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THE LATTICE SCHWINGER MODEL WITH SLAC FERMIONS*

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ABSTRACT

The lattice Schwinger model with SLAC fermions is analyzed with two methods: a Hartree-Fock calculation of the ground state wavefunction, and a weak coupling approximation involving a truncated SLAC derivative. It is shown that a Goldstone boson of chiral symmetry is actually present, but in the weak coupling limit it acquires infinite velocity.

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1. Introduction

When one tries to naively translate a Dirac Hamiltonian into a lattice Hamiltonian, one finds that there are unwanted low energy states with momenta close to π; these states can be interpreted in the continuum limit as new species of fermions. Different methods have been proposed to overcome this difficulty. Wilson's method[1] consists in adding to the Hamiltonian a term of the form:

$$r\bar{\psi}\nabla\psi \tag{1.1}$$

where ∇ is the lattice laplacian. This term raises the energy of the unwanted states, and, being of the order of the lattice spacing, vanishes in the continuum limit. A different approach, of Susskind[2] , consists in starting with a single component fermion, and then exploit the doubling in order to recover all the desired spin and flavor components. In both methods chiral symmetry is explicitly broken, and it is recovered only in the continuum limit. Therefore the Goldstone bosons associated with spontaneous chiral symmetry breaking are not massless for any finite coupling. This is an undesirable feature, especially in strong coupling calculations. In contrast, the SLAC group[3] proposed a fermionic Hamiltonian with a long range derivative, which avoids the doubling completely, and maintains full chiral symmetry for any couplings. There is, however a new difficulty: the current associated to the U(1) axial symmetry is both conserved and gauge invariant. Therefore, in all circumstances in which chiral symmetry breaking is expected, one would expect to find a massless excitation associated with the axial current. Of course, such excitation would have little to do with the continuum limit theory.

The nature of this problem is particularly transparent in $1+1$ dimensional electrodynamics with a single Dirac fermion, the lattice version of the Schwinger model. There the anomaly has a very simple physical interpretation, which will be illustrated in Section 2, for the continuum limit case, as well as for the various types of lattice derivative. There it is shown that in the case of SLAC fermions, unlike the case of Susskind's or Wilson's fermions, the electromagnetic coupling, because of the anomaly, must cause nontrivial dynamical effects at the "bottom" of the Fermi sea. In other words, the usual assumption, that only low energy states participate to the dynamics of the model, fails because of the anomaly.

In Section 3, a variational calculation of the ground state wave-function of the lattice Schwinger model with SLAC fermions is presented. There it is shown that fermionic states with momenta around π, are in fact excited, because of the long range nature of the electrodynamic forces. The relation between lattice and continuum operators is discussed.

In Section 4 a physically intuitive picture of the continuum Schwinger model is given; the same picture is applied to a lattice version of the Schwinger model with a truncated SLAC derivative, and an exact solution of the model in the weak coupling regime is found. It is shown that the theory has the correct continuum limit, but for any finite coupling the spectrum is sharply modified: at small momenta, the spectral branch of the massive excitation turns into the Goldstone boson of the lattice chiral symmetry. The relevance of this solution to the full SLAC derivative Schwinger model is discussed.

2. The anomaly in the Schwinger model.

The anomaly in $1+1$ dimensions has a very simple physical interpretation* Consider a $1+1$ dimensional Dirac fermion in the continuum limit. The spectrum is represented in Fig. 1(a). There are two spectral branches, one with positive slope (right movers) and one with negative slope (left movers). In the ground state all the negative energy states will be filled. Suppose we turn on an external electromagnetic field, pointing to the right, for a time Δt; all the right movers will acquire momentum $\Delta k = gE\Delta t$, and energy $\Delta\epsilon = \Delta k$; left movers will acquire the same amount of momentum, but will lose an amount of energy equal to Δk,

* I have been unable to find out the origin of this argument. See ref. 4 for references on the subject.

since they move against the electric field. After a time Δt the state of the system will be as represented in Fig. 1(b). Since the number of states in a momentum interval Δk is $L\Delta k/h$, where L is the total length of the system, the number of right movers will exceed the number of left movers of an amount

$$\Delta n_R - \Delta n_L = 2\frac{L\Delta k}{h} = L\frac{g}{\pi}E\Delta t. \tag{2.1}$$

But $n_R - n_L$ is the chiral charge of the system; therefore we find

$$\frac{\Delta Q_5}{\Delta t} = L\frac{g}{\pi}E. \tag{2.2}$$

This is precisely the anomaly equation.

Consider now a Susskind fermion on a lattice; the Hamiltonian is:

$$H = \frac{1}{2i}\sum_j (a^\dagger_{j+1}a_j - a^\dagger_j a_{j+1}) \tag{2.3}$$

and the spectrum is represented in Fig. 2(a). The low energy sector of the spectrum, has states with momenta close to 0 and positive velocity, and states with momenta close to π and negative velocity. Since momentum conservation modulo 2π implies momentum conservation modulo π, if we redefine the momentum to be conserved modulo π, the continuum limit of the model will represent both chiral components of a single Dirac fermion. Under an external field the state of the system will evolve into the state represented in Fig. 2(b). If we define the chiral charge to be the difference between the right and left movers we recover Eq. (2.2). It is clear that the role played by the 'bottom' of the Fermi sea, is to convert right movers into left movers when an electric field is applied. No such conversion is operating in the continuum case; simply, particles sink and emerge in the infinitely deep sea. The mechanism operating in the continuum case is only possible when the cut-off is strictly infinite. A lattice theory has therefore to provide a different mechanism that produces the same effect, as in the case of Susskind fermions. For Wilson's fermions the energy eigenstates are chiral diagonal only for small momenta; the conversion of left into right movers involves mixing of the two chiral components, and only if the conversion proceeds slowly enough Eq. (2.2) is recovered. This has been analyzed in detail by Ambjorn et al[4] . Consider now the case of SLAC fermions. In chiral basis the Hamiltonian is:

$$H = i \sum_{j,k} \delta'(j-k)\psi_j^\dagger \sigma_z \psi_k \tag{2.4}$$

with $\delta'(j-k)$ given by:

$$\delta'(j-k) = \begin{cases} \dfrac{(-)^{j-k}}{j-k}, & for \quad j \neq k \\ 0, & for \quad j = k \end{cases} \tag{2.5}$$

if we define:

$$\psi_i = \begin{vmatrix} a_i \\ b_i \end{vmatrix} \tag{2.6}$$

we obtain:

$$H = \sum_{j,k} i\delta'(j-k)(a_j^\dagger a_k - b_i^\dagger b_j) = \sum_{-\pi}^{\pi} k(a_k^\dagger a_k - b_k^\dagger b_k) \tag{2.7}$$

The spectrum is represented in Fig. 3(a). If we apply an electric field for a short time, the system evolves into the state depicted in Fig. 3(b). The exact lattice chiral charge, defined as $\sum_i (a_i^\dagger a_i - b_i^\dagger b_i)$ is conserved. There is therefore a sharp difference with respect to Susskind's or Wilson's fermions. One can define a point-split chiral charge, which only counts states close to the Fermi surface (see ref.5) and thereby obeys the correct anomaly equation. The above argument, however, seems to indicate that there is nontrivial dynamics taking place at the bottom of the Fermi sea. It has been argued that the continuum theory could therefore be sick. In the next Section I will analyze the dynamics of the SLAC lattice schwinger model in the Hartree-Fock approximation.

3. The lattice Schwinger model in the Hartree-Fock approximation

The Schwinger model Hamiltonian in Coulomb gauge is:

$$\begin{aligned} H &= \overline{\psi}\gamma_1 i\partial_1\psi - \frac{g^2}{4}\int dx\, dy\, \rho(x)\,|x-y|\,\rho(y) \\ \rho(x) &= :\overline{\psi}(x)\psi(x): \end{aligned} \tag{3.1}$$

this translates into the following lattice Hamiltonian:

$$H = i\sum_{i,j}\delta'(i-j)\,\psi_i^\dagger\sigma_z\psi_j - \frac{g^2}{4}\sum_{i,j}\rho_i|i-j|\,\rho_j \tag{3.2}$$

which in momentum space becomes:

$$\begin{aligned} H &= \sum_k \psi_k^\dagger\sigma_z\psi_k\epsilon(k) + \frac{g^2}{4}N\sum_q \rho_{-q}\,\rho_q\,\frac{1}{1-\cos q} \\ \rho_q &= \frac{1}{N}\sum_k \psi_{q+k}^\dagger\psi_k \\ \epsilon(k) &= k \quad \text{for } -\pi \le k \le \pi, \text{ periodic with period } 2\pi. \\ N &= \text{number of sites.} \end{aligned} \tag{3.3}$$

All the functions of momenta are considered to be periodic functions with period 2π; momentum variables are to be viewed as angular variables. Momentum sums can be converted to momentum integrals according to the replacement rule:

$$\sum_k \to N\int\frac{dk}{2\pi} \tag{3.4}$$

We finally get for the Hamiltonian:

$$\begin{aligned} \frac{H}{N} &= \int\frac{dk}{2\pi}\,\psi_k^\dagger\,\sigma_z\,\psi_k\,\epsilon(k) + \frac{g^2}{4}\int\frac{dq}{2\pi}\,\rho_{-q}\,\rho_q\,\frac{1}{1-\cos q} \\ \rho_q &= \int\frac{dk}{2\pi}\,\psi_{k+q}^\dagger\,\psi_k \end{aligned} \tag{3.5}$$

Now define:

$$\psi_i = \begin{vmatrix} a_i \\ b_i \end{vmatrix} \tag{3.6}$$

I will minimize the expectation value of the Hamiltonian for the most general

Hartree-Fock state which conserves momentum and charge. This means that only the following expectation values of fermionic bilinears will be allowed:

$$\begin{array}{ll} \langle a_k^\dagger\, a_k\rangle = \Gamma_k^a & \langle a_k\, a_k^\dagger\rangle = 1-\Gamma_k^a \\ \langle b_k^\dagger\, b_k\rangle = \Gamma_k^b & \langle b_k\, b_k^\dagger\rangle = 1-\Gamma_k^b \\ \langle a_k^\dagger\, b_k\rangle = \Lambda_k & \langle b_k^\dagger\, a_k\rangle = \Lambda_k^* \end{array} \tag{3.7}$$

The expectation values Γ_k^a, Γ_k^b, and Λ_k, are not completely arbitrary. Positivity of the Hilbert space requires that the matrix:

$$\begin{pmatrix} \Gamma_k^a & \Lambda_k & 0 & 0 \\ \Lambda_k^* & \Gamma_k^b & 0 & 0 \\ 0 & 0 & 1-\Gamma_k^a & -\Lambda_k^* \\ 0 & 0 & -\Lambda_k & 1-\Gamma_k^b \end{pmatrix} \tag{3.8}$$

is positive semidefinite (this follows from the requirement that any linear combination of $a_k^\dagger\,|\mathrm{Vac}\rangle$, $b_k^\dagger\,|\mathrm{Vac}\rangle$, $a_k\,|\mathrm{Vac}\rangle$, $b_k\,|\mathrm{Vac}\rangle$ has positive norm). We thus obtain the following constraints:

$$\begin{array}{ll} \Gamma_k^a\Gamma_k^b \geq \left|\Lambda_k^2\right|, & (1-\Gamma_k^a)(1-\Gamma_k^b) \geq \left|\Lambda_k^2\right| \\ 1 \geq \Gamma_k^a \geq 0, & 1 \geq \Gamma_k^b \geq 0 \end{array} \tag{3.9}$$

We can now calculate the expectation value of $\rho_q\,\rho_{-q}$:

$$\langle \rho_q \rho_{-q}\rangle = \frac{1}{N^2}\sum_{k,k'}\langle (a_k^\dagger a_{k+q} + b_k^\dagger b_{k+q} - \delta_{k,k+q})(a_{k'+q}^\dagger a_{k'} + b_{k'+q}^\dagger b_{k'} - \delta_{k',k'+q})\rangle \tag{3.10}$$

the constant $\delta_{k,k+q}$ in the definition of ρ is required by the normal ordering prescription. We obtain:

$$\begin{aligned} \langle \rho_q\,\rho_{-q}\rangle &= \left(\frac{1}{N}\sum_k(\Gamma_k^a + \Gamma_k^b - 1)\right)^2 \delta_{q,0} \\ &+\frac{1}{N^2}\sum_k(\Gamma_k^a(1-\Gamma_{k+q}^a) + \Gamma_k^b(1-\Gamma_{k+q}^b) - \Lambda_k\Lambda_{k+q}^* - \Lambda_{k+q}\Lambda_k^*) \end{aligned} \tag{3.11}$$

In order for the potential energy to be finite, this has to vanish at $q=0$; the second term of Eq. (3.11) at $q=0$ becomes:

$$\frac{1}{N^2}\sum_k\left[\frac{\Gamma^a_k(1-\Gamma^a_k)+\Gamma^b_k(1-\Gamma^b_k)}{2}-|\Lambda_k|^2\right] \tag{3.12}$$

Multiplying the first two inequalities of (3.9)we obtain:

$$\sqrt{[\Gamma^a_k(1-\Gamma^a_k)][\Gamma^b_k(1-\Gamma^b_k)]}\geq|\Lambda_k|^2 \tag{3.13}$$

Since the geometric average of two numbers is never larger then the arithmetic average, (3.12) can vanish only if (3.13) is saturated; the first two bounds in (3.9) are therefore both saturated. It is easy to show that in this case we must have:

$$\begin{aligned}\Gamma^a_k &= 1-\Gamma^b_k\\ \Gamma^a_k\Gamma^b_k &= |\Lambda_k|^2\end{aligned} \tag{3.14}$$

This also implies that the first term in (3.11) vanishes. We can now parametrize the Γ and Λ in the following way:

$$\begin{aligned}\Gamma^a_k &= \sin^2\theta_k\\ \Gamma^b &= \cos^2\theta_k\\ \Lambda_k &= \sin\theta_k\,\cos\theta_k\,e^{i\phi_k}\\ 0\leq\theta_k &\leq\frac{\pi}{2}\end{aligned} \tag{3.15}$$

Substituting in (3.11) we obtain:

$$\begin{aligned}\langle\rho_q\,\rho_{-q}\rangle=\frac{1}{N^2}\sum_k(&\sin^2\theta_k\,\cos^2\theta_{k+q}+\cos^2\theta_k\,\sin^2\theta_{k+q}-\\ &2\sin\theta_k\,\cos\theta_k\,\sin\theta_{k+q}\,\cos\theta_{k+q}\,\cos(\phi_k-\phi_{k+q}))\end{aligned} \tag{3.16}$$

The expectation value of the Hamiltonian (3.3) can now be written as:

$$\begin{aligned}\langle H\rangle=&\sum_k(\sin^2\theta_k-\cos^2\theta_k)\epsilon(k)+\\ &\frac{g^2}{4N}\sum_q\sum_k(\sin^2\theta_k\,\cos^2\theta_{k+q}+\cos^2\theta_k\,\sin^2\theta_{k+q}-\\ &2\sin\theta_k\,\cos\theta_k\sin\theta_{k+q}\,\cos\theta_{k+q}\,\cos(\phi_k-\phi_{k+q}))\frac{1}{1-\cos q}\end{aligned} \tag{3.17}$$

Minimizing with respect to ϕ_k immediately gives $\phi_k = constant$. This arbitrary constant corresponds to the freedom of performing global chiral rotations on the system. The Hamiltonian becomes simply:

$$\langle H \rangle = \sum_k (2sin^2\theta_k - 1)\epsilon(k) + \frac{g^2}{4N}\sum_q\sum_k \frac{\sin^2(\theta_k - \theta_{k+q})}{1-\cos q} \tag{3.18}$$

For $g^2 = 0$ the minimum is at $\sin^2\theta_k = 1$ for $-\pi < k < 0$, $\sin^2\theta_k = 0$ for $0 < k < \pi$, as plotted in Fig. 4(a). For small g this solution will give an infinite potential energy, originating from the discontinuity of Γ_k^a at $k = 0$ and $k = \pi$. In fact, in this case, the charge-charge correlation is:

$$\langle \rho_q\, \rho_{-q} \rangle = \frac{1}{N^2}\sum_k \sin^2(\theta_k - \theta_{k+q}) = \frac{1}{N}\frac{|q|}{\pi} \quad \text{for } -\pi < q < \pi \tag{3.19}$$

extended to be periodic with period 2π in the variable q; the potential in (3.17) is therefore logarithmically divergent at $q = 0$. We would then expect that for small couplings Γ_k^a will become something like Fig. 4(b), with the theta function type singularities smoothed out; we can estimate the parameters h and ℓ, representing the size of the k intervals respectively around 0 and around π, in which Γ_k^a differs from the free fermions solution (Fig. 4(b)), in the following way: the kinetic energy behaves as $Ah^2 + B\ell$, where A and B are constants of order 1, while the potential energy behaves like $Cg^2|\log h| + Dg^2|\log \ell|$, because of the logarithmic singularities at $h = 0$ and $\ell = 0$; therefore:

$$H \sim Ah^2 + B\ell + Cg^2|\log h| + Dg^2|\log \ell| \tag{3.20}$$

and minimizing with respect to h and ℓ we obtain:

$$\begin{aligned} h^2 &\sim g^2 \\ \ell &\sim g^2 \end{aligned} \tag{3.21}$$

Let us now examine the behavior of the charge-charge correlation $\langle \rho_q\, \rho_{-q} \rangle$; for small q, $\sin^2(\theta_k - \theta_{k+q})$ vanishes for most values of k, except for $k \sim 0$ or $k \sim \pi$;

therefore we have:

$$\frac{1}{N^2}\sum_{k}\sin^2(\theta_k+\theta_{k+q}) = \frac{1}{N^2}\sum_{k\sim 0}\sin^2(\theta_k-\theta_{k+q})+\frac{1}{N^2}\sum_{k\sim\pi}\sin^2(\theta_k-\theta_{k+q}) \tag{3.22}$$

In the free field limit, both sums on the right hand side of (3.22) equal $|q|/(2\pi N)$, and we recover (3.19) . In the weak coupling regime, the $k \sim 0$ part of the correlation function will be smoothed out in a region of size $h \sim g$, while the $k \sim 0$ part will be smoothed out in a region of size $\ell \sim g^2$. It is easy to see that the whole variational Hamiltonian can be split into a part involving momenta of order zero, plus a part involving momenta of order π; this is because, in this limit, the potential coming from the intermediate q region, is highly insensitive to the behavior of θ_k for $k \sim 0$ and $k \sim \pi$; the part of the Hamiltonian involving lattice momenta close to 0, coincides with what we would have obtained if we performed a variational calculation of the continuum Schwinger model. In the case of the continuum Schwinger model we expect:

$$\langle \rho_q \, \rho_{-q} \rangle = q^2 \langle E_q \, E_{-q} \rangle \propto \frac{q^2}{\sqrt{q^2 + g^2/\pi}} \tag{3.23}$$

This is not what we obtain in the $g^2 \to 0$ limit of the SLAC lattice Schwinger model. Instead, we have:

$$\begin{aligned}\langle \rho_q \, \rho_{-q} \rangle &= \frac{1}{N^2}\sum_{k\sim 0}\sin^2(\theta_k - \theta_{k+q}) + \sum_{k\sim\pi}\sin^2(\theta_k - \theta_{k+q}) = \\ &= \frac{1}{N2\pi}|q| f_0(q) + \frac{1}{N2\pi}|q| f_\pi(q)\end{aligned} \tag{3.24}$$

where $f_0(q) \to 1$ for $q \gg g$, $f_\pi(q) \to 1$ for $q \gg g^2$. Only $f_0(q)$ is related to the continuum limit physics; therefore, we expect it to be an approximation to the function

$$\frac{|q|}{\sqrt{q^2 + g^2/\pi}}$$

for small g.

The variational problem has been solved by sampling the function θ_k with up to 100 points, and then minimizing (3.17) numerically. In Fig. 5(a), the function $\sin^2\theta_k$ is plotted for two different values of the coupling constant, and the corresponding excitation mass, evaluated by calculating $q/f_0(q)$, and fitting it with a form $\sqrt{q^2+m^2}$, is given. The excitation mass is seen to approach the continuum value from above as g^2 gets smaller, as plotted in Fig. 5(b); in particular, for $g^2 = .25$, the calculated lattice model mass overestimates the exact continuum mass by about 10%* . The separation of the $\ell \sim 0$ states can be performed more rigorously by defining a cut-off (point-split) electromagnetic charge density:

$$\rho_{k\sim 0} = \frac{1}{N}\sum_k F(k+q/2)\,\psi^\dagger_{k+q}\psi_k \tag{3.25}$$

where $F(k)$ is a smooth function such that $F(0) = 1$, $f(\pi) = 0$, $F(k) = F(-k)$. If we define:

$$\rho_{k\sim\pi}(q) = \frac{1}{N}\sum_k (1 - F(k+q/2))\,\psi^\dagger_{k+q}\psi_k \tag{3.26}$$

we have $\rho = \rho_{k\sim 0} + \rho_{k\sim\pi}$, and one can verify that for small q:

$$\langle\rho_q\,\rho_{-q}\rangle = \langle\rho_{k\sim 0}(q)\,\rho_{k\sim 0}(-q)\rangle + \langle\rho_{k\sim\pi}(q)\,\rho_{k\sim\pi}(-q)\rangle \tag{3.27}$$

From the Hartree-Fock calculation performed in this section, we can draw the following conclusions:

(a) The model contains a massive excitation corresponding to the continuum Schwinger model excitation.

(b) The lattice charge density does not corresponds to the continuum charge density. One can define a charge density with the correct continuum limit by cutting off states with momenta close to π. This is essentially a point splitting definition of the charge[5] .

* When I started this work, Michael Peskin informed me that he actually performed a Hartree-Fock calculation of the continuum Schwinger model. He claims the mass of the excitation he obtained differs from the exact mass by 5%.

(c) The part of the charge density involving momenta close to π, has nontrivial dynamics, in the sense that $\langle \rho_{k\sim\pi}(q)\, \rho_{k\sim\pi}(-q) \rangle$ has nontrivial behavior as $q \to 0$. This behavior manifest itself at scales $q \sim g^2$, unlike the point split charge density, which has nontrivial behavior at scales $q \sim g$. The dynamics associated with fermions of momenta around π does not have a Lorentz invariant continuum limit.

A few questions remain unanswered in this analysis. First of all, we obtain for small q that $\langle \rho_{k\sim 0}(q)\, \rho_{k\sim\pi}(-q) \rangle = 0$ in our Fock state. This is because we did not assume any expectation value for $\langle \psi_k^\dagger\, \psi_{k+\pi} \rangle$. The Hartree-Fock calculation becomes much more involved in this case. A second problem is to determine the spectrum of the excitation associated with $\rho_{k\sim\pi}$. We cannot follow the same procedure used for $\rho_{k\sim 0}$, because in that case we relayed on Lorentz invariance, which is necessary to extract a mass parameter from an equal time correlation function, while the dynamical effects associated with $k \sim \pi$ fermions are clearly not Lorentz invariant. A third problem is the following: we know that the lattice ρ^5 is both conserved and gauge invariant. Is there a Goldstone particle associated with it? We cannot rely on Goldstone theorem in this case, because the SLAC derivative is long range when applied to objects with momentum π, and the full lattice ρ^5 does contain such objects.

In the next section I will analyze these problems in a simplified contest, that is to say, in the case of a truncated SLAC derivative; in this contest the analysis will turn out to be quite simple, so that the physics of the model becomes transparent.

4. The truncated SLAC derivative Schwinger model.

There is a fairly simple physical argument that allows to solve exactly the Schwinger model. The argument goes as follows: we would expect that in the Schwinger model only fermions with momenta of order g will be excited; if we divide space into elements $\Delta L \ll 1/g$, we would expect that the local Fermi sea will not differ much from the free Fermi sea, except that we may have local charge density fluctuations, that would be represented by shifts of the 'left' and 'right' Fermi surface (see Fig. 6). Since the number of states in the momentum interval Δk is $\Delta k\, \Delta L/h$, we can obtain the following right and left charge densities:

$$\begin{aligned} \rho_R &= \frac{\Delta k_R}{2\pi} \\ \rho_L &= \frac{\Delta k_L}{2\pi} \end{aligned} \tag{4.1}$$

The kinetic energy contributed by the element ΔL is:

$$T_{\Delta L} = \frac{1}{2}\Delta k_R\, \rho_R\, \Delta_L + \frac{1}{2}\Delta k_L\, \rho_L\, \Delta L = \frac{\pi}{2}(\rho^2 + {\rho^5}^2)\, \Delta L \tag{4.2}$$

Since $\rho = \partial_1 E$, $\rho^5 = \partial_0 E$,* and the electromagnetic energy density is given by $g^2 E^2/2$, we obtain for the total energy density:

$$H = \frac{\pi}{2}((\partial_1 E)^2 + (\partial_0 E)^2) + \frac{g^2}{2} E^2 \tag{4.3}$$

which represents a free excitation with mass g^2/π.

The above intuitive argument gives the exact answer; a more rigorous derivation is given in the Appendix.

We will consider now a lattice version of the Schwinger model with a truncated SLAC derivative†. Whichever truncation mechanism is chosen, the spectrum will look like Fig. 7. The slope of the spectrum at momenta close to π

* The above definition of the E field differs by a factor of g from the conventional one.

† Truncated versions of the SLAC derivative have in fact been used in lattice calculations, see Ref. 6.6.

is of the order of the range of the derivative; as the slope goes to infinity, we recover the full SLAC derivative. We can define the charge densities n_0^a, n_0^b, to be the densities of fermions of type a or b with momenta close to 0, and n_π^a, n_π^b, the densities of fermions a and b with momenta close to π. As before, we can calculate the contribution to the kinetic energy coming from fermions with momenta close to 0 to be equal to:

$$\begin{aligned} &\frac{\pi}{2}(\rho_0^2 + {\rho_0^5}^2) \\ &\rho_0 = n_0^a + n_0^b \\ &\rho_0^5 = n_0^a - n_0^b \end{aligned} \tag{4.4}$$

and the kinetic energy contributed by fermions with momenta close to π:

$$\begin{aligned} &c\frac{\pi}{2}(\rho_\pi^2 + {\rho_\pi^5}^2) \\ &\rho_\pi = n_\pi^a + n_\pi^b \\ &\rho_\pi^5 = n_\pi^a - n_\pi^b \end{aligned} \tag{4.5}$$

We can now introduce two scalar fields χ and ϕ such that:

$$\begin{aligned} \rho_0 &= \partial_1\phi \\ \rho_\pi &= \partial_1\chi \end{aligned} \tag{4.6}$$

Since ρ_0^5 is the charge current of fermions with momenta close to 0 we have:

$$\rho_0^5 = \partial_0\phi \tag{4.7}$$

similarly:

$$\rho_\pi^5 = -\frac{1}{c}\partial_0\chi \tag{4.8}$$

The total electric field is $\phi + \chi$, and therefore the total Hamiltonian is:

$$H = \frac{\pi}{2}(\,(\partial_0\phi)^2 + (\partial_1\phi)^2) + c\frac{\pi}{2}(\,(\partial_1\chi)^2 + \frac{1}{c^2}(\partial_0\chi)^2) + \frac{g^2}{2}(\phi + \chi)^2 \tag{4.9}$$

We can now define canonical fields:

$$\widetilde{\phi} = \sqrt{\pi}\phi$$
$$\widetilde{\chi} = \sqrt{\frac{\pi}{c}}\chi \tag{4.10}$$

and the Hamiltonian becomes:

$$H = \frac{1}{2}\left((\partial_0\widetilde{\phi})^2 + (\partial_0\widetilde{\chi})^2\right) + \frac{1}{2}(\partial_1\widetilde{\phi})^2 + \frac{1}{2}(c\partial_1\widetilde{\chi})^2 + \frac{g^2}{2\pi}(\widetilde{\phi} + \sqrt{c}\widetilde{\chi})^2 \tag{4.11}$$

The independent modes of this Hamiltonian can be obtained by diagonalizing the matrix:

$$\begin{vmatrix} k^2 + \frac{g^2}{\pi} & \sqrt{c}\frac{g^2}{\pi} \\ \sqrt{c}\frac{g^2}{\pi} & c^2k^2 + c\frac{g^2}{\pi} \end{vmatrix} \tag{4.12}$$

and the energy of each mode will be the square root of the eigenvalues. First, for $k = 0$, we have the eigenvalues 0 and $(c+1)g^2/(2\pi)$; the corresponding eigenvectors are:

$$v_1 = \frac{\widetilde{\phi}_0\sqrt{c} - \widetilde{\chi}_0}{\sqrt{1+c}}$$
$$v_2 = \frac{\widetilde{\phi}_0 + \sqrt{c}\widetilde{\chi}_0}{\sqrt{1+c}} \tag{4.13}$$

where the subscript '0' indicates the 0 momentum Fourier component of $\widetilde{\phi}$ and $\widetilde{\chi}$. The variable conjugated to the null eigenstate is:

$$\partial_0\widetilde{\phi}_0\sqrt{c} - \partial_0\widetilde{\chi}_0 \propto \partial_0\phi_0 - \partial_0\chi_0/\sqrt{c} \propto Q_0^5 + Q_\pi^5 = Q^5 \tag{4.14}$$

The null eigenstate is therefore the Goldstone mode associated with the lattice ρ^5. For small k the evolution of the spectrum is easily obtained from first order perturbation theory:

$$\lambda_1 = v_1^\dagger \begin{vmatrix} k^2 & 0 \\ 0 & c^2k^2 \end{vmatrix} v_1 = ck^2$$
$$\lambda_2 = (c+1)\frac{g^2}{\pi} + v_2^\dagger \begin{vmatrix} k^2 & 0 \\ 0 & c^2k^2 \end{vmatrix} v_2 = (c+1)\frac{g^2}{\pi} + \left(\frac{1+c^3}{1+c}\right) \tag{4.15}$$

On the other hand, for k values such that $c^2k^2 \gg cg^2/\pi$, there will be very little mixing, and the eigenvalues will be:

$$\lambda_1 = k^2 + \frac{g^2}{\pi}$$
$$\lambda_2 = c^2k^2 + c\frac{g^2}{\pi} \tag{4.16}$$

The two spectral branches are plotted in Fig. 8. From the figure it is clear that mixing between fermionic states with momenta close to 0 and those with momenta close to π becomes negligible for $k \gg g/\sqrt{c}$; in the same limit the upper branch of the spectrum reaches the free field limit. In order for the approximations involved to be consistent, the part of the spectrum involving fermions with momenta around π must reach the free field limit for $k \ll 1/c$, because only in this range the fermion spectrum is actually linear with slope c; we therefore obtain the bound:

$$g\sqrt{c} \ll 1 \, . \tag{4.17}$$

Because of the above inequality we cannot take the strict $c \to \infty$ limit, but we can consider a double limit $g \to 0$, $c \to \infty$, $\sqrt{c}\,g = \text{constant} \ll 1$. In this limit the high energy spectral branch acquires infinite energy, and the lower branch will converge to the continuum limit spectrum, except at $k = 0$. We still expect that for $\sqrt{c}g \sim 1$ the qualitative behavior indicated in Fig. 8 will persist; in this case, the spectrum of the lattice theory does not match the continuum limit spectrum unless $k \gg g/\sqrt{c} \sim g^2$. The Hartree-Fock calculation, carried out with the full SLAC derivative, leads to the same conclusion; therefore, we would expect that the spectrum will be qualitatively represented by Fig. 8, except that it may not have a finite slope at the origin, and the deviation from the continuum limit spectrum would extend into a region$k \lesssim g^2$.

5. Conclusions.

We have seen that the continuum limit spectrum of the SLAC lattice Schwinger model does match the exact continuum solution; however, the operators of the lattice model do not correspond directly to continuum limit operators. Only point-split operators have a sensible continuum limit, and this holds for the chiral charge as well as for the electromagnetic charge. One can view lattice operators as the sum of two pieces, one (the point-split one) involving fermions with momenta close to 0, and the other involving fermions with momenta close to π; although one would naively expect that states with momenta close to π are not altered by the presence of an interaction, because of the high kinetic energies involved, the anomaly forces some nontrivial dynamics in this region. These dynamical effects manifest themselves in the two point functions of the fermions bilinears, for small values of the momenta. This is consistent with the results obtained by Rabin in $3+1$ QED, see ref. 7; he shows that in perturbation theory extra subtractions are needed in SLAC lattice QED in order to recover the continuum limit.

The spectrum of the theory also exhibits sharp differences from the continuum spectrum in the low momentum region; In particular, a zero energy excitation, corresponding to the conserved and gauge invariant chiral charge, is present in the spectrum at zero momentum. Its velocity, for small couplings, is much larger than 1; it is not, therefore, a Lorentz invariant particle with a sensible continuum limit. For sufficiently large momenta, the spectrum of this excitation approaches the spectrum of the continuum limit massive particle.

Although the results quoted above have been derived for the simple case of the Schwinger model, there are good reasons to believe that they will persist, at least qualitatively, in higher dimensional theories. First of all, long range behavior of current-current correlation functions, caused by fermionic states with at least one component of the momentum close to π, should persist in higher dimensions; this implies that, in a confining theory, the states with momenta lying close to the $k_i = \pi$ planes, will acquire nontrivial dynamics. A second reason, is that the chiral anomaly in $3+1$ dimensions has a remarkable connection to the Schwinger model anomaly, (see for example ref. 4); this connection originates from the fact that free fermions, in an external magnetic field, behave like different copies of one dimensional Dirac fermions, one of these copies being massless. It is in fact the Schwinger model anomaly of this zero mode that, in presence of an electric field parallel to the magnetic field, generates the $3+1$ dimensional anomaly.

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APPENDIX

Consider a fermion in a one dimensional lattice; assume that the spectrum has negative energy states for $-\pi < k < 0$, and positive energy for $0 < k < \pi$. We can define a cut-off charge density associated with the fermions with momenta around 0 as:

$$\rho_q = a^\dagger_{k+q} a_k \, f(k+q) \, f(k) \tag{A1}$$

where $f(k)$ is constant equal to 1 for $k \sim 0$, and it vanishes smoothly at $k \sim \pm\frac{\pi}{2}$. For small q we have:

$$[\rho_q, \rho_{q'}] = \sum_k a^\dagger_{k+q} a_{k-q'} \frac{1}{2} (q' - q) \left(\frac{d}{dk} f(k)^4 \right) + O(q^2, q'^2) \, . \tag{A2}$$

The derivative of $f(k)$ vanishes in a region around $k = 0$; therefore the fermionic bilinear in the sum acts only on states of high energy. Assuming that a weak interaction is active, only states at low energy will be altered, so that we can replace: $a^\dagger_{k+q} a_{k+q'} \to \delta_{q_q'} \theta(k)$, and Eq. (A2) becomes:

$$[\rho_q, \rho_{q'}] = -\delta_{q+q'} \, q \, \frac{N}{2\pi} \, , \qquad \text{N = number of lattice sites} \tag{A3}$$

for a left mover we would have obtained:

$$[\rho_q, \rho_{q'}] = \delta_{q+q'} \, q \, \frac{N}{2\pi} \, . \tag{A4}$$

The same conclusion holds for a cut-off charge density involving $k \sim \pi$ states. With a similar argument one can show that:

$$\sum_q \rho_q \, \rho_{-q} = \pm \frac{N}{\pi} \sum_k a^\dagger_k a_k + \text{constant} \times \sum_k a^\dagger_k a_k \, . \tag{A5}$$

when the plus sign holds for right movers, the minus sign for left movers, and the sums involve only small values of q and k. Equation (A5) relates the energy and the square of the charge density in the same way as the argument of Section 4, and the commutators (A3), (A4) can be used to show that the variables $\tilde{\phi}$ and $\tilde{\chi}$, defined in Eq. (4.11) have canonical commutation relations with $\partial_0 \tilde{\phi}$ and $\partial_0 \tilde{\chi}$.

FIGURE CAPTIONS

1. Anomaly in continuum 1+1 dimensional Dirac fermions.

2. Anomaly for Susskind fermions.

3. Anomaly for SLAC fermions.

4. Expectation value for the occupation number of right movers in the free and interacting Hartree-Fock vacuum of the lattice Schwinger model with SLAC fermions

5. (a) numerical results for the expectation value of the occupation number of right movers for $g^2 = 2$ and $g^2 = 1$,
(b) numerically determined mass to continuum limit mass ratio plotted as a function of g^2.

6. The local fermi sea in the Schwinger model.

7. Spectrum associated with a truncated SLAC derivative.

8. Spectrum of the excitations in the lattice Schwinger model with a truncated SLAC derivative.

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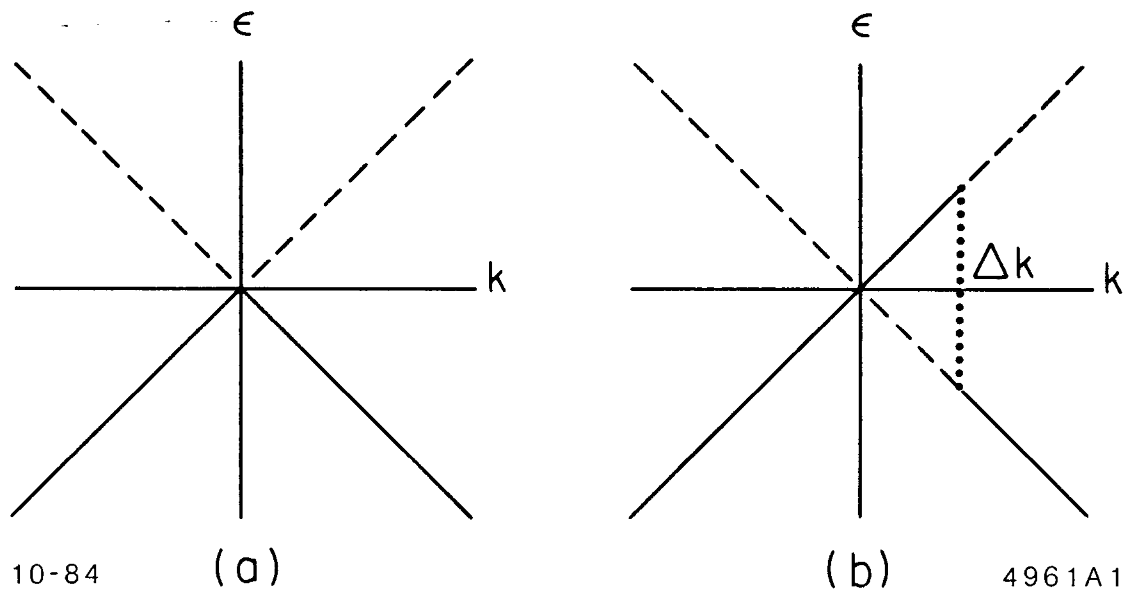


Fig. 1

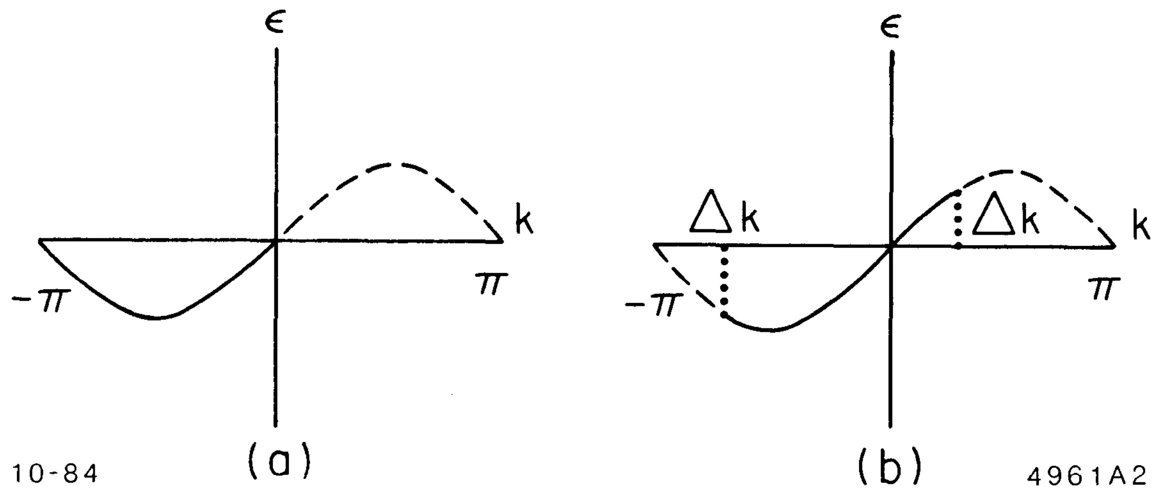
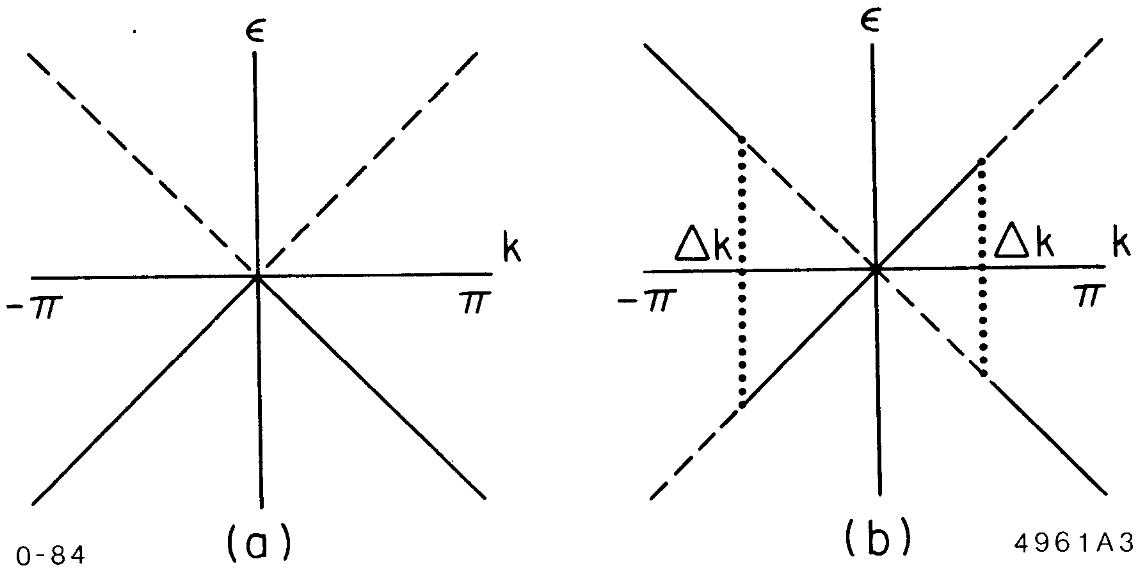


Fig. 2



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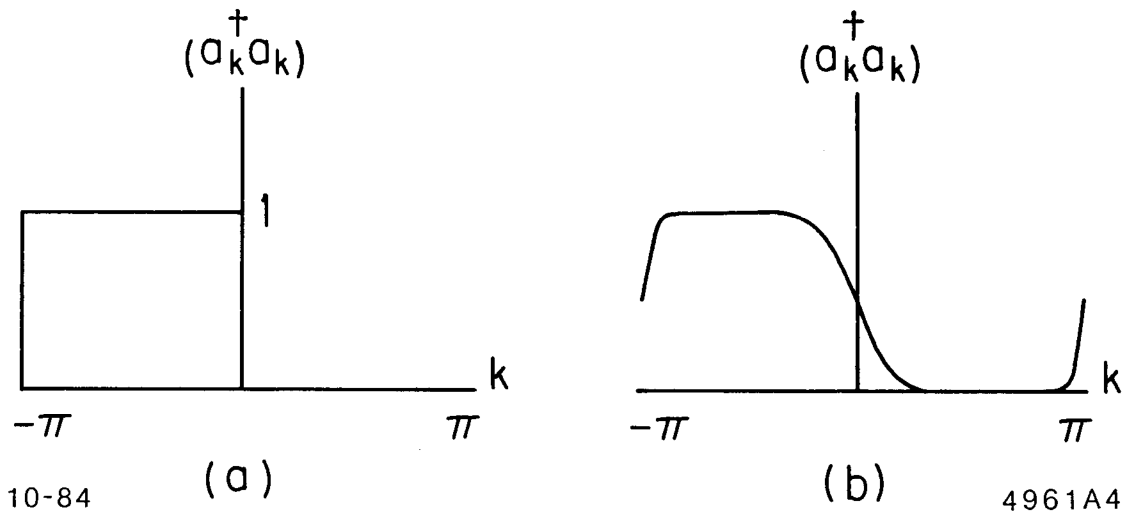

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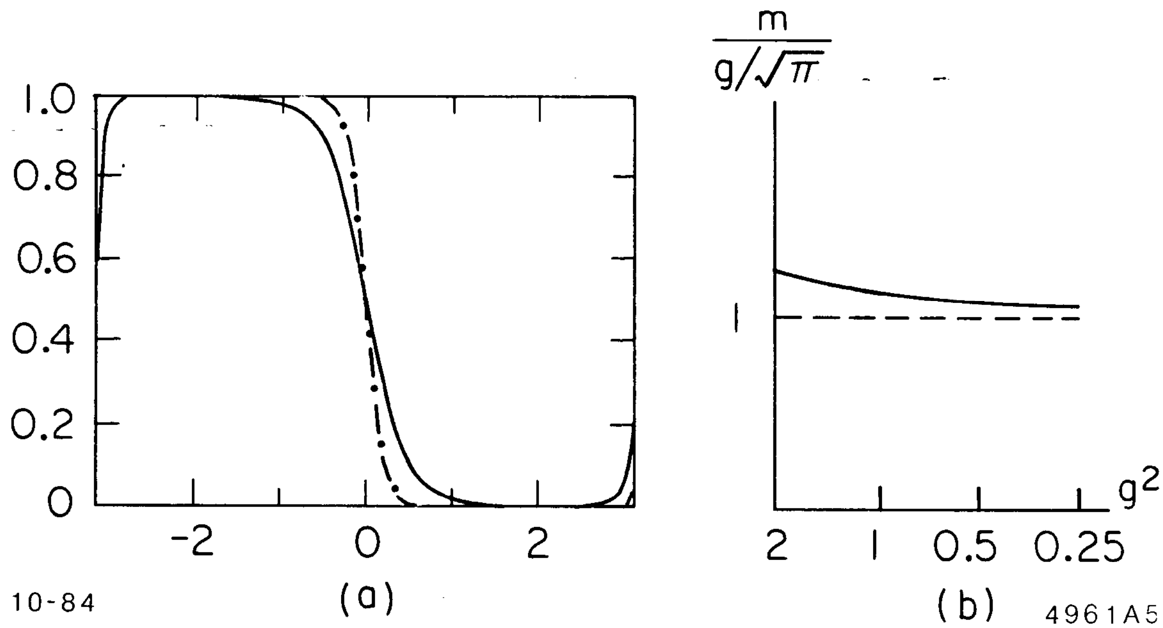

Fig. 5

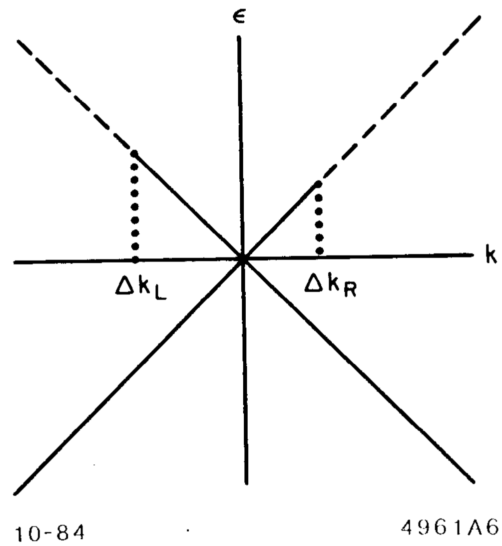

Fig. 6

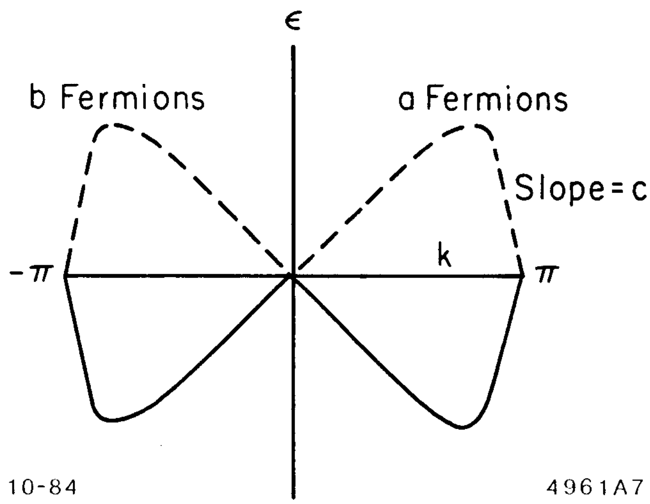

Fig. 7

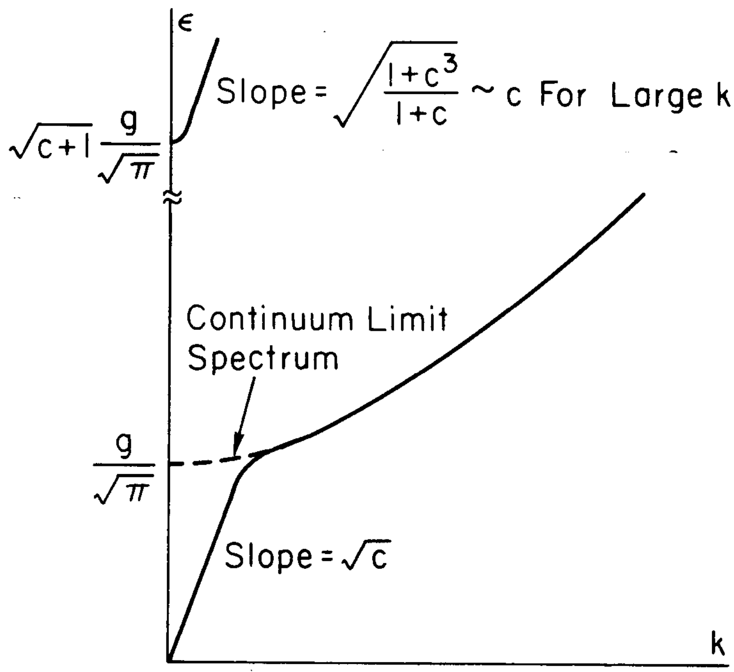

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Fig. 8