\frac{\omega r}{c}\right)\right]\,\hat{\phi}\ ; \tag{48}$$

$$\vec{B}(r,t) = -\frac{\mu_0 M_0 \omega^2}{4c^2}\left[\sin(\omega t) N_0\left(\frac{\omega r}{c}\right) + \cos(\omega t) J_0\left(\frac{\omega r}{c}\right)\right]\,\hat{z}\ ; \tag{49}$$

$$\mathcal{P} = \frac{\mu_0 M_0^2 \omega^3}{8c^2}\left[1 - \sin(2\omega t_0)\right]\ . \tag{50}$$

These results are strikingly similar to those for the infinite wire carrying a longitudinal sinusoidal current, with the roles of $\vec{E}$ and $\vec{B}$ reversed.[18]

To obtain the "Larmor" formula for the power radiated in this configuration, we proceed as before. Let

$$u = (\rho - r)/c\ , \tag{51}$$

and remember that

$$t_0 = t - r/c \tag{52}$$

is to be held constant as $r \to \infty$. Expanding the integrand in (45), and keeping only the term in $1/\sqrt{r}$, we find

$$A_{\text{rad}}(r,t) = \frac{\mu_0}{2\pi}\,\frac{1}{\sqrt{2cr}}\int_0^\infty \frac{\dot{M}(t_0 - u)}{\sqrt{u}}\,du\ . \tag{53}$$

It follows that

$$\vec{E}_{\text{rad}}(r,t) = -\frac{\mu_0}{2\pi\sqrt{2cr}}\,\hat{\phi}\int_0^\infty \frac{\ddot{M}(t_0 - u)}{\sqrt{u}}\,du\ , \tag{54}$$

$$\vec{B}_{\text{rad}}(r,t) = -\frac{\mu_0}{2\pi c\sqrt{2cr}}\,\hat{z}\int_0^\infty \frac{\ddot{M}(t_0 - u)}{\sqrt{u}}\,du = \frac{1}{c}\left(\hat{r} \times \vec{E}_{\text{rad}}\right)\ , \tag{55}$$

and the power radiated at time t_0 is, as before,

$$\mathcal{P} = \frac{\mu_0}{4\pi} \left[Q(t_0)\right]^2 , \tag{56}$$

where in this case

$$Q(t) = \frac{1}{c} \int_0^{\infty} \frac{\ddot{M}(t-u)}{\sqrt{u}} \, du . \tag{57}$$

Equation (56) is the analog to the Larmor formula for radiation from an infinitely long solenoid of infinitesimal diameter. The reader is invited to check that it reproduces the final results in Examples 1-4. Notice that a steady current ($\dot{M} = 0$) does not radiate (of course); nor (surprisingly) does a current which increases linearly ($\ddot{M} = 0$) for all time.[19]

Suppose now that our infinite solenoid has a nonzero radius R, and carries a sinusoidal surface current:

$$K(t) = \left(\frac{M_0}{\pi R^2}\right) \sin(\omega t) . \tag{58}$$

Putting this into Eq. (38), and performing the z-integral:[9]

$$\begin{aligned} A(r,t) &= \frac{\mu_0 M_0}{2\pi^2 R} \int_0^{2\pi} \left\{ \int_{\xi}^{\infty} \frac{\sin \omega(t-\rho/c)}{\sqrt{\rho^2 - \xi^2}} \, d\rho \right\} \cos\phi \, d\phi \\ &= -\frac{\mu_0 M_0}{2\pi R} \int_0^{\pi} \left[\sin(\omega t) N_0 \left(\frac{\omega \xi}{c}\right) + \cos(\omega t) J_0 \left(\frac{\omega \xi}{c}\right) \right] \cos\phi \, d\phi . \end{aligned} \tag{59}$$

This integral can be done exactly;[20] the result is

$$A(r,t) = -\frac{\mu_0 M_0}{2R} J_1 \left(\frac{\omega R}{c}\right) \left[\sin(\omega t) N_1 \left(\frac{\omega r}{c}\right) + \cos(\omega t) J_1 \left(\frac{\omega r}{c}\right) \right] . \tag{60}$$

Apart from an overall constant factor $(2c/\omega R) J_1(\omega R/c)$ — which reduces to 1 in the limit $R \to 0$ — this is identical to the corresponding result for the solenoid of infinitesimal radius [Eq. (47)]. The fields are likewise multiplied by this factor, and we may read the power radiated, at time t_0, from Eq. (50):

$$\mathcal{P} = \frac{\mu_0 \omega}{8} \left[\frac{M_0}{R} \, J_1 \, \left(\frac{\omega R}{c} \right) \right]^2 [1 - \sin(2\omega t_0)] \ . \tag{61}$$

(The same result holds for $M(t) = M_0 \cos(\omega t)$, except that the minus sign becomes a plus sign.) Evidently the solenoid will not radiate if the current is sinusoidal with a frequency such that $(\omega R/c)$ is a zero of $J_1(z)$:

$$\omega_j = \lambda_j (c/R) \ , \quad j = 0, 1, 2, \dots \ , \quad \text{where } J_1(\lambda_j) = 0 \ . \tag{62}$$

(Note that $z = 0$ is a zero of $J_1(z)$ — we'll call it λ_0.) At these special frequencies *the fields are precisely zero everywhere outside the solenoid.* It follows that any current of the form

$$M(t) = \sum_{j=0}^{\infty} \left[a_j \cos(\omega_j t) + b_j \sin(\omega_j t) \right] \ , \tag{63}$$

with ω_j given by Eq. (62), will generate no external fields and produce no radiation.

Once again, this raises an interesting mathematical question: what is the most general function $M(t)$ that can be expanded in such a "Fourier-like" series? As before, it turns out that the functions $\{\cos(\omega_j t),\ \sin(\omega_j t)\}$ are complete[14] on the interval $(-R/c,\ R/c)$: *any* well-behaved function on this interval can be written in the form (63). In a related paper[15] we show how to evaluate the coefficients a_j and b_j, and examine the behavior of the series outside the interval $(-R/c,\ R/c)$.[21]

4. THE INFINITE PLANE

Suppose the y-z plane carries a time-dependent surface current

$$\vec{K}(t) = K(t)\ \hat{z}\ . \tag{64}$$

The plane is electrically neutral, so the scalar potential is zero, and the vector potential at a distance x from the plane is given by

$$\vec{A}(x,t) = \frac{\mu_0}{4\pi}\ \hat{z} \int_0^\infty \frac{K(t-\rho/c)}{\rho}\ 2\pi r\ dr = \frac{\mu_0}{2}\ \hat{z} \int_x^\infty K(t-\rho/c)\ d\rho\ , \tag{65}$$

where (see Fig. 4)

$$\rho = \sqrt{x^2 + r^2}\ . \tag{66}$$

The fields are

$$\vec{E}(x,t) = -\frac{\partial A}{\partial t}\ \hat{z}\ ; \quad \vec{B}(x,t) = -\frac{\partial A}{\partial x}\ \hat{y}\ . \tag{67}$$

(As before, we write $\vec{A} = A\hat{z}$.) The Poynting vector is

$$\vec{S}(x,t) = -\frac{1}{\mu_0}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial x}\right)\ \hat{x} \tag{68}$$

and the energy per unit time passing through a surface of area a at a height x above the plane is

$$P_x(t) = -\frac{1}{\mu_0}\left(\frac{\partial A}{\partial t}\right)\left(\frac{\partial A}{\partial x}\right)\ a\ . \tag{69}$$

The same energy, of course, passes through a symmetrically located area below the plane, so the total power radiated per unit area, at time t_0 is

$$\mathcal{P} = \frac{2}{a} \lim_{x\to\infty} P_x\left(t_0 + \frac{x}{c}\right) , \quad \text{with } t_0 \text{ held constant} . \tag{70}$$

Example 1. Suppose a constant current K_0 is turned on abruptly at time $t = 0$:

$$K(t) = \begin{cases} 0, & t \leq 0 \\ K_0, & t > 0 \end{cases} .$$

For $t > x/c$, we find

$$A(x,t) = \frac{\mu_0 K_0}{2}\,(ct - x) \; ; \quad \vec{E}(x,t) = -\frac{\mu_0 K_0 c}{2}\,\hat{z} \; ;$$

$$\vec{B}(x,t) = \frac{\mu_0 K_0}{2}\,\hat{y} \; ; \quad \mathcal{P} = \frac{\mu_0 K_0^2 c}{2} .$$

Example 2. If the current increases linearly,

$$K(t) = \begin{cases} 0, & t \leq 0 \\ \alpha t, & t \geq 0 \end{cases} ,$$

then, for $t > x/c$

$$A(x,t) = \frac{\mu_0 \alpha}{4c}\,(ct - x)^2 \; ; \quad \vec{E}(x,t) = -\frac{\mu_0 \alpha}{2}\,(ct - x)\hat{z} \; ;$$

$$\vec{B}(x,t) = \frac{\mu_0 \alpha}{2c}\,(ct - x)\hat{y} \; ; \quad \mathcal{P} = \frac{\mu_0 c}{2}\,(\alpha t_0)^2 .$$

Example 3. If

$$K(t) = \beta\,\delta(t) ,$$

then

$$A(x,t) = \frac{\mu_0 \beta c}{2}\,\theta(ct - x) \; ; \quad \vec{E}(x,t) = -\frac{\mu_0 \beta c^2}{2}\,\delta(ct - x)\hat{z} \; ;$$

$$\vec{B}(x,t) = \frac{\mu_0 \beta c}{2}\,\delta(ct - x)\hat{y} \; ; \quad \mathcal{P} = \frac{\mu_0 \beta^2 c}{2}\,[\delta(t_0)]^2 .$$

(The latter is of formal interest, at best, since the square of a delta-function is undefined.)

Example 4. To obtain the vector potential for a sinusoidal current, we are obliged to turn the current off at some time in the distant past:

$$K(t) = \begin{cases} K_0 \sin(\omega t), & t > -T \\ 0, & t \leq -T \end{cases} \quad . \tag{71}$$

Then

$$A(x,t) = \frac{\mu_0 K_0 c}{2\omega} \left[\cos(\omega T) - \cos\omega(t - x/c)\right] \ . \tag{72}$$

Because the cutoff appears only as an additive constant, it does not affect the fields:

$$\vec{E}(x,t) = -\frac{\mu_0 K_0 c}{2} \sin\omega(t - x/c)\,\hat{z} \ ;$$

$$\vec{B}(x,t) = \frac{\mu_0 K_0}{2} \sin\omega(t - x/c)\hat{y} \ ; \tag{73}$$

$$\mathcal{P} = \frac{\mu_0 K_0^2 c}{2} \sin^2(\omega t_0) \ . \tag{74}$$

To derive the "Larmor" formula for radiation from a plane, we make the usual substitution

$$u = (\rho - x)/c \tag{75}$$

and remember that

$$t_0 = t - x/c \tag{76}$$

will be held constant as $x \to \infty$. In terms of these variables, Eq. (65) becomes

$$A(x,t) = \frac{\mu_0 c}{2} \int_0^\infty K(t_0 - u)\, du \ . \tag{77}$$

It follows that

$$\vec{E}(x,t) = -\frac{\mu_0 c}{2}\,\hat{z}\int_0^\infty \frac{\partial}{\partial t}\,[K(t_0-u)]\,du\ .$$

Now

$$\frac{\partial}{\partial t}\,K(t_0-u) = \frac{\partial}{\partial t_0}\,K(t_0-u) = -\frac{\partial}{\partial u}\,K(t_0-u)\ . \tag{78}$$

The integral can now be done, and we are left with the surprisingly simple result[22]

$$\vec{E}(x,t) = -\frac{\mu_0 c}{2}\,K(t_0)\,\hat{z}\ . \tag{79}$$

Similarly,

$$\vec{B}(x,t) = \frac{\mu_0}{2}\,K(t_0)\,\hat{y} = \frac{1}{c}\,(\hat{x}\times\vec{E})\ . \tag{80}$$

Thus the Poynting vector is

$$\vec{S}(x,t) = \frac{\mu_0 c}{4}\,[K(t_0)]^2\,\hat{x}\ , \tag{81}$$

and the power radiated, per unit area, is

$$\mathcal{P} = \frac{\mu_0 c}{2}\,[K(t_0)]^2\ . \tag{82}$$

This is the "Larmor" formula for radiation from an infinite plane. Notice that — unlike the spherical and cylindrical cases — we never had to take the limit ($x \to \infty$ with t_0 held constant); Eqs. (77), (79), (80), and (81) are *exact*, and the *same* power (82) passes through every surface, on its way out to infinity. The reader is invited to check these formulas against the results in Examples 1-4.

Consider now a *pair* of planes, one at $x = R$ and one at $x = -R$, each carrying a surface current $K(t)/2$. (This is the analog to the spherical shell, the pipe, and the solenoid of nonzero radius.) We simply replace $K(t_0)$, in Eqs. (79) and (80), by $\frac{1}{2}\,[K(t_0 + R/c) + K(t_0 - R/c)]$. In particular, if the current is sinusoidal,

$$K(t) = K_0 \, \sin(\omega t) \; , \tag{83}$$

we find that

$$\begin{aligned} \vec{E}(x,t) &= -\frac{\mu_0 K_0 c}{2} \, \cos\left(\frac{\omega R}{c}\right) \sin(\omega t_0) \, \hat{z} \; ; \\ \vec{B}(x,t) &= \frac{\mu_0 K_0}{2} \, \cos\left(\frac{\omega R}{c}\right) \sin(\omega t_0) \, \hat{y} \; ; \end{aligned} \tag{84}$$

$$\mathcal{P} = \frac{\mu_0 K_0^2 c}{2} \, \cos^2\left(\frac{\omega R}{c}\right) \sin^2(\omega t_0) \; . \tag{85}$$

These results are identical to those for a single plane (Example 4), except for the factors of $\cos(\omega R/c)$. (For a current $K(t) = K_0 \cos(\omega t)$ the sines in (84) and (85) are simply replaced by cosines.) Evidently the double plane will not radiate if the current is sinusoidal with a frequency such that $(\omega R/c)$ is a zero of $\cos z$:

$$\omega_j = \left(j + \frac{1}{2}\right) \pi \left(\frac{c}{R}\right) \; , \quad j = 0, 1, 2, \ldots \; . \tag{86}$$

In fact, for these special frequencies ***the exterior fields are precisely zero.*** It follows that any current of the form

$$K(t) = \sum_{j=0}^{\infty} \left[a_j \, \cos(\omega_j t) + b_j \, \sin(\omega_j t)\right] \; , \tag{87}$$

with ω_j given by Eq. (86), will generate no fields, and produce no radiation. As we know from Fourier analysis, any well-behaved function on the interval

$(-R/c,\ R/c)$ can be written in the form (87), with the familiar procedure for evaluating the coefficients. *Outside* this interval the series is periodic, with alternating signs.[23]

5. GOEDECKE'S CONDITION

Some time ago, Goedecke[5] derived a stunningly simple test for the absence of radiation. Goedecke's criterion amounts to the condition that the Fourier transform of the current density

$$J^{\mu}(k) = \frac{1}{4\pi^2} \int e^{i(k^{\nu} x_{\nu})} j^{\mu}(x)\ d^4x \tag{88}$$

vanish whenever its argument is lightlike:[24]

$$J^{\mu}(k) = 0 \qquad \text{when} \qquad k^{\nu} k_{\nu} = 0\ . \tag{89}$$

In this section we check that our results are consistent with Goedecke's condition.

For the "pipe" configuration, with an axially symmetric sinusoidal current flowing in the z-direction, we have[25]

$$j^{\mu}(x) = \left(0,\ 0,\ 0,\ j(r)\, e^{i\omega t}\right)\ . \tag{90}$$

The Fourier transform is

$$J^{\mu}(k) = (0,\ 0,\ 0,\ J(k))\ , \tag{91}$$

where

$$J(k) = \frac{c}{4\pi^2} \int e^{i(-k^{o}ct + \vec{k}\cdot\vec{r})}\, j(r)\ e^{i\omega t}\, dt\ d^3r\ . \tag{92}$$

In cylindrical coordinates,

$$\vec{k} \cdot \vec{r} = k_x r \cos\phi + k_y r \sin\phi + k_z z \ , \quad \text{and} \quad d^3 r = r\, dr\, d\phi\, dz \ . \tag{93}$$

Carrying out the t, ϕ, and z integrals, we obtain

$$J(k) = 2\pi\delta(k^o - \omega/c)\,\delta(k_z) \int_0^\infty j(r)\, J_0(k_r r)\, r\, dr \ , \tag{94}$$

where $k_r = \sqrt{k_x^2 + k_y^2}$ is the radial component of $\vec{k}$, and k_z is its z-component. Because of the delta-functions, $J^\mu(k)$ is *automatically* zero except when $k_z = 0$ and $k^o = \omega/c$; if k^μ is lightlike, this leaves $k_r = \omega/c$. So Goedecke's condition reduces in this case to

$$\int_0^\infty j(r)\ J_0\left(\frac{\omega r}{c}\right) r\, dr = 0 \ . \tag{95}$$

In particular, if the current is confined to the surface of the pipe, so that

$$j(r) = \frac{I_0}{2\pi R}\ \delta(r - R) \ , \tag{96}$$

there will be no radiation provided

$$J_0\left(\frac{\omega R}{c}\right) = 0 \ , \tag{97}$$

which is precisely what we found before [Eq. (34)].[26]

For the "solenoid" configuration we have

$$j^\mu(x) = (0,\ -\sin\phi,\ \cos\phi,\ 0)\ j(r)\ e^{i\omega t}\ , \tag{98}$$

and the Fourier transform is

$$J^\mu(k) = (0,\ -k_y,\ k_x,\ 0)\ J(k)\ , \tag{99}$$

where

$$J(k) = \frac{2\pi i}{k_r}\ \delta(k^o - \omega/c)\,\delta(k_z) \int_0^\infty j(r)\, J_1(k_r r)\, r\, dr\ . \tag{100}$$

In this case, then, Goedecke's condition reduces to the constraint

$$\int_0^\infty j(r)\ J_1\left(\frac{\omega r}{c}\right) r\, dr = 0\ . \tag{101}$$

If the current is confined to the surface of the solenoid, so that

$$j(r) = \frac{M_0}{\pi R^2}\ \delta(r - R) \tag{102}$$

then there is no radiation provided

$$J_1\left(\frac{\omega R}{c}\right) = 0\ , \tag{103}$$

confirming our previous result [Eq. (64)].[27]

For the "plane" configuration, we have

$$j^{\mu}(x) = \left(0,\ 0,\ 0,\ j(x)\, e^{i\omega t}\right) \ , \tag{104}$$

so that

$$J^{\mu}(k) = (0,\ 0,\ 0,\ J(k)) \ , \tag{105}$$

with

$$J(k) = 2\pi\delta(k^o - \omega/c)\ \delta(k_y)\ \delta(k_z) \int_{-\infty}^{\infty} j(x)\ e^{ik_x x}\ dx \ . \tag{106}$$

In this case Goedecke's condition reduces to

$$\int_{-\infty}^{\infty} j(x)\ e^{i\omega x/c}\, dx = 0 \ . \tag{107}$$

If the current is confined to two parallel planes, so that

$$j(x) = \frac{K_0}{2}\ [\delta(x+R) + \delta(x-R)] \ , \tag{108}$$

there will be no radiation if

$$\cos\left(\frac{\omega R}{c}\right) = 0 \ , \tag{109}$$

confirming Eq. (86).[28]

6. RADIATION REACTION

Ordinarily, the emission of radiation is accompanied by a "radiation reaction" — a recoil force attributable to the fields acting back on the source. Indeed, one would suppose that the work done against this radiative recoil force (by whatever agency it is that moves the charge) must equal the energy radiated, for which it is ultimately responsible. By the same token, if there is no radiation, there should be no radiation reaction. However, the connection between emission of radiation and the radiation reaction force is a subtle one, as we can see by comparing the Larmor formula for the power radiated from a point charge[1]

$$P = \frac{1}{4\pi\epsilon_0}\,\frac{2e^2}{3c^3}\,a^2$$

with the Abraham-Lorentz formula for the radiation reaction on such a charge[29]

$$F_{\text{rad}} = \frac{1}{4\pi\epsilon_0}\,\frac{2e^2}{3c^3}\,\dot{a}\ .$$

Observe that the particle *radiates* whenever it accelerates, but it experiences a radiation reaction force only when its acceleration *changes*. The explanation for this apparent violation of conservation of energy is that the *nearby* fields function as a "reservoir," in which energy can be stored, so that work done against the radiation reaction need not show up directly in the form of radiation.[30] Nevertheless, for *periodic* motion it is certainly the case that the work done against the radiation reaction force *in one full cycle* must equal the energy radiated *during one full cycle*, since the energy stored in the nearby fields is the same at the end of the interval as at the beginning.

For example, in the hollow pipe configuration the power necessary to drive a current $I(t)$ is given by[31]

$$\mathcal{P}_d = -IEL \; , \tag{110}$$

where E is the electric field at the surface of the pipe, and L is the length of the segment. Referring to our results in Section 2 [see Eq. (32)] for a current $I(t) = I_0 \sin(\omega t)$, we find that the driving power per unit length is

$$\mathcal{P}_d = \frac{\mu_0 I_0^2 \omega}{4} J_0\left(\frac{\omega R}{c}\right) \sin(\omega t) \left[\sin(\omega t) J_0\left(\frac{\omega R}{c}\right) - \cos(\omega t) N_0\left(\frac{\omega R}{c}\right)\right] \; . \tag{111}$$

This is plainly *not* equal to the power radiated [Eq. (33)], but the work done per unit length in one full cycle

$$\mathcal{W} = \int_0^{2\pi/\omega} \mathcal{P}_d \, dt = \frac{\mu_0 \pi I_0^2}{4} \left[J_0\left(\frac{\omega R}{c}\right)\right]^2 \tag{112}$$

does equal the total energy radiated. Similarly, for the hollow solenoid the power needed to drive a surface current $K(t)$ is

$$\mathcal{P}_d = -2\pi R \, KEL \; , \tag{113}$$

and we find, for the current in Eq. (58):

$$\mathcal{P}_d = \frac{\mu_0 M_0^2 \omega}{R^2} J_1\left(\frac{\omega R}{c}\right) \sin(\omega t) \left[\sin(\omega t) J_1\left(\frac{\omega R}{c}\right) - \cos(\omega t) N_1\left(\frac{\omega R}{c}\right)\right] \; . \tag{114}$$

Again, this differs from the power radiated [Eq. (61)], but both yield the same energy when integrated over a full cycle:

$$\mathcal{W} = \frac{\mu_0 \pi M_0^2}{R^2} \left[J_1 \left(\frac{\omega R}{c} \right) \right]^2 . \tag{115}$$

Finally, for the parallel planes, the power required to sustain a surface current $K(t)$ is

$$P_d = -KEa , \tag{116}$$

where a is the area of the section in question. For the sinusoidal current $K_0 \sin(\omega t)$, the driving power per unit area [see Eq. (84)] is

$$P_d = \frac{\mu_0 K_0^2 c}{2} \cos\left(\frac{\omega R}{c}\right) \sin(\omega t) \sin\omega\left(t - \frac{R}{c}\right) , \tag{117}$$

which is not the same as the power radiated [Eq. (85)], but they both yield the same energy in one full cycle:[32]

$$\mathcal{W} = \frac{\pi \mu_0 K_0^2 c}{\omega} \cos^2\left(\frac{\omega R}{c}\right) . \tag{118}$$

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1. J. D. Jackson, *Classical Electrodynamics*, 2nd Edition, John Wiley, New York (1975), p. 659. The Larmor formula assumes that $v^2 \ll c^2$. However, the relativistic generalization (Liénard's formula — Jackson, p. 660) shares the feature that P is proportional to a^2.

2. This result, originally due to G. A. Schott (Philosophical Magazine **7**, 752 (1933)) is discussed in detail by P. Pearle in his excellent review "Classical Electron Models," which appears as Chapter 7 in *Electromagnetism: Paths to Research*, D. Teplitz, ed., Plenum, New York (1982). The sphere is taken to be rigid in the lab frame. Pearle (Foundations of Physics **7**, 931 (1977)) has shown that a shell which is rigid in its own instantaneous rest frame admits no nontrivial radiationless modes.

3. Because the Poynting vector is quadratic in the fields, power does not, in general, obey the Superposition Principle. However, in this case the radiation fields themselves vanish, so the linear combination in Eq. (3) remains radiationless.

4. This result, originally due to A. Sommerfeld (Verh. d. III Internat. Mathem. Kongr. Heidelberg, 1904) is discussed more recently by J. Daboul and J. H. D. Jensen, Z. Physik **265**, 455 (1973).

5. G. Goedecke, Phys. Rev. **135B**, 281 (1964).

6. Because I is independent of z, the charge density $\lambda(z)$ is necessarily constant in time, and as far as the radiation is concerned there is no loss of generality in taking $\lambda = 0$.

7. D. J. Griffiths, *Introduction to Electrodynamics*, Prentice-Hall, Englewood

Cliffs (1981), Eqs. (5.26) and (9.8). Because the current extends to infinity, the vector potential may not exist (for a steady current, in particular, it is logarithmically divergent). In such cases we may, in principle, truncate the integral at very large $|z|$, calculate the fields, and then let the cutoff go to infinity. In this paper, however, it will never be necessary to perform this operation explicitly.

8. We thank Prof. L. Schecter for bringing this example to our attention.

9. The necessary integrals may be found in I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Corrected and Enlarged Edition), Academic Press, New York (1981), §3.753.

10. If we average $P_r(t)$ over one full cycle [Ref. 9, §8.477] the r-dependence drops out: $\langle P_r(t)\rangle = \mu_0\omega I_0^2 L/8$. Thus the same energy crosses each cylindrical surface — as is to be expected, for a periodic system.

11. In the case of localized charge/current distributions one looks for fields which go like $1/r$, so that the Poynting vector goes like $1/r^2$; integration over a spherical surface then yields a constant. Here we are integrating over a *cylinder*, so we want $S \sim 1/r$, and hence $E,\ B \sim 1/\sqrt{r}$.

12. A careful justification of this procedure will be found in T. A. Abbott, senior thesis, Reed College, 1984 (unpublished).

13. Reference 9, §6.684. Incidentally, the potential *inside* the wire is given by Eq. (32) with r and R interchanged.

14. R. E. A. C. Paley and N. Wiener, *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications Vol. XIX, New York (1934), Chapter VI.

15. R. Mayer, T. A. Abbott and D. J. Griffiths, to be published.

16. For an arbitrary longitudinal current density $j(r)$, we simply replace I_0 [in Eq. (32)] by $j(r)2\pi r dr$, and integrate. Thus, if the current is uniform over the cross-section [Ref. 9, §6.561], $J_0(\omega R/c) \to (2c/\omega R)\, J_1(\omega R/c)$. In this case there is no radiation if $(\omega R/c)$ is a zero of the *first*-order Bessel function.

17. In this case we found it easiest to go back to Eq. (38) and do the z-integral first, instead of working from Eq. (45). See Ref. 9.

18. As before (Ref. 10), if we average $P_r(t)$ over a full cycle the r-dependence drops out: $\langle P_r(t)\rangle = \mu_0 M_0^2 \omega^3 L/8c^2$.

19. This actually holds for a solenoid of arbitrary diameter, as the reader can check, using Eq. (38). See also Ref. 12. Incidentally, a spherical shell which rotates with constant angular acceleration is likewise radiationless — see Ref. 4, p. 462.

20. Integrate by parts, and use Ref. 9, §6.684.

21. For an arbitrary solenoidal current density $j(r)$, we replace M_0 [in Eq. (60)] by $j(r)\pi r^2 dr$, and integrate. Thus, for a uniformly charged solid cylinder which rocks back and forth about its axis (Ref. 9, §6.561), $J_1(\omega R/c) \to (4c/\omega R) J_2(\omega R/c)$. In this case there is no radiation if $(\omega R/c)$ is a zero of the *second*-order Bessel function.

22. We assume here that the current goes to zero in the remote past, as is implicitly required already in Eq. (77). As we found in Example 4, however, currents which do *not* go to zero can sometimes be handled with a suitable cutoff procedure, and the final results [(79)–(82)] still hold. An exception is the *constant* current $K(t) = K_0$, for which, of course, $E = 0$ and $\mathcal{P} = 0$.

23. For an arbitrary symmetrical current density $j(x)$, we replace K_0 [in Eq. (84)] by $j(x)\,dx$, and integrate. Thus, if the current is uniform over a slab of thickness $2R$, $\cos(\omega R/c) \rightarrow (c/\omega R)\sin(\omega R/c)$. In this case there is no radiation if $(\omega R/c)$ is a zero of the sine function.

24. Goedecke's criterion was generalized and perfected by Pearle (Ref. 2), to a necessary *and sufficient* condition: $k^o J^\mu(k) = k^\mu J^o(k)$ for lightlike k. Because our sources are neutral, $J^o(k) = 0$; in dropping the factor k^o we lose at most the case $\omega = 0$, which is trivially radiationless.

25. For simplicity we use $e^{i\omega t}$ here, instead of $\sin(\omega t)$ or $\cos(\omega t)$. Really, the terms $\delta(k^o - \omega/c)$ should be replaced by $\frac{1}{2i}\left[\delta(k^o - \omega/c) - \delta(k^o + \omega/c)\right]$ or $\frac{1}{2}\left[\delta(k^o - \omega/c) + \delta(k^o + \omega/c)\right]$, respectively, in Eqs. (94), (100) and (106). But the rest of the argument is unaffected.

26. For a current uniformly distributed over the cross-section of the wire, Eq. (95) yields $J_1(\omega R/c) = 0$, as we found in Ref. 16.

27. For a uniformly charged rotating cylinder, Eq. (101) yields $J_2(\omega R/c) = 0$, as we found in Ref. 21.

28. For current uniformly distributed over a slab, Eq. (107) yields $\sin(\omega R/c) = 0$, as we found in Ref. 23.

29. Reference 7, p. 380.

30. The term "radiation reaction" is in fact something of a misnomer — it should really be called the "field reaction."

31. $P = Fv = qEv = (\lambda L)Ev = IEL$. We include a minus sign because we want the work done *against*, not *by*, the field.

32. Interestingly, for those special frequencies which do not radiate, the ra-

diation reaction vanishes identically. The same is true for a nonrotating spherical shell (see Pearle, Ref. 2). For further discussion of the relation between radiationlessness and the radiation reaction, see Ref. 4 and T. Erber, Fortschr. Physik **9**, 343 (1961).

FIGURE CAPTIONS

1. The infinite straight wire; geometry for Eq. (5).

2. The infinite pipe of radius R; geometry for Eq. (27).

3. "Fourier-like" expansion [Eq. (35)] of the step function [Eq. (36)]. Horizontal (time) axis in units of R/c. The coefficients are

$$a_j = 0 \,, \qquad b_j = \frac{4}{\pi J_1(\lambda_j)} \sum_{k=0}^{\infty} \frac{(-\lambda_j^2)^k}{[(2k+1)!!]^2} \,.$$

4. The infinite plane; geometry for Eq. (65).

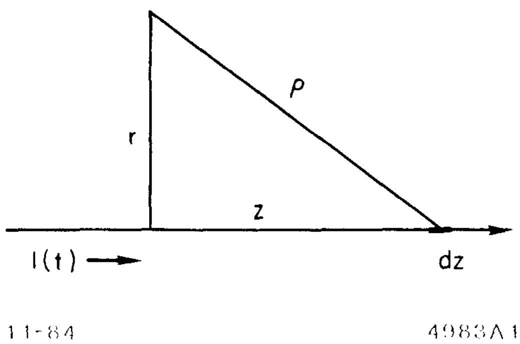

Fig. 1

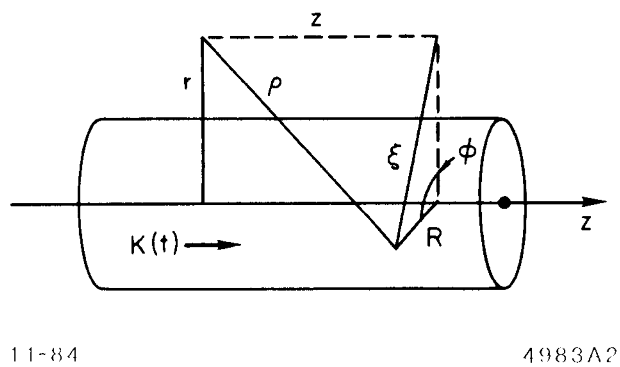

Fig. 2

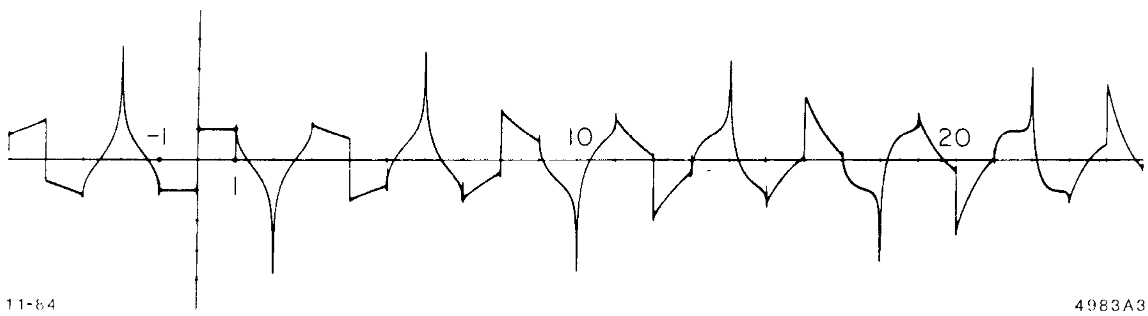

Fig. 3

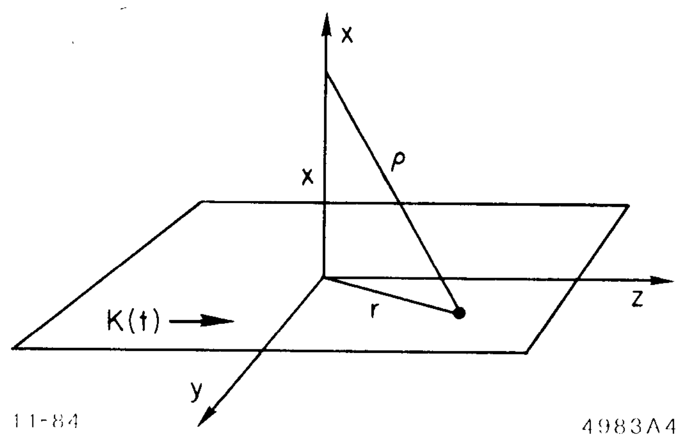

Fig. 4