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CRITICAL DIMENSION OF STRING THEORIES IN CURVED SPACE*

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ABSTRACT

The critical dimension of string theories in which the background metric is a product of Minkowski space and an $SU(N)$ or $O(N)$ group manifold is derived. A consistent string theory can be constructed only in the presence of a Wess-Zumino term associated with the compactified dimension. This implies that the compactified radius is quantized in units of the string tension. A generalization to the supersymmetric case is discussed.

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The quantum theory of a string is very different from that of a point particle. A consistent theory for a point particle can be defined in any number of dimensions, whereas studies with string theories show that the dimension of space time cannot take any arbitrary value. In a flat background the bosonic string theory is known to be a consistent quantum theory only in 26 dimension.[1] The fermionic string theory of Neveu, Schwarz and Ramond[2] and the superstrings of Green and Schwarz[3] requires that space time has 10 dimensions.

If string theories are to provide us with renormalizable (or even finite) theory which unifies all interactions including gravity, it is clearly necessary to study string theories on manifolds with some of the dimensions compactified.[4] A Kaluza Klein like string theory[4,5] may therefore turn out to be important in reducing the theory (compactification) down to four dimension. With this is mind we have started to study string theories in a curved background. In this letter we present the calculation of the critical dimension of theories where the background metric is a product of Minkowski space and the group manifold $SO(N)$ or $SU(N)$. Our analysis is based on the fact that in a curved background the string action provides us with a two dimensional nonlinear sigma model. An important feature of a string theory is its reparametrization invariance. In terms of the two dimensional field theory this reflects itself as a conformal invariance[6] i.e. the σ model field theory must have a zero β-function. A non linear σ model we know to have $\beta = 0$ is the one discussed by Witten.[7] In order to have a conformal invariant theory a Wess-Zumino term has to be added into the theory. Since $\pi_2(SO(N)) = \pi_2(SU(N)) = 0$ and $\pi_3(SO(N)) = \pi_3(SU(N)) = Z$ a two dimensional non linear theory which resides on the group manifold of either $SO(N)$ or $SU(N)$ admits a Wess-Zumino term. Witten has shown that for a particular relation between the coupling constant of the sigma model and the coefficient of the Wess-Zumino term the sigma model is conformally invariant. In the string theory this relation between the coupling constant and the coefficient of the Wess-Zumino term corresponds to a relation between the string tension and the size of the compact dimensions.

As an example of a string theory with non flat background we study a string moving on a product of a three dimensional sphere with radius R and the d-dimensional Minkowski space. This case corresponds to the group manifold of $SU(2)$. Our analysis can easily be generalized to $SO(N)$ or $SU(N)$. Because of reparametrization invariance we are free to choose a gauge. In the orthonormal gauge[8] it is enough to consider only the transverse directions. The string action for the spherical part can be written as

$$A = \frac{1}{4\pi\alpha'} \int d\sigma\, d\tau\ \left[\dot{x}^2 - x'^2 + \lambda[x^2 - R^2]\right] \tag{1}$$

where λ is a Lagrange multiplier and α' is related to the string tension by $T = (2\pi\alpha')^{-1}$. In Eq. (1) we have used the notation

$$x^{i'} = \frac{\partial x^i}{\partial \sigma} \qquad \dot{x}^i = \frac{\partial x^i}{\partial \tau}\ . \tag{2}$$

Rescaling the string position variable x^i and defining

$$g = (x^0\,\mathbf{1} + i\,\vec{x}\cdot\vec{\sigma}\,)/\sqrt{2} \tag{3}$$

where σ^a are the Pauli matrices, Eq. (1) can be written in a more compact form

$$A = \frac{R^2}{4\pi\alpha'} \int d\sigma\, d\tau\, \mathrm{Tr}\, \left[\partial_\tau g\, \partial_\tau g^{-1} - \partial_\sigma g\, \partial_\sigma g^{-1}\right]\ . \tag{4}$$

This action is not conformally invariant. Witten[7] has shown that conformal invariance can be restored if a Wess-Zumino term

$$\frac{K}{12\pi}\int d^3y\, \epsilon^{abc}\mathrm{Tr}\,\left[g^{-1}\partial_a g\, g^{-1}\partial_b g\, g^{-1}\partial_c g\right] \tag{5}$$

is added to the action. Conformal invariance and hence reparametrization invariance is restored if the integer coefficient K of the Wess-Zumino term and the coupling constant $4\pi\alpha'/R^2$ of the sigma model satisfy the relation[7]

$$\frac{\alpha'}{R^2}=\frac{2}{|K|}\ . \tag{6}$$

This means that the radius of the compactified dimensions gets quantized in units of the string tension. Furthermore, when K approaches infinity one recovers the flat space limit.

To analyze the string theory we take advantage of Witten's work[7] on two dimensional sigma models. The currents of the sigma model are most easily expressed using light cone coordinates $u=\sigma+\tau$ and $v=\sigma-\tau$. When Eq. (6) is satisfied the algebra of the currents

$$\begin{aligned} J_+(u) &= \frac{iK}{\pi}\, g^{-1}\partial_u g \\ J_-(v) &= -\frac{iK}{\pi}\, \partial_v g\, g^{-1} \end{aligned} \qquad K>0 \tag{7}$$

takes a very simple form.

$$\begin{aligned} &[J^a_-(v),\, J^b_-(v')] = 2i\, f^{abc}J^c_-(v)\delta(v-v') + \frac{iK}{\pi}\,\delta'(v-v')\delta^{ab} \\ &[J^a_+(u),\, J^b_+(u')] = 2i\, f^{abc}J^c_+(u)\,\delta(u-u') + \frac{iK}{\pi}\,\delta'(u-u')\delta^{ab} \\ &[J^a_+(u),\, J^b_-(v)] = 0\ . \end{aligned} \tag{8}$$

This light cone algebra is known as the Kac-Moody algebra with a central extension.[9] This central term is the generalization of the well known Schwinger term of current algebra. The equal time version of this algebra and its relation to the light cone algebra has been discussed in Refs. 7 and 10. The Kac-Moody algebra has well behaved unitary representation if K is an integer as in the case of the Wess-Zumino term. Furthermore, the irreducible representations of the Kac-Moody algebra are conformally invariant.

The generators L_n of the conformal transformation satisfy a Virasoro type of algebra

$$[L_n, L_m] = (n-m)\, L_n + \frac{c}{12}\,(n^3 - n)\,\delta_{n,-m} \tag{9}$$

where m and n range over the integers and c is a real number, the central element of the algebra. The central charge is intimately connected with the trace anomaly of the energy momentum tensor in a background gravitational field.[6] This algebra was extensively studied. In particular, Friedan, Qiu and Shenker[11] have studied the critical behavior of two-dimensional theories by characterizing the allowed (under certain conditions) values of c. In a very recent preprint, Goddard and Olive[12] used a group theoretical method for constructing new unitary representations of the Virasoro algebra.

For our case the coefficient c in Eq. (9) can be determined from the current algebra of Eq. (8). The value of c obtained is in agreement with the more general characterization given in Ref. 12. The Fourier components of the energy momentum tensor represents the generators of reparametrization (conformal) invariance and they generate the Virasoro algebra (9). Hence to determine the Virasoro algebra we need the commutation relations of the energy momentum tensor. These commutation relations can be calculated from the current algebra (8) since our theory is a Sugawara type of theory with a current current interaction. The energy momentum can be expressed in terms of the σ model currents of Eq. (7).[♮1]

$$
\begin{aligned}
\theta_{00} &\propto \mathrm{Tr}\,[J_+^2(u) + J_-^2(v)] \\
\theta_{01} &\propto \mathrm{Tr}\,[J_+^2(u) - J_-^2(v)]\ .
\end{aligned}
\tag{10}
$$

The central charge in Eq. (9) can be determined from the δ''' Schwinger term of the commutation relation of the energy momentum tensor. When we take the Fourier transform, δ''' will give rise to the n^3 term in the Virasoro algebra. We do not expect to get the $-n$ part of the central charge from the Schwinger term since it arises from the normal ordering of L_0. However, δ''' is enough to determine the central charge. To calculate the δ''' Schwinger term we need to normal order the energy momentum tensor.[13] This is a well defined procedure since at the particular point of Eq. (6) our currents are free. The currents $J_+(u)$ and $J_-(v)$ can, therefore, be divided into positive and negative frequency parts

$$
\begin{aligned}
J_+^a(u) &= \left(\frac{K}{2\pi}\right)^{1/2} \sum_{n=1}^{\infty} \left[a_n^a\, e^{-inu} + a_n^{+a}\, e^{in\cdot u}\right] \\
J_-^a(v) &= \left(\frac{K}{2\pi}\right)^{1/2} \sum_{n=1}^{\infty} \left[b_n^a\, e^{-in\cdot v} + b_n^{+a}\, e^{in\cdot v}\right]
\end{aligned}
\ .
\tag{11}
$$

The operators a and b satisfy the following commutation relations.

$$
\begin{aligned}
[a_n^a, a_m^b] &= \frac{i}{\sqrt{K}}\, f^{abc} a_{n+m}^c + n\, \delta_{n+m,0}\, \delta^{ab} \\
[b_n^a, b_m^b] &= \frac{i}{\sqrt{K}}\, f^{abc} b_{n+m}^c + n\, \delta_{n+m,0}\, \delta^{ab}
\end{aligned}
\tag{12}
$$

with $a_n^+ \equiv a_{-n}$. All the other commutation relations vanish. The commutation relations (12) follow immediately from the current algebra (8). To calculate the

♮1 The energy momentum tensor is normalized properly in Eq. (13) to give the correct equation of motion for the currents.

commutation relation of the energy momentum tensor it is useful to define

$$\theta_+(u) = \frac{1}{A} \ : \ [J_+(u)]^2 \ :$$
$$\theta_-(v) = \frac{1}{B} \ : \ [J_-(v)]^2 \ : \tag{13}$$

The normalization constants A and B arise because of normal ordering. Note that θ_+ is only a function of u and θ_- of v respectively. The integrals of these densities generate translations in u and v directions, respectively. The normalization constants A and B in Eq. (13) can be determined from the commutation relation

$$[: J_+^2(u) \ : \ , \ J_+^b(u')] = \frac{2i}{\pi}\,(k+2)\,\delta'(u-u')\,J_+^b(u) \ . \tag{14}$$

From Eq. (13) and (14) it immediately follows that

$$\left[\int \theta_+(u')du',\, J_+(u)\right] = \partial_u J_+(u)$$

if

$$A = \frac{\pi}{2(k+2)} \ . \tag{15}$$

Similarly one can determine B with the result

$$B \ = \ A \ . \tag{16}$$

Normal ordering in θ_+ was essential for getting the correct normalization constants. For the Virasoro algebra we need

$$[: J_+(u)^2 \ : ,: \ J_+(u')^2 :] = 2i[J_+(u),\, J_+(u')]\,\delta'(u-u') - \frac{i}{6\pi}\,\frac{3K}{K+2}\,\delta'''(u-u') \ . \tag{17}$$

The normal ordering of the currents was again essential for getting the correct δ''' Schwinger term. The Virasoro algebra can now be easily determined. Let's define

$$L_n = \frac{1}{2} \int_{-\pi}^{\pi} d\lambda \, e^{in\lambda} \, [\theta_+(\lambda)]^2 \; . \tag{18}$$

After a straightforward algebra we find that the L_n's satisfy

$$[L_n, \, L_m] = (n - m) \, L_{m+n} + \frac{3}{12} \, \frac{K}{K+2} \, n^3 \delta_{n,-m} \; . \tag{19}$$

Formula (19) is only valid for $K > 0$ (recall that in the definition of the current in Eq. (9) we restricted ourselves to positive K). However, our analysis can be easily extended to negative K. The current algebra of Eq. (8) will remain the same with K replaced by $|K|$. Therefore the central charge of the Virasoro algebra for both positive and negative K can be written as

$$c = \frac{3|K|}{|K|+2} \; . \tag{20}$$

Note that c is not proportional to K, i.e., it is not the value one would naively expect for a free fermionic theory. As we already remarked $K \to \infty$ corresponds to the flat space limit. In this limit we indeed recover the result of flat space dimensions with $c = 3$. From Eq. (20) we can also determine the critical dimension for the flat Minkowski space

$$d_c + \frac{3|K|}{|K|+2} = 26 \; . \tag{21}$$

Note that the formula is symmetric under $K \to -K$. Equation (21) tells us that by compactifying part of the transverse dimensions we can reduce the critical dimensionality of the flat Minkowski space. For $K = 1$ the critical dimension is

25 and for $K = 4$ it is 24. This formula can be easily generalized to arbitrary $SO(N)$ or $SU(N)$. The Virasoro algebra for these group manifolds has the form

$$[L_n, L_m] = (n - m)\, L_{n+m} + \frac{D}{12}\, n^3 \delta_{n,-m} \tag{22}$$

where D is related to the critical dimension d_c by

$$D = 26 - d_c = \begin{cases} \frac{(N^2-1)K}{N+K} & \text{for } SU(N) \\ \frac{\frac{1}{2}N(N-1)K}{K+N-2} & \text{for } SO(N)\ . \end{cases} \tag{23}$$

Both formulas can be written in terms of the dimension d_G and rank r of the group

$$D = \begin{cases} \frac{d_G K}{r+1+K} & \text{for } SU(N) \\ \frac{d_G K}{K+2r-2+\delta} & \text{for } SO(N) \end{cases} \tag{24}$$

where $\delta = \frac{1}{2}\left(1-(-1)^N\right)$. Note that while the isometry group is $SU(N) \times SU(N)$ (or $O(N) \times O(N)$), d_G actually refers to the dimension of the manifold. Although the critical dimension d_c of the Minkowski space is reduced down from 26, the total dimension, i.e. $d_c + d_G$, is larger than 26. This can be verified immediately from Eq. (24) since the factor multilying d_G is smaller than one.

The critical dimension of the string theory can also be calculated using Polyakov's method.[6] As we already mentioned the central charge of the Virasoro algebra is related to the trace anomaly of the energy momentum tensor. Our computation gives the coefficient of the trace anomaly in the external gravitational field for the compactified dimensions. Combining this with the free field conformal anomaly associated with the flat dimensions as calculated by Polyakov, gives the full trace anomaly for the product space $M^d \times m(SU(N)$ or $SO(N))$.

Calculating the trace anomaly is equivalent to calculating the effective action in an external background gravitational field. The effective action can be calculated using Feynman graphs. In Fig. 1 we have shown the two leading orders in the large K expansion. Figure 1a is of order 1 whereas 1b is of order $1/|K|$. Note that 1b does not depend on the sign of K. The three point coupling in Fig. 1b comes form the Wess-Zumino term, whereas the four point coupling comes from the sigma model. Figure 1a corresponds to the leading order of the expansion of the central charge (20) $c = 3\left(1 - \frac{2}{K}\right) + O(1/K^2)$ and Fig. 1b to the $1/K$ part. From the graphical approach, it is clear that for any manifold G/H the coefficient c will be proportional to the number of Goldstone boson, i.e. to the dimension of the manifold.

In the large K limit one may try to find a semiclassical approach to solving the present theory. Furthermore the large K limit may provide a way to regularize the limiting flat theory at the critical point.

Our analysis can be extended to the Neveu Schwarz Ramond model.[2] The supersymmetric generalization of the two dimensional Wess-Zumino term has been constructed by Rohm[14] and by Curtwright and Zachos.[15] From the analysis of Ref. 14 it follows that at the critical point the fermions remain free. It seems therefore that the fermions will provide the usual reduction of the critical dimension from 26 down to 10, while the Wess-Zumino term will work just as in the bosonic string. We therefore expect the critical dimensions to be determined by

$$d_c + \frac{2}{3}D + \frac{1}{3}d_G = 10 \tag{25}$$

where D is given in Eq. (24).

The reduction down from ten dimensions raises the question of the realization of supersymmetry.[16] To analyze this point carefully one would like to have the spectrum of the theory. One should note that the supersymmetry discussed here is the two dimensional one associated with the fermionic string of Ref. 2. An important problem which we have just started to investigate is the possible construction of a model with a space time supersymmetry corresponding to the new string theory of Green and Schwarz.[3]

The theories discussed in this letter provide us with a new class of string theories. It would be extremely interesting to construct the representation of the Virasoro algebra associated with these theories[11,12,14] and to construct the vertex function. We hope to report on progress in these directions in the near future.

While finishing this paper we received a new preprint with the very interesting work of Friedan, Qiu and Shenker[17] generalizing their previous work[11] to the supersymmetric case. We have also learned[18] that the whole program of compactification of string theories has been extensively studied by D. Friedan, Z. Qiu, and S. Shenker. As part of their work they have also derived the critical dimensions.

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FIGURE CAPTIONS

1. (a) The leading order ($K \to \infty$) contribution to the trace anomaly in external gravitational field.

   (b) The next-to-leading ($O(1/K)$) contribution to the truce anomaly in external gravitational field. The black dot represents a vertex coming from the Wess-Zumino term.

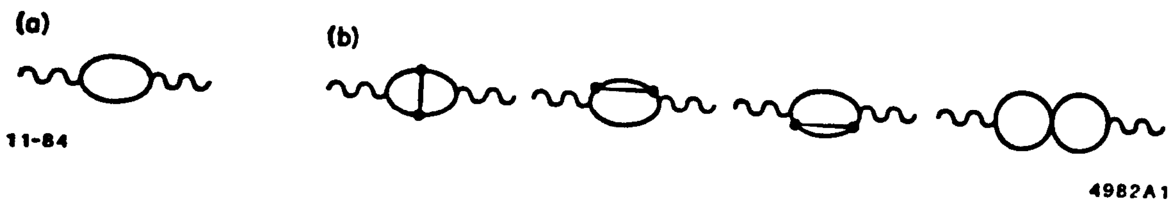

FIG. 1