SLAC – PUB – 3461
September 1984
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SUPERSYMMETRIC SIGMA MODELS*

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Lectures given at the

Bonn–NATO Advanced Study Institute on Supersymmetry

Bonn, West Germany

August 20-31, 1984

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515

ABSTRACT

We begin to construct the most general supersymmetric Lagrangians in one, two and four dimensions. We find that the matter couplings have a natural interpretation in the language of the nonlinear sigma model.

1. INTRODUCTION

The past few years have witnessed a dramatic revival of interest in the phenomenological aspects of supersymmetry [1,2]. Many models have been proposed, and much work has been devoted to exploring their experimental implications. A common feature of all these models is that they predict a variety of new particles at energies near the weak scale. Since the next generation of accelerators will start to probe these energies, we have the exciting possibility that supersymmetry will soon be found.

While we are waiting for the new experiments, however, we must continue to gain a deeper understanding of supersymmetric theories themselves. One vital task is to learn how to construct the most general possible supersymmetric Lagrangians. These Lagrangians can then be used by model builders in their search for realistic theories. For $N = 1$ rigid supersymmetry, it is not hard to write down the most general possible supersymmetric Lagrangian. For higher N, however, and for all local supersymmetries, the story is more complicated.

In these lectures we will begin to discuss the most general matter couplings in $N = 1$ and $N = 2$ supersymmetric theories. We will start in $1 + 1$ dimensions, where we will construct the most general supersymmetric couplings of the massless spin $(0, \frac{1}{2})$ matter multiplet [3,4]. We will consider $N = 1$, $N = 2$ and $N = 4$ supersymmetric theories, and we shall find that as N increases, the matter couplings become more and more restricted.

We shall then drop to $0 + 1$ dimensions, where we will discuss supersymmetric quantum mechanics. Although supersymmetric quantum mechanics might seem irrelevant, it has important mathematical and physical consequences. For example, it has been used to demonstrate dynamical supersymmetry breaking [5], to prove the Atiyah-Singer index theorem [6], and even to invent a new branch of Morse theory [7].

After introducing supersymmetric quantum mechanics, we shall climb back to $3+1$ dimensions, where we will remain for the rest of the lectures. We will first consider $N=1$ supersymmetry, both rigid and local. We will construct the most general Lagrangian containing the spin $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$ supersymmetry multiplets [8–13]. The resulting Lagrangian will be long and complicated, but it will have a very simple geometrical interpretation. In the case of local supersymmetry, we shall see that global topology places important restrictions on the matter couplings. A surprising result is that in some cases, global consistency requires that Newton's constant be quantized in units of the scalar self-coupling [11].

For our final topic, we will begin to discuss the most general couplings of the massless spin $(0, \frac{1}{2})$ multiplet in $N=2$ supersymmetry. We will not attempt to include masses, potentials or gauge fields, but we will still find a striking result: Matter couplings that are allowed in $N=2$ rigid supersymmetry are forbidden in $N=2$ local supersymmetry, and vice versa [14]. Furthermore, we shall see that the reduction from $N=2$ to $N=1$ is not trivial.

2. SUPERSYMMETRY IN ONE AND TWO DIMENSIONS

The Supersymmetric Nonlinear Sigma Model

As we shall see throughout these lectures, supersymmetric matter couplings generally induce complicated terms in the interaction Lagrangian. Fortunately, these terms have a relatively simple description in the language of the nonlinear sigma model. The most familiar sigma model is the famous $O(3)$ model introduced by Gell-Mann and Lévy [15]. In this model, three pion fields π^a are constrained to lie on the sphere S^3. The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} \;=\; -\,\frac{1}{2}\, g_{ab}(\pi^c)\, \partial_\mu \pi^a \partial^\mu \pi^b \ , \tag{2.1}$$

where $g_{ab}(\pi^c)$ is the metric on S^3,

$$g_{ab}(\pi^c) \;=\; \left[(f^2 - \vec{\pi}\cdot\vec{\pi})\delta_{ab} + \pi_a \pi_b\right] \Big/ (f^2 - \vec{\pi}\cdot\vec{\pi}) \ . \tag{2.2}$$

From this example it is clear how to generalize the sigma model to other Riemannian manifolds $\mathcal{M}$. One simply views the scalar fields ϕ^a as maps from spacetime into $\mathcal{M}$,

$$\phi^a \ : \quad \text{spacetime} \ \longrightarrow \ \mathcal{M} \ . \tag{2.3}$$

The bosonic Lagrangian is given by

$$\mathcal{L} \;=\; -\,\frac{1}{2}\, g_{ab}(\phi^c)\, \partial_\mu \phi^a \partial^\mu \phi^b \ , \tag{2.4}$$

where $g_{ab}(\phi^c)$ is the metric on the manifold $\mathcal{M}$. Note that for each value of μ, $\partial_\mu \phi^a$ is a vector in the tangent space of $\mathcal{M}$. The Lagrangian $\mathcal{L}$ is constructed entirely from geometrical objects, and transforms as a scalar under coordinate transformations in $\mathcal{M}$. Furthermore, $\mathcal{L}$ is left invariant under the various isometries of $\mathcal{M}$.

In $1+1$ dimensions, it is easy to find a supersymmetric extension of (2.4) [3,4]. The superspace Lagrangian is given by [♯1]

$$\mathcal{L} \;=\; \int d^2\theta \; g_{ab}(\Phi^c)\, \overline{D}\Phi^a D\Phi^b \; . \tag{2.5}$$

Here Φ^a is a real scalar superfield in $1+1$ dimensions [17],

$$\Phi^a(x,\theta) \;=\; \phi^a(x) \;+\; \overline{\theta}\chi^a(x) \;+\; \frac{1}{2}\overline{\theta}\theta\, F^a(x) \;, \tag{2.6}$$

and $g_{ab}(\Phi^c)$ is again the metric. The component fields ϕ^a and F^a are real and bosonic, and χ^a is a two-component Majorana spinor. The spinor derivative

$$D \;=\; \frac{\partial}{\partial\overline{\theta}} \;+\; (\overline{\theta}\gamma^\mu)\partial_\mu \tag{2.7}$$

and the superfield Φ^a both contain the real Grassmann parameter $\overline{\theta} = (-\theta_2, \theta_1)$. The derivative D anticommutes with the spinor supercharge Q, so the Lagrangian (2.5) is manifestly supersymmetric.

To see that (2.5) corresponds to the sigma model (2.4), we must expand Φ^a in terms of component fields. Inserting (2.6) in (2.5), taking the highest component, and eliminating the auxiliary fields F^a, we find the following component Lagrangian [4],

$$\begin{aligned}\mathcal{L} \;=\; &-\frac{1}{2}\, g_{ab}(\phi^c)\, \partial_\mu\phi^a\partial^\mu\phi^b \;-\; \frac{1}{2}\, g_{ab}(\phi^c)\, \overline{\chi}^a\gamma^\mu \mathcal{D}_\mu\chi^b \\ &+\; \frac{1}{12}\, R_{abcd}\, (\overline{\chi}^a\chi^c)(\overline{\chi}^b\chi^d) \; .\end{aligned} \tag{2.8}$$

The covariant derivative

$$\mathcal{D}_\mu\chi^b \;=\; \partial_\mu\chi^b \;+\; \Gamma^b{}_{cd}\partial_\mu\phi^c\chi^d \tag{2.9}$$

ensures that χ^b transforms in the tangent space of $\mathcal{M}$. The connection $\Gamma^b{}_{cd}$ is the Christoffel symbol on the (Riemannian) manifold $\mathcal{M}$, and R_{abcd} is the usual Riemann curvature.

♯1 Our metric and spinor conventions are those of Ref. [16].

The Lagrangian (2.8) is manifestly supersymmetric because it was derived from a superspace formalism. However, it is useful to check that it is invariant (up to a total derivative) under the following supersymmetry transformations:

$$\begin{aligned} \delta\phi^a &= \bar{\epsilon}\chi^a , \\ \delta\chi^a &= \gamma^\mu \partial_\mu \phi^a \epsilon - \Gamma^a{}_{bc}\delta\phi^b \chi^c . \end{aligned} \tag{2.10}$$

The term with the connection coefficient is required to ensure that supersymmetry transformations commute with the coordinate transformations, or diffeomorphisms, of the manifold $\mathcal{M}$.

MODELS WITH EXTENDED SUPERSYMMETRY

The Lagrangian (2.8) demonstrates that $N = 1$ supersymmetric sigma models exist for all Riemannian manifolds $\mathcal{M}$. We would now like to know which manifolds give rise to sigma models with extended supersymmetry ($N > 1$). In this section we shall see that extra supersymmetries imply strong restrictions on the manifolds $\mathcal{M}$ [4,8].

To discover exactly what restrictions follow from additional supersymmetries, we take the most general ansatz for the transformation laws, consistent with dimensional arguments and Lorentz and parity invariance,

$$\begin{aligned} \delta\phi^a &= I^a{}_b \bar{\epsilon}\chi^b \\ \delta\chi^a &= H^a{}_b \gamma^\mu \partial_\mu \phi^b \epsilon - S^a{}_{bc}(\bar{\epsilon}\chi^b)\chi^c + V^a{}_{bc}(\bar{\epsilon}\gamma^\mu\chi^b)\gamma_\mu\chi^c \\ &\quad + P^a{}_{bc}(\bar{\epsilon}\gamma_5\chi^b)\gamma_5\chi^c . \end{aligned} \tag{2.11}$$

Here I, H, S, V and P are all functions of the dimensionless field ϕ^a. Since the transformations (2.11) must commute with diffeomorphisms, I, H, V and P must all be tensors.

We now demand that the Lagrangian (2.8) be invariant (up to a total derivative) under the transformations (2.11). Cancellation of the $\bar{\epsilon}\chi$ terms requires

$$g_{ac}I^c{}_b = g_{bc}I^c{}_a , \qquad \nabla_c I^a{}_b = \nabla_c H^a{}_b = 0 , \tag{2.12}$$

while cancellation of the $(\bar{\epsilon}\chi)(\bar{\chi}\chi)$ terms implies

$$V^a{}_{bc} = P^a{}_{bc} = 0 , \qquad S^a{}_{bc} = \Gamma^a{}_{cd}I^d{}_b , \qquad I^a{}_b H^b{}_c = \delta^a{}_c . \tag{2.13}$$

The conditions (2.12) and (2.13) lead automatically to the cancellation of the $(\bar{\epsilon}\chi)(\bar{\chi}\chi)(\bar{\chi}\chi)$ terms in the variation of $\mathcal{L}$.

The conditions (2.12) and (2.13) tell us that each additional supersymmetry requires the existence of a covariantly constant tensor $I^a{}_b$, such that

$$g_{ab} I^a{}_c I^b{}_d \; = \; g_{cd} \; . \tag{2.14}$$

For each such tensor, there is a fermionic transformation law given by

$$\begin{aligned} \delta\phi^a \; &= \; I^a{}_b \bar{\epsilon} \chi^b \; , \\ \delta\chi^a \; &= \; (I^{-1})^a{}_b \gamma^\mu \partial_\mu \chi^b \epsilon \; - \; \Gamma^a{}_{bc} \delta\phi^b \chi^c \; . \end{aligned} \tag{2.15}$$

Note, however, that the transformations (2.15) are not quite supersymmetries. To be supersymmetries, they must also obey the supersymmetry algebra

$$\{ Q^{(A)}, \overline{Q}^{(B)} \} \; = \; 2\gamma^\mu P_\mu \, \delta^{AB} \; , \tag{2.16}$$

where $A, B = 1, \ldots, N$. If there are several supersymmetries, each with its own covariantly constant tensor $I^{(A)a}{}_b$, then (2.16) requires

$$I^{(A)} I^{(B)-1} \; + \; I^{(B)} I^{(A)-1} \; = \; 2\delta^{AB} \; , \tag{2.17}$$

where we have switched to matrix notation. Let us suppose that $A = 1$ or $B = 1$ denotes the first supersymmetry, so that $I^{(1)a}{}_b = \delta^a{}_b$. Then, when $A \neq 1$ and $B = 1$, (2.17) implies

$$I^{(A)} \; = \; - \, I^{(A)-1} \; , \tag{2.18}$$

or

$$I^{(A)a}{}_c I^{(A)c}{}_b \; = \; - \, \delta^a{}_b \; , \qquad I^{(A)}{}_{ab} \; = \; - \, I^{(A)}{}_{ba} \; , \tag{2.19}$$

(for $A \neq 1$). We see that each extra supersymmetry requires the existence of a covariantly constant tensor whose square is -1. Furthermore, equation (2.17) implies a Clifford algebra structure for all supersymmetries beyond the first,

$$I^{(A)} I^{(B)} \; + \; I^{(B)} I^{(A)} \; = \; - \, 2\delta^{AB} \; , \tag{2.20}$$

where $A, B \neq 1$.

The preceeding analysis tells us that any Riemannian manifold gives rise to a supersymmetric sigma model in $1+1$ dimensions. However, a second supersymmetry exists only if one can define a covariantly constant tensor field $I^a{}_b$, satisfying

$$I^a{}_c I^c{}_b \;=\; -\,\delta^a{}_b\,, \qquad g_{ab} I^a{}_c I^b{}_d \;=\; g_{cd}\,. \tag{2.21}$$

Such a tensor field is called a *complex structure*; it denotes "multiplication by i" in the tangent space. A manifold $\mathcal{M}$ that admits such a tensor field is called a *Kähler manifold.* In $1+1$ dimensions, $N=2$ supersymmetry requires that $\mathcal{M}$ be Kähler [4,8].

Further supersymmetries require the existence of additional parallel complex structures satisfying (2.20) and (2.21). Note that if two such structures exist, their product automatically generates a third, so $N=3$ supersymmetry automatically implies $N=4$. A manifold with three parallel complex structures satisfying (2.20) and (2.21) is called a *hyperkähler manifold.* In $1+1$ dimensions, $N=4$ supersymmetry requires that $\mathcal{M}$ be hyperkähler [4].

The various manifolds discussed above can be described very elegantly in terms of their holonomy groups G [18]. The *holonomy group* of a connected n-dimensional Riemannian manifold is the group of transformations generated by parallel transporting all vectors around all possible closed curves in $\mathcal{M}$. If the parallel transport is done with respect to the Riemannian connection, the transported vector V'^a will be related to the original vector V^a by a rotation in the tangent space. In infinitesimal form, we have

$$[\nabla_a, \nabla_b]\,V^c \;=\; R_{ab}{}^c{}_d V^d\,, \tag{2.22}$$

and we see that the holonomy group G is generated by the Riemann curvature tensor of the manifold.

In general, an n-dimensional Riemannian manifold has holonomy group $O(n)$, provided the parallel transport is done with respect to the Riemannian connection. If, however, the manifold admits a parallel complex structure, the holonomy group is not all of $O(n)$. Since the complex structure $I^{(A)a}{}_b$ is parallel, it commutes with the holonomy group,

$$[\nabla_a, \nabla_b]\, I^{(A)c}{}_d \;=\; 0 \qquad \Rightarrow \qquad I^{(A)c}{}_f\, R_{ab}{}^f{}_d \;-\; R_{ab}{}^c{}_f\, I^{(A)f}{}_d \;=\; 0\,. \tag{2.23}$$

Manifolds $\mathcal{M}$ whose holonomy group G leaves invariant one complex structure are called Kähler. They necessarily have dimension $2n$, and their holonomy group is not all of $O(2n)$, but rather $G \subseteq U(n) \subseteq O(2n)$. Manifolds $\mathcal{M}$ whose holonomy group leaves invariant the quaternionic structure (2.20), (2.21) are called hyperkähler. They necessarily have dimension $4n$ and holonomy group $G \subseteq Sp(n) \subseteq O(4n)$. The relation between supersymmetry, sigma model manifolds and holonomy groups is summarized in Table 1.

Table 1
Rigid Supersymmetry in 1 + 1 Dimensions

N	manifold $\mathcal{M}$	holonomy G
1	n-dimensional Riemannian	$\subseteq O(n)$
2	$2n$-dimensional Kähler	$\subseteq U(n)$
4	$4n$-dimensional hyperkähler	$\subseteq Sp(n)$

SUPERSYMMETRIC QUANTUM MECHANICS

With these results, it is easy to specialize to $0+1$ dimensions and discuss supersymmetric quantum mechanics [5]. One simply assumes that the fields ϕ^a and χ^a do not depend on the spatial coordinate x. Thus the action for supersymmetric quantum mechanics is given by

$$\begin{aligned} \mathcal{A} \; = \; \int dt \Big[- \frac{1}{2} g_{ab}(\phi^c)\, \dot{\phi}^a \dot{\phi}^b \; - \; \frac{1}{2} g_{ab}(\phi^c) \overline{\chi}^a \gamma^0 \frac{\mathcal{D}}{\mathcal{D}t} \chi^b \\ + \; \frac{1}{12} R_{abcd}(\overline{\chi}^a \chi^c)(\overline{\chi}^b \chi^d) \Big] \; , \end{aligned} \tag{2.24}$$

where, as before, $g_{ab}(\phi^c)$ is the metric on the manifold $\mathcal{M}$, R_{abcd} is the Riemann curvature tensor, $\mathcal{D}\chi^b/\mathcal{D}t = \dot{\chi}^b + \Gamma^b{}_{cd}\dot{\phi}^c\chi^d$ is the covariant derivative, and all fields depend only on time. One can think of this Lagrangian as describing an object with a tower of spins (up to spin $\frac{1}{2}n$) moving on the n-dimensional manifold $\mathcal{M}$. The action (2.24) can be used to derive the Atiyah-Singer index theorem for the de Rahm and signature complexes [6].

The Lagrangian (2.24) can be further restricted to have $N = \frac{1}{2}$ (but not $N = 1$) supersymmetry. To do this, one identifies the components of each Majorana spinor, ${\chi_1}^a = {\chi_2}^a = \chi^a$, $\epsilon_1 = \epsilon_2 = \epsilon$. Under this restriction, the curvature term in (2.24) vanishes, while the kinetic terms remain as before. The new Lagrangian has $N = \frac{1}{2}$ supersymmetry because the supersymmetry parameter ϵ contains half the number of degrees of freedom. The $N = \frac{1}{2}$ Lagrangian can be used to derived the Atiyah-Singer index theorem for the Dirac spin complex [6].

3. $N = 1$ SUPERSYMMETRY IN FOUR DIMENSIONS

$N = 1$ Supersymmetry and Kähler Geometry

Lest we be accused of spending too much time in too few dimensions, we shall now turn to the important question of supersymmetric matter couplings in $3+1$ dimensions. We shall start by considering the most general coupling of spin $(0, \frac{1}{2})$ multiplets in $N = 1$ rigid supersymmetry. We expect our $3+1$ dimensional results to be related to the $1+1$ dimensional results discussed earlier. This is because $(0, \frac{1}{2})$ multiplets exist in both $1+1$ and $3+1$ dimensions, and $N = 1$ supersymmetry in $3+1$ dimensions reduces to $N = 2$ supersymmetry in $1+1$ dimensions. Indeed, we will find that the most general $(0, \frac{1}{2})$ matter coupling in $3+1$ dimensions may be described by a sigma model. For $N = 1$ rigid supersymmetry, we will see that the scalar fields ϕ^i must be the coordinates of a Kähler manifold $\mathcal{M}$.

However, in $3+1$ dimensions this result arises in a different way than in $1+1$ dimensions. This is because in $3+1$ dimensions, spin $(0, \frac{1}{2})$ multiplets are described by *chiral* superfields Φ^i, $\overline{D}\Phi^i = 0$. The lowest component of a chiral superfield is a *complex* scalar field, so the natural way to describe a Kähler manifold in $3+1$ dimensions is in terms of complex coordinates.

Therefore, before actually constructing the sigma model, let us take a moment to discuss complex manifolds, in general, and Kähler manifolds, in particular [19,20]. An n-dimensional *complex manifold* is a $2n$-dimensional real Riemannian manifold whose $2n$ real coordinates can be regarded as n complex coordinates z^i together with their n conjugates z^{*j}. Consistency requires that neighboring coordinate patches be linked by holomorphic (analytic) transition functions, $z'^i = f^i(z^j)$.

In terms of the $2n$ real coordinates $x^a = \{\mathrm{Re}\, z^i, \mathrm{Im}\, z^j\}$, a complex manifold has a globally defined tensor field $I^a{}_b$, such that

$$I^a{}_b I^b{}_c \;=\; -\,\delta^a{}_c\,. \tag{3.1}$$

As before, the field $I^a{}_b$ defines multiplication by i in the tangent space, and is called an almost complex structure. All complex manifolds are endowed with an almost complex structure. However, not all manifolds with an almost complex structure are, in fact, complex. The necessary and sufficient condition for this to hold is that the Nijenhuis tensor must vanish:

$$I^d{}_a \partial_d I^b{}_c \;-\; I^d{}_c \partial_d I^b{}_a \;-\; I^b{}_d \partial_a I^d{}_c \;+\; I^b{}_d \partial_c I^d{}_a \;=\; 0\,. \tag{3.2}$$

Equation (3.2) is the integrability condition for the existence of complex coordinates. When (3.2) is satisfied, $I^a{}_b$ is called a *complex structure.*

Having defined complex manifolds, we shall now restrict our attention to hermitian manifolds. A *hermitian manifold* is a complex manifold on which the line element ds^2 takes the following form:

$$ds^2 \;=\; g_{ij^*}\, dz^i\, dz^{*j} \;. \tag{3.3}$$

The matrix g_{ij^*} is hermitian, so ds^2 is real. For $\mathcal{M}$ to be hermitian, the metric g_{ab} must be invariant under the complex structure,

$$g_{ab} \;=\; g_{cd}\, I^c{}_a I^d{}_b \;. \tag{3.4}$$

In this case $I_{ab} = g_{ac} I^c{}_b$ is an antisymmetric tensor.

On any hermitian manifold, the tensor I_{ab} defines a fundamental two-form Ω,

$$\begin{aligned}\Omega \;&=\; \frac{1}{2}\, I_{ab}\, dx^a \wedge dx^b \\ &=\; \frac{i}{2}\, g_{ij^*}\, dz^i \wedge dz^{*j} \;.\end{aligned} \tag{3.5}$$

The two-form Ω is known as the Kähler form; it is both real and nondegenerate. A *Kähler manifold* is a hermitian manifold on which the Kähler form is closed,

$$d\Omega \;=\; 0 \;. \tag{3.6}$$

This is equivalent to saying that the complex structure is parallel with respect to the Riemann connection

$$\nabla_a I^b{}_c \;=\; 0 \;. \tag{3.7}$$

Equations (3.1), (3.4), and (3.7) are the same as equations (2.12), (2.14), and (2.19) of Section 2.

The fact that the Kähler form is closed leads to important consequences. In terms of complex coordinates, equation (3.6) implies

$$\frac{\partial}{\partial z^k}\, g_{ij^*} \;-\; \frac{\partial}{\partial z^i}\, g_{kj^*} \;=\; 0 \;, \quad \frac{\partial}{\partial z^{*k}}\, g_{ij^*} \;-\; \frac{\partial}{\partial z^{*j}}\, g_{ik^*} \;=\; 0 \;. \tag{3.8}$$

This means that *locally* the metric g_{ij^*} is the second derivative of real scalar function $K(z^i, z^{*j})$:

$$g_{ij^*} \;=\; \frac{\partial}{\partial z^i}\, \frac{\partial}{\partial z^{*j}}\, K(z^k, z^{*\ell}) \;. \tag{3.9}$$

The function K is known as the Kähler potential. It is important to note that (3.9) does not determine K uniquely. The metric g_{ij^*} is invariant under shifts of the Kähler potential,

$$K(z^i, z^{*j}) \;\rightarrow\; K(z^i, z^{*j}) \;+\; F(z^i) \;+\; F^*(z^{*j})\;, \tag{3.10}$$

where $F(z^i)$ is a holomorphic function of the coordinates. Although the metric g_{ij^*} is defined over the entire Kähler manifold, the potential K is in general defined only locally.

The Kähler condition (3.8) severely restricts the connection coefficients of the metric connection. In complex coordinates, the only nonvanishing components are

$$\Gamma^i{}_{jk} \;=\; g^{i\ell^*}\,\frac{\partial}{\partial z^k}\,g_{j\ell^*}\;, \qquad \Gamma^{i^*}{}_{j^*k^*} \;=\; g^{\ell i^*}\,\frac{\partial}{\partial z^{*k}}\,g_{\ell j^*}\;. \tag{3.11}$$

From these it is a simple exercise to compute the nonvanishing components of the curvature tensor,

$$R_{ij^*k\ell^*} \;=\; -\,R_{j^*ik\ell^*} \;=\; -\,R_{ij^*\ell^*k} \;=\; R_{j^*i\ell^*k}\;, \tag{3.12}$$

where

$$R_{ij^*k\ell^*} \;=\; -\,g_{mj^*}\,\frac{\partial}{\partial z^{*\ell}}\,\Gamma^m{}_{ik}\;. \tag{3.13}$$

We are now ready to discuss the $N = 1$ matter coupling. In $3+1$ dimensions, the most general coupling of chiral superfields is given by

$$\mathcal{L} \;=\; \int d^4\theta\, K(\Phi^i, \Phi^{*j}) \;+\; \int d^2\theta\, W(\Phi^i)\;. \tag{3.14}$$

The function K is real, and W is analytic. For the moment let us set W to zero.

To find the component Lagrangian, one must expand Φ^i in components,

$$\begin{aligned}\Phi^i(x,\theta,\overline{\theta}) \;=\; & \phi^i(x) \;+\; \sqrt{2}\,\overline{\theta}_R\chi_L{}^i \;+\; \overline{\theta}_R\theta_L\,F^i \;-\; \overline{\theta}_L\gamma^\mu\theta_L\partial_\mu\phi^i \\ & +\; \frac{1}{\sqrt{2}}\,\overline{\theta}_R\theta_L\;\overline{\theta}_L\gamma^\mu\partial_\mu\chi_L{}^i \;+\; \frac{1}{4}\,\overline{\theta}_R\theta_L\,\overline{\theta}_R\theta_L\,\partial^2\phi^i\;,\end{aligned} \tag{3.15}$$

where $\chi_{L,R} = (1\pm\gamma_5)\chi$. One must do the θ-integrals and eliminate the auxiliary fields F^i. It turns out that the component Lagrangian has a geometrical interpretation precisely when the function K is identified with the Kähler potential [8]. In this case the component Lagrangian takes the following form

$$\begin{aligned} \mathcal{L} \ = \ & - g_{ij^*}\, \partial_\mu \phi^i \partial^\mu \phi^{*j} \ - \ \frac{1}{2} \overline{\chi}^i\, \widehat{g_{ij^*}}\, \gamma^\mu \mathcal{D}_\mu \chi^j \\ & + \ \frac{1}{32} R_{ij^*kl^*}\, \overline{\chi}^i (1+\gamma_5)\gamma_\mu \chi^j\, \overline{\chi}^k (1+\gamma_5)\gamma^\mu \chi^l \ . \end{aligned} \tag{3.16}$$

The hat notation is from Ref. [9]; it is basically a fancy way to write two-component spinors in four-component notation ($\widehat{A} = \mathrm{Re}\, A + i\gamma_5\, \mathrm{Im}\, A$). The scalar fields ϕ^i parametrize a Kähler manifold $\mathcal{M}$. The spinors χ^i are four-component Majorana spinors; they lie in the complexified tangent bundle over $\mathcal{M}$. The covariant derivative is given by

$$\mathcal{D}_\mu \chi^i \ = \ \partial_\mu \chi^i \ + \ \widehat{\Gamma^i{}_{jk} \partial_\mu \phi^j \chi^k} \ , \tag{3.17}$$

where the connection coefficients are given in (3.11). The Kähler curvature tensor $R_{ij^*kl^*}$ is exactly that given in (3.13). Note that $R_{ij^*kl^*}$ has precisely the right symmetries to appear in the four-fermion term.

The first thing to remark about (3.16) is that it is rigidly supersymmetric. This is guaranteed because (3.16) is derived from a superspace formalism. One can check, however, that $\mathcal{L}$ is invariant (up to a total derivative) under the following supersymmetry transformations:

$$\begin{aligned} \delta \phi^i \ &= \ \frac{1}{2} \sqrt{2}\, \overline{\epsilon} (1+\gamma_5) \chi^i \\ \delta \chi^i \ &= \ \sqrt{2}\, \gamma^\mu\, \widehat{\partial_\mu \phi^{*i}} \epsilon \ - \ \frac{1}{2} \sqrt{2}\, (\gamma_5 \chi^k)(\overline{\chi}^j\, \widehat{\Gamma^i{}_{jk}}\, \gamma_5 \epsilon) \\ & \quad - \ \frac{1}{2} \sqrt{2}\, \chi^k (\overline{\chi}^j \widehat{\Gamma^i{}_{jk}}\, \epsilon) \ . \end{aligned} \tag{3.18}$$

As in (2.15), the term with the connection coefficients is required to ensure that supersymmetry transformations commute with diffeomorphisms.

The Lagrangian $\mathcal{L}$ also possesses the Kähler invariance (3.10). This is easy to verify directly. It may also be seen in superspace, provided one uses the fact that

$$\int d^4\theta\, F(\Phi^i) \ = \ \int d^4\theta\, F^*(\Phi^{*j}) \ = \ 0 \ . \tag{3.19}$$

The Kähler invariance is necessary, but not quite sufficient, to prove that the Lagrangian $\mathcal{L}$ is well-defined over the whole Kähler manifold $\mathcal{M}$. One must still worry about problems that might arise globally. For the case of rigid supersymmetry, one may readily show that there are no such problems [8].

Having constructed the sigma model, we are now ready to couple it to gravity [21]. The first surprise is that in supergravity, the superspace sigma model Lagrangian is not

$$\int d^4\theta \, E \, K(\Phi^i, \Phi^{*j}) \ , \tag{3.20}$$

but rather

$$\mathcal{L} \ = \ - \, 3 \int d^4\theta \, E \, \exp\left[-\frac{1}{3} K(\Phi^i, \Phi^{*j}) \right] \ , \tag{3.21}$$

where E is the superdeterminant of the superspace vielbein [9–11].[♮2] Equation (3.21) is the choice that leads, after Weyl rescaling and eliminating the auxiliary fields, to a component Lagrangian with the correct normalizations for the Einstein action and for the matter kinetic energies. To make (3.21) more plausible, let us restore the factors of Newton's constant, $\kappa^2 = 8\pi G_N$, and expand the exponential. As $\kappa \to 0$, we find

$$\mathcal{L} \ \to \ - \, \frac{3}{\kappa^2} \int d^4\theta \, E \ + \ \int d^4\theta \, E \, K(\Phi^i, \Phi^{*j}) \ + \ \mathcal{O}(\kappa^2) \ . \tag{3.22}$$

The first term contains the Einstein and Rarita-Schwinger actions, which are frozen out as $\kappa \to 0$. The second term is simply (3.20), the flat sigma model Lagrangian. Higher order terms vanish as $\kappa \to 0$, so (3.21) has the correct flat-space limit. The final justification for (3.21) comes, however, from expanding in components and examining the normalizations of the various kinetic terms.

Precisely how to pass from the superspace Lagrangian (3.21) to the correct form of the component Lagrangian is beyond the scope of these lectures [22]. There are many different methods on the market [16,17,23], and each suffers from its own drawbacks. None is particularly easy to use. Suffice it

♮2 In these lectures, we consider the "$n = -1/3$" version of supergravity. Other versions are treated in Ref. [42].

to say that the component Lagrangian corresponding to (3.21) is given by

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}\, eR \;-\; e\, g_{ij^*}\, \partial_\mu\phi^i \partial^\mu \phi^{*j} \\
& -\; \frac{1}{2}\, \epsilon^{\mu\nu\rho\sigma}\, \overline{\psi}_\mu \gamma_5 \gamma_\nu \mathcal{D}_\rho \psi_\sigma \;-\; \frac{1}{2}\, e\, \overline{\chi}^i\, \widehat{g_{ij^*}}\, \gamma^\mu \mathcal{D}_\mu \chi^j \\
& +\; \frac{1}{2}\, \sqrt{2}\, e\, \overline{\chi}^i \widehat{g_{ij^*}\partial_\nu} \phi^{*j} \gamma^\mu \gamma^\nu \psi_\mu \\
& +\; \frac{1}{16}\, e\, g_{ij^*} \overline{\chi}^i (1+\gamma_5)\gamma_\sigma \chi^j \left[\epsilon^{\mu\nu\rho\sigma}\, \overline{\psi}_\mu \gamma_\nu \psi_\rho \;-\; \overline{\psi}_\rho \gamma_5 \gamma^\sigma \psi^\rho \right] \\
& -\; \frac{1}{64}\, e \left[g_{ij^*} g_{k\ell^*} \;-\; 2R_{ij^*k\ell^*} \right] \overline{\chi}^i (1+\gamma_5)\gamma_\mu \chi^j \overline{\chi}^k (1+\gamma_5) \gamma^\mu \chi^\ell \, .
\end{aligned}
\tag{3.23}
$$

Again g_{ij^*} is the Kähler metric and $R_{ij^*k\ell^*}$ is the Kähler curvature.

It is important to note that, in supergravity, the covariant derivatives contain several new pieces:

$$
\begin{aligned}
\mathcal{D}_\mu \chi^i \;&=\; \partial_\mu \chi^i \;+\; \frac{1}{2}\, \omega_{\mu\alpha\beta}\, \sigma^{\alpha\beta} \chi^i \;+\; \widehat{\Gamma^i{}_{jk} \partial_\mu \phi^j} \chi^k \\
& \qquad -\; \frac{i}{2}\, \mathrm{Im} \left(\frac{\partial K}{\partial \phi^j}\, \partial_\mu \phi^j \right) \gamma_5\, \chi^i \\
\mathcal{D}_\mu \psi_\nu \;&=\; \partial_\mu \psi_\nu \;+\; \frac{1}{2}\, \omega_{\mu\alpha\beta}\, \sigma^{\alpha\beta} \psi_\nu \;+\; \frac{i}{2}\, \mathrm{Im} \left(\frac{\partial K}{\partial \phi^j}\, \partial_\mu \phi^j \right) \gamma_5\, \psi_\nu \, .
\end{aligned}
\tag{3.24}
$$

In the second-order formalism, the spin connection $\omega_{\mu\alpha\beta}$ is a function of the vierbein $e_{\mu\alpha}$ and the gravitino ψ_μ. It is present because of the fact that spacetime is curved. The Kähler connection is also present in the covariant derivative of χ^i. It just says that the χ^i transform in the (complexified) tangent bundle over $\mathcal{M}$. The striking new term is the $U(1)$ connection $\frac{i}{2}\,\mathrm{Im}\,[(\partial K/\partial\phi^j)\partial_\mu\phi^j]$. It occurs in the covariant derivatives of χ^i and ψ_ν. We shall see that this term gives rise to dramatic consequences for the matter coupling [11].

The Lagrangian (3.23) is manifestly supersymmetric because it is derived from a superspace formalism. It is not obvious, however, that (3.23) possesses the Kähler invariance (3.10). This must be verified if $\mathcal{L}$ is to be well-defined over the entire Kähler manifold. To do this, let us imagine that we cover our Kähler manifold by open sets $\mathcal{O}_A$. We assume that our covering is fine enough so that each intersection region $\mathcal{O}_A \cap \mathcal{O}_B$ is simply connected. On each open set we have a Kähler potential K_A. On the overlap regions $\mathcal{O}_A \cap \mathcal{O}_B$, K_A does not equal K_B, but rather

$$
K_A \;-\; K_B \;=\; F_{AB} \;+\; F^*_{AB} \, . \tag{3.25}
$$

Here F_{AB} is an analytic function, and $F_{AB} = -F_{BA}$. Note that (3.25) does not quite uniquely define the F_{AB}. If we let

$$F_{AB} \rightarrow F_{AB} + i\,C_{AB} , \tag{3.26}$$

where the C_{AB} are *real constants*, then F_{AB} is still analytic and (3.25) is still obeyed.

As we see from the covariant derivatives (3.24), the Lagrangian (3.23) in $\mathcal{O}_A \cap \mathcal{O}_B$ is invariant under Kähler transformations (3.25) only if the Kähler transformations are accompanied by chiral rotations of the Fermi fields:

$$\begin{aligned} \chi^i_{(A)} &= \exp\left[\frac{i}{2}\,\mathrm{Im}\,F_{AB}\,\gamma_5\right]\chi^i_{(B)} , \\ \psi_{\nu(A)} &= \exp\left[-\frac{i}{2}\,\mathrm{Im}\,F_{AB}\,\gamma_5\right]\psi_{\nu(B)} . \end{aligned} \tag{3.27}$$

The combined Kähler/chiral invariance is necessary, but not sufficient, to ensure that $\mathcal{L}$ is well-defined over the whole Kähler manifold $\mathcal{M}$. We must still worry about problems that might arise globally. In particular, we must worry about the consistency of (3.27). Inconsistency might arise in the triple intersection regions $\mathcal{O}_A \cap \mathcal{O}_B \cap \mathcal{O}_C$, since (3.27) relates $\chi^i_{(A)}$ to $\chi^i_{(B)}$, $\chi^i_{(B)}$ to $\chi^i_{(C)}$, and $\chi^i_{(C)}$ back to $\chi^i_{(A)}$. The consistency condition is

$$1 = \exp\left[\pm\frac{1}{4}\left(F_{AB} + F_{BC} + F_{CA} - F^*_{AB} - F^*_{BC} - F^*_{CA}\right)\right] . \tag{3.28}$$

It is this condition that leads to the quantization of Newton's constant in certain $N = 1$ supergravity theories [11].

To understand the consistency condition, note that the identity $(K_A - K_B) + (K_B - K_C) + (K_C - K_A) = 0$ implies

$$F_{AB} + F_{BC} + F_{CA} = -\left(F_{AB} + F_{BC} + F_{CA}\right)^* . \tag{3.29}$$

The left-hand side of (3.29) is an analytic function that equals minus its own complex conjugate. This implies that if we denote

$$2\pi i\,C_{ABC} = F_{AB} + F_{BC} + F_{CA} , \tag{3.30}$$

then the C_{ABC} are *real constants*. Note that the C_{ABC} are not quite uniquely defined. In view of (3.26), we are free to shift

$$C_{ABC} \rightarrow C_{ABC} + C_{AB} + C_{BC} + C_{CA} . \tag{3.31}$$

The consistency condition requires that we choose the C_{AB} such that the C_{ABC} are *even integers*.

On a general Kähler manifold, it is not always possible to choose the C_{ABC} to be integers. It is only possible to do this if $\mathcal{M}$ is a Kähler manifold of restricted type, or a *Hodge* manifold [11,20,24,25]. What, precisely, is a Hodge manifold? A mathematician would tell us that a Hodge manifold is a Kähler manifold on which it is possible to define a complex line bundle whose first Chern form is proportional to the Kähler form. How can we understand the consistency condition in this mathematical language?

First of all, the transformations (3.27) tell us that the χ^i are sections—not only of the tangent bundle—but also of a complex line bundle $\mathcal{E}$. The transition functions in this bundle are given by the $U(1)$ elements $\exp(\pm\frac{i}{2}\,\mathrm{Im}\,F)$. Furthermore, the covariant derivatives (3.24) imply that the connection ω on $\mathcal{E}$ is given by

$$\omega \;=\; \frac{i}{4}\left(\frac{\partial K}{\partial\phi^i}\,d\phi^i \;-\; \frac{\partial K}{\partial\phi^{*i}}\,d\phi^{*i}\right)\;. \tag{3.32}$$

The corresponding curvature tensor $\Omega = dw$ may be readily seen to be

$$\Omega \;=\; \frac{i}{2}\,\frac{\partial^2 K}{\partial\phi^i\partial\phi^{*j}}\,d\phi^i \wedge d\phi^{*j}\;. \tag{3.33}$$

This is just the Kähler form (3.5). Now, the first Chern form of any line bundle is proportional to the curvature Ω. Hence the first Chern form of our line bundle $\mathcal{E}$ is indeed the Kähler form, and our manifold $\mathcal{M}$ must be Hodge. The consistency condition (3.28) simply says that the first Chern form integrated over any closed two-cycle must give an (even) integer. When this is true, the complex line bundle $\mathcal{E}$ exists, and the Lagrangian $\mathcal{L}$ is well-defined over the whole Kähler manifold.

To see how the consistency condition (3.28) can lead to the quantization of Newton's constant, let us consider a simple example. For $\mathcal{M}$ we take CP^1, the ordinary two-dimensional sphere. This is a Kähler manifold, the Riemann sphere S^2. Deleting the point at infinity, we stereographically project S^2 onto the complex plane (see Fig. 1).

In terms of the complex coordinates $(z,\,z^*)$, an appropriate choice for the Kähler potential is $K(z,z^*) = n\log(1+z^*z)$. In this case the bosonic part of the Lagrangian (3.23) is given by

$$\mathcal{L} \;=\; \Lambda^2\sqrt{-g}\left[-\,\frac{1}{2}R \;-\; n\partial_\mu z\,\partial^\mu z^*/(1+z^*z)^2\right]\;, \tag{3.34}$$

where a constant, Λ^2, has been restored. The conventional form of the sigma

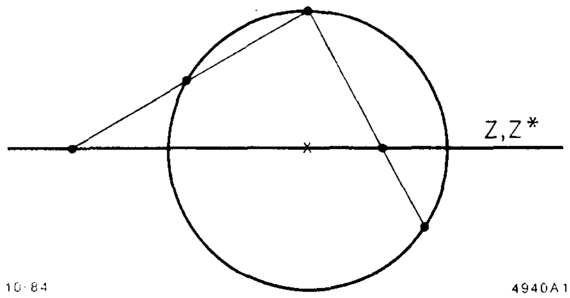

Figure 1: Complex projective coordinates for the sphere S^2

model would be

$$\mathcal{L} = \sqrt{-g}\left[-\frac{1}{16\pi G_N} R - f^2 \partial_\mu z \partial^\mu z^* / (1 + z^* z)^2\right] . \qquad (3.35)$$

Comparing (3.34) and (3.35), we see that G_N and f are related by

$$G_N = n/8\pi f^2 . \qquad (3.36)$$

As we shall see, the quantization condition for S^2 is that n must be an even integer. Equation (3.36) implies that Newton's constant must be quantized in units of the scalar self-coupling.

The sphere S^2 is so simple that we can work out the quantization directly, without resorting to the cocycle condition (3.28). The complex z-plane spans all of S^2 except the point at infinity. Letting $w = 1/z$, we see that the w-plane spans the entire sphere except for the point $z = 0$. In terms z, we take $K_z = n\log(1 + z^* z)$; for w, we take $K_w = n\log(1 + w^* w)$. The difference is

$$K_z - K_w = n \log z + n \log z^* . \qquad (3.37)$$

Therefore we identify F as

$$F = n \log z . \qquad (3.38)$$

The problem with this is that F is multi-valued—it cannot be defined consistently on the whole plane $z \neq 0$, $z \neq \infty$. However, in relating the description ~~on~~ the z-plane to the description of the w-plane, all we need is the single-valuedness of $\exp \frac{1}{4}(F - F^*)$. This is obeyed if and only if n is an (even) integer.

The quantization condition discussed here is exactly the same as the Dirac condition for a magnetic monopole. For the magnetic monopole, one must worry about consistently defining a $U(1)$ bundle over S^2. The integral of the first Chern form over S^2 measures the monopole charge, and consistency demands it be quantized with respect to the electric charge. The (even) integer n introduced above is the supersymmetric analogue of the "monopole charge" for the CP^1 sigma model.

Thus we have seen that in rigid supersymmetry, sigma models exist for all Kähler manifolds $\mathcal{M}$. Only a subclass of these models may be coupled to supergravity—those whose scalar fields lie on a Hodge manifold. For topologically nontrivial Hodge manifolds, this leads to the quantization of Newton's constant in terms of the scalar self-coupling.

4. SUPERPOTENTIAL AND GAUGE COUPLINGS

THE SUPERPOTENTIAL

in this section we shall finish our discussion of the $N = 1$ matter couplings. We will first include the superpotential $W(\Phi^i)$, and then we will gauge the (holomorphic) isometries of $\mathcal{M}$.

As mentioned in Section 3, the superpotential $W(\Phi^i)$ is an analytic function of the chiral superfield Φ^i. In flat superspace, it gives rise to an extra term in the superfield Lagrangian

$$\Delta \mathcal{L} \;=\; \int d^2\theta \; W(\Phi^i) \; . \tag{4.1}$$

This term can and should be included in the most general coupling of chiral superfields. In terms of components, equations (4.1) induces the following additional terms in the Lagrangian (3.16):

$$\Delta \mathcal{L} \;=\; -\;\frac{1}{2}\overline{\chi}^i \left[\nabla_i \frac{\partial W}{\partial \phi^j}\right] \chi^j \;-\; g^{ij^*} \left(\frac{\partial W}{\partial \phi^i}\right) \left(\frac{\partial W}{\partial \phi^j}\right)^* \;, \tag{4.2}$$

where ∇_i is the covariant derivative (3.11).

The first thing to note is that if W is quadratic, the terms in $\Delta\mathcal{L}$ give mass to the fields ϕ^i and χ^i. In supersymmetric theories, it is not usually so easy to give masses to fields, because massive and massless multiplets contain different numbers and types of fields. In $3+1$ dimensions, the $(0,\frac{1}{2})$ multiplet is unique in its ability to represent both the massless and massive supersymmetry algebra.

The second thing to note about (4.2) is that the scalar potential $\mathcal{V} = g^{ij*}(\partial W/\partial\phi^i)\;(\partial W/\partial\phi^j)^*$ is positive semidefinite. Supersymmetry is spontaneously broken if and only if $\langle \partial W/\partial\phi^i \rangle \neq 0$ for some value of i.

In local supersymmetry, it is also possible to include a superpotential. The superfield Lagrangian is given by

$$\Delta \mathcal{L} = \int d^2\theta\, \widetilde{\mathcal{E}}\, W(\Phi^i) + \text{h.c.}\,, \tag{4.3}$$

where $\widetilde{\mathcal{E}}$ is the chiral superspace density. Expanding Φ^i in components, doing the θ-integral, eliminating the auxiliary fields, and Weyl rescaling, one finds the following addition to the Lagrangian (3.23) [9–11,22]:

$$\begin{aligned} e^{-1}\Delta\mathcal{L} \; = \; & \exp(K/2)\Big\{\overline{\psi}_\mu \sigma^{\mu\nu} \widehat{W}^* \psi_\nu \; + \; \frac{1}{2}\sqrt{2}\,\overline{\psi}\cdot\gamma\, \widehat{D_i W}\, \chi^i \\ & - \frac{1}{2}\overline{\chi}^i \,\widehat{\left[\mathcal{D}_j(D_i W)\right]}\, \chi^j \Big\} \\ & + \exp(K)\left[3\,|W|^2 \; - \; (D_i W)(D_j W)^* g^{ij^*}\right] . \end{aligned} \tag{4.4}$$

Here $D_i W = \partial W/\partial\phi_i + (\partial K/\partial\phi^i)W$, and $\mathcal{D}_j D_i W = \partial\, D_i W/\partial\phi^j - \Gamma^k_{ji} D_k W + (\partial K/\partial\phi^j)\, D_i W$.

From the transformation law for χ^i,

$$\delta\chi^i \; = \; -\,\sqrt{2}\, e^{\frac{1}{2}K} \widehat{g^{ij^*}(D_j W)^*}\, \epsilon \; + \; \dots \,, \tag{4.5}$$

we see that χ^i transforms by a shift whenever $\left\langle D_j W\right\rangle \neq 0$. This identifies $\left\langle D_j W\right\rangle \neq 0$ as the criterion for spontaneous supersymmetry breaking, and χ^i as the associated Goldstone fermion. Note that $\left\langle D_j W\right\rangle \neq 0$ reduces to $\left\langle \partial W/\partial\phi^j\right\rangle \neq 0$ as $\kappa \to 0$.

From the potential $\mathcal{V}$,

$$\mathcal{V} \; = \; e^K\left[(D_i W)(D_j W)^* g^{ij^*} \; - \; 3|W|^2\right] \tag{4.6}$$

we see that in local supersymmetry, it is possible to have spontaneously broken supersymmetry and *zero* cosmological constant. This requires that $\langle W\rangle \sim 1/\kappa^2$. Note that spontaneously broken supersymmetry gives no restrictions on the cosmological constant. Unbroken supersymmetry, however, requires that the cosmological constant be zero or negative [21,26]. In particular, this implies that unbroken supersymmetry cannot exist in de Sitter space.

The Lagrangian (4.4) possesses the Kähler invariance (3.10), but only if we also transform $W \to W\exp(-F)$. This means that if $\mathcal{M}$ is covered by open sets $\mathcal{O}_A$, on each of which we choose a superpotential W_A, then we must require

$$W_A \;=\; W_B \exp(-F_{AB}) \tag{4.7}$$

on $\mathcal{O}_A \cap \mathcal{O}_B$ for the two descriptions to match. The consistency condition is that the C_{ABC} must again be integers.

As before, all this has a standard mathematical interpretation. The W_A are sections of a holomorphic line bundle $\mathcal{F}$ over $\mathcal{M}$. On $\mathcal{F}$, there is a natural metric given by $\| W \|^2 = WW^* \exp(K)$. It is invariant under the Kähler transformations (3.10) and (4.7). On $\mathcal{F}$, there is also a holomorphic connection

$$\omega \;=\; i\,\frac{\partial K}{\partial \phi^i}\, d\phi^i \; . \tag{4.8}$$

As before, the first Chern form of this connection is proportional to the Kähler form. The consistency condition simply says that the first Chern form integrated over any closed two-cycle must give an even integer [11].

The potential (4.6) looks somewhat different than the potential usually used in supergravity model-building [9,10]. However, it is the form (4.6) that makes the geometrical structure manifest. To recover the formulae used by model-builders, one need only choose the gauge $W = 1$. In this gauge, the two formalisms are identical.

THE GAUGE INVARIANT SUPERSYMMETRIC NONLINEAR SIGMA MODEL

We would now like to gauge the bosonic symmetries of the Lagrangian $\mathcal{L}$. These symmetries are given by the *isometries* of the sigma model manifold. In this section, we shall gauge the (holomorphic) isometries of the manifold $\mathcal{M}$ [12].

To that end, let us assume that $\mathcal{M}$ admits a d-dimensional isometry group $\mathcal{G}$. The group $\mathcal{G}$ has d linearly independent generators,

$$\delta\phi^i \;=\; V^{(a)i}(\phi^j)\;, \qquad a = 1,\ldots,d\;. \tag{4.9}$$

Since we would like the symmetry rotations to preserve the complex structure, we require that the $V^{(a)i}$ be holomorphic,

$$\frac{\partial V^{(a)i}}{\partial \phi^{*j}} \;=\; \frac{\partial V^{(a)*i}}{\partial \phi^{j}} \;=\; 0\;. \tag{4.10}$$

Furthermore, since the $V^{(a)i}$ must generate isometries, they must also obey

Killing's equation. On a complex manifold, Killing's equation has two parts,

$$\begin{aligned} \nabla_i V_j{}^{(a)} + \nabla_j V_i{}^{(a)} &= 0 , \\ \nabla_i V_{j^*}{}^{(a)} + \nabla_{j^*} V_i{}^{(a)} &= 0 . \end{aligned} \tag{4.11}$$

The first is satisfied automatically; it is a consequence of (4.10) and of the fact that $\nabla_i g_{jk^*} = 0$. The second implies that *locally* there exist d real scalar functions $D^{(a)}(\phi^i, \phi^{*j})$, such that

$$g_{ij^*} V^{(a)*j} = i \frac{\partial D^{(a)}}{\partial \phi^i} . \tag{4.12}$$

The functions $D^{(a)}$ are called *Killing potentials* because their gradients give the *Killing vectors* $V^{(a)i}$.

It is important to note that (4.12) defines the potentials $D^{(a)}$ only up to arbitrary integration constants $C^{(a)}$,

$$D^{(a)} \rightarrow D^{(a)} + C^{(a)} . \tag{4.13}$$

We will see that this freedom is related to the Fayet-Iliopoulos D-term in abelian gauge theories [27].

The Killing vectors $V^{(a)i}$ are well-defined over the entire Kähler manifold $\mathcal{M}$. The potentials $D^{(a)}$, however, are defined only locally. For example, if $\mathcal{M}$ is not simply connected and $\mathcal{G}$ contains a $U(1)$ factor, the $D^{(a)}$ may not be globally well-defined. We shall see that the global existence of the $D^{(a)}$'s is the necessary and sufficient condition to gauge the group $\mathcal{G}$.

Since the Killing vectors $V^{(a)i}$ generate a Lie group, they must obey the usual Lie bracket relations:

$$\left[V^{(a)}, V^{(b)}\right]^i = V^{(a)j} \frac{\partial}{\partial \phi^j} V^{(b)i} - V^{(b)j} \frac{\partial}{\partial \phi^j} V^{(a)i} = f^{abc} V^{(c)i} . \tag{4.14}$$

If the $D^{(a)}$ exist, they can be chosen to transform in the adjoint representation of the (compact) gauge group,

$$\left[V^{(a)i} \frac{\partial}{\partial \phi^i} + V^{(a)*i} \frac{\partial}{\partial \phi^{*i}}\right] D^{(b)} = f^{abc} D^{(c)} . \tag{4.15}$$

Note that (4.15) fixes the constants $C^{(a)}$ for nonabelian groups. For each $U(1)$ factor, however, there is an undetermined constant C. This will turn out to be the reason why Fayet-Iliopoulos terms only arise in abelian theories.

Locally, the Killing vector fields $V^{(a)i}$ generate the following motions:

$$\begin{aligned} \delta\phi^i &= \epsilon^{(a)} V^{(a)i} , \\ \delta\chi^i &= \epsilon^{(a)} \widehat{\frac{\partial V^{(a)i}}{\partial\phi^j}} \chi^j . \end{aligned} \tag{4.16}$$

These motions are isometries, so they leave the metric invariant. They do, however, shift the Kähler potential,

$$\delta K = \delta\phi^i \frac{\partial K}{\partial\phi^i} + \delta\phi^{*i} \frac{\partial K}{\partial\phi^{*i}} . \tag{4.17}$$

It is easy to check that (4.17) is a Kähler transformation, $\delta K = F + F^*$, provided

$$F = \epsilon^{(a)} \left[V^{(a)i} \frac{\partial K}{\partial\phi^i} + iD^{(a)} \right] \tag{4.18}$$

The function F is analytic because of (4.10) and (4.12). This shows explicitly that the isometries (4.9) leave invariant the Lagrangian (3.14).

We are now ready to gauge the group $\mathcal{G}$. This corresponds to setting $\epsilon^{(a)} \to \epsilon^{(a)}(x)$ in (4.16), exactly as in ordinary Yang-Mills theory. We would like to do this using superfields, in order to ensure that our resulting Lagrangian is supersymmetric. However, it is only easy to gauge that subgroup $\mathcal{H} \subseteq \mathcal{G}$ that leaves K invariant (a general transformation in $\mathcal{G}/\mathcal{H}$ shifts K by a Kähler transformation, $K \to K + F + F^*$) [28].

In flat superspace, the Lagrangian invariant under $\mathcal{H}$ is given by

$$\mathcal{L} = \int d^4\theta \ K \left[\Phi^i, (\Phi^+ e^{2gV})^{*j} \right] . \tag{4.19}$$

To this one must add the kinetic terms for the vector multiplet V,

$$\mathcal{L} = \int d^2\theta \ \mathrm{Tr}\, \overline{W}_R W_L + \text{h.c.} \tag{4.20}$$

From here it is straightforward to work out the component Lagrangian. One can then guess its extension to a new Lagrangian invariant under $\mathcal{G}$,

$$
\begin{aligned}
\mathcal{L} \;=\; & - g_{ij^*} \mathcal{D}_\mu \phi^i \mathcal{D}^\mu \phi^{*j} \;-\; \frac{1}{2} g^2 D^{(a)2} \\
& - \frac{1}{2} \overline{\chi}^i \widehat{g_{ij^*}} \gamma^\mu \mathcal{D}_\mu \chi^j \;-\; \frac{1}{4} F_{\mu\nu}{}^{(a)} F^{\mu\nu(a)} \\
& - \frac{1}{2} \overline{\lambda}^{(a)} \gamma^\mu \mathcal{D}_\mu \lambda^{(a)} \;-\; \sqrt{2}\, g\, \overline{\chi}^i \widehat{g_{ij^*} V^{(a)*j}}\, \lambda^{(a)} \\
& + \frac{1}{32} R_{ij^*k\ell^*} \overline{\chi}^i (1+\gamma_5) \gamma_\mu \chi^j \, \overline{\chi}^k (1+\gamma_5) \gamma^\mu \chi^\ell \,,
\end{aligned}
\tag{4.21}
$$

where

$$
\begin{aligned}
\mathcal{D}_\mu \phi^i \;&=\; \partial_\mu \phi^i \;-\; g A_\mu{}^{(a)} V^{(a)i} \\
\mathcal{D}_\mu \chi^i \;&=\; \partial_\mu \chi^i \;+\; \widehat{\Gamma^i{}_{jk} \mathcal{D}_\mu \phi^j}\, \chi^k \;-\; g A_\mu{}^{(a)} \widehat{\frac{\partial V^{(a)i}}{\partial \phi^j}}\, \chi^j \\
\mathcal{D}_\mu \lambda^{(a)} \;&=\; \partial_\mu \lambda^{(a)} \;-\; g\, f^{abc} A_\mu{}^{(b)} \lambda^{(c)} \,.
\end{aligned}
\tag{4.22}
$$

The covariant derivatives (4.22) are coordinate and gauge covariant, as is evident from the transformations (4.16). The Lagrangian (4.21) is also gauge invariant under the isometry group $\mathcal{G}$. What is not evident, however, is that (4.21) is still supersymmetric. That must be checked by hand, using the following transformation laws:

$$
\begin{aligned}
\delta \phi^i \;&=\; \frac{1}{2} \sqrt{2}\, \overline{\epsilon} (1+\gamma_5) \chi^i \\
\delta \chi^i \;&=\; \sqrt{2}\, \gamma^\mu \widehat{\mathcal{D}_\mu \phi^{*i}} \epsilon \;-\; \frac{1}{2} \sqrt{2}\, (\gamma_5 \chi^k) (\overline{\chi}^j \widehat{\Gamma^i_{jk}} \gamma_5 \epsilon) \\
& \quad - \; \frac{1}{2} \sqrt{2}\, \chi^k \, (\overline{\chi}^j \widehat{\Gamma^i_{jk}}\, \epsilon) \\
\delta A_\mu{}^{(a)} \;&=\; -\, \overline{\epsilon} \gamma_\mu \lambda^{(a)} \\
\delta \lambda^{(a)} \;&=\; F_{\alpha\beta}{}^{(a)} \sigma^{\alpha\beta} \epsilon \;+\; i\, g\, D^{(a)} \gamma_5 \epsilon \,.
\end{aligned}
\tag{4.23}
$$

The Lagrangian (4.21) gives what might be called the gauge invariant supersymmetric nonlinear sigma model [12].[♮3]

♮3 Equations (4.21)–(4.23) are written as if $\mathcal{G}$ were a simple group. The generalization to an arbitrary group is trivial.

The Lagrangian (4.21) does not contain a superpotential W, which has been omitted for simplicity. Yet it still contains a scalar potential,

$$\mathcal{V} \;=\; \frac{1}{2}\, g^2\, D^{(a)2} \;. \tag{4.24}$$

The potential (4.24) is a sigma model generalization of the so-called "D-term" familiar to model builders. Supersymmetry is spontaneously broken if $\langle D^{(a)} \rangle \neq 0$, for some value of $a = 1, \ldots, d$. The spinor $\lambda^{(a)}$ is the corresponding Goldstone fermion.

For $U(1)$ factors in the gauge group $\mathcal{G}$, the relations (4.12) and (4.15) do not completely determine the corresponding potentials D. They leave the D's undetermined up to additive constants C,

$$D \;\rightarrow\; D \;+\; C \;. \tag{4.25}$$

However, each D appears in the potential $\mathcal{V}$, and supersymmetry is spontaneously broken whenever any D develops a nonzero vacuum expectation value. By choosing the C's appropriately, it is possible to ensure that $\langle D \rangle \neq 0$. This is known as the Fayet-Iliopoulos mechanism for spontaneous supersymmetry breaking [27].

THREE EXAMPLES

To get a better feeling for the formalism developed above, let us consider three examples. For the first we take $\mathcal{M}$ to be the complex z-plane, and we choose to gauge the rotations about the origin. We set $K = z^*z + d$ and $D = z^*z + c$. Then the metric g_{zz^*} is 1, and $R_{zz^*zz^*} = 0$. The Killing vector V is simply $-iz$, so the covariant derivatives are

$$\begin{aligned} \mathcal{D}_\mu z \;&=\; \partial_\mu z \;+\; igA_\mu z \;, \\ \mathcal{D}_\mu \chi \;&=\; \partial_\mu \chi \;+\; igA_\mu \chi \;. \end{aligned} \tag{4.26}$$

We see that the first example corresponds to a renormalizable $U(1)$ gauge theory. The potential $\mathcal{V}$ is

$$\mathcal{V} \;=\; \frac{1}{2}\, g^2 (zz^* \;+\; c)^2 \;. \tag{4.27}$$

For $c > 0$, supersymmetry is spontaneously broken via the Fayet-Iliopoulos mechanism. This example can be generalized for any renormalizable gauge theory. One takes $\mathcal{M} = C^n$, and one gauges an appropriate subgroup of $U(n)$.

For our second example, we again take $\mathcal{M}$ to be C^1. This time, however, we choose to gauge translations in the y-direction on $\mathcal{M}$. (Note that we could have chosen to gauge translations in the x-direction, but because of (4.15), we cannot gauge both simultaneously.) As before, we take $K = z^*z + d$, so $g_{zz^*} = 1$ and $R_{zz^*zz^*} = 0$. For D we take $D = m(z + z^*)$, so $V = -im$. The covariant derivatives are

$$\begin{aligned} \mathcal{D}_\mu z &= \partial_\mu z \; + \; imgA_\mu \; , \\ \mathcal{D}_\mu \chi &= \partial_\mu \chi \; . \end{aligned} \tag{4.28}$$

The field z gives a gauge invariant mass to the vector A_μ. It is a compensating field, analogous to the compensators introduced in conformal supergravity [21].

For our final example, we take $\mathcal{M} = CP^1 = S^2 = SU(2)/U(1)$. It is a Kähler manifold as well as a homogeneous space $\mathcal{G}/\mathcal{H}$. As in Section 3, we use projective coordinates z and z^*. In these coordinates, we take $K = \log(1 + zz^*)$. We choose to gauge the entire isometry group $\mathcal{G} = SU(2)$. The isotropy group $\mathcal{H}$ is $U(1)$, and the functions D are as follows:

$$D^{(1)} \; = \; \frac{1}{2}\,\frac{z + z^*}{(1 + z^*z)}\,, \quad D^{(2)} \; = \; -\,\frac{i}{2}\,\frac{z - z^*}{(1 + z^*z)}\,, \quad D^{(3)} \; = \; -\,\frac{1}{2}\,\left(\frac{1 - z^*z}{1 + z^*z}\right) . \tag{4.29}$$

From here one can work out the covariant derivatives $\mathcal{D}_\mu z$ and $\mathcal{D}_\mu \chi$. Because we have gauged the full $SU(2)$, we may go to the "unitary gauge" where $z = z^* = 0$. This gauge exhibits the particle content of the theory,

$$\begin{aligned} \mathcal{L} \; = \; & - \; \frac{1}{4}\,F^{(a)}_{\mu\nu}F^{\mu\nu(a)} \; - \; \frac{1}{2}\overline{\lambda}^{(a)}\gamma^\mu \mathcal{D}_\mu \lambda^{(a)} \; - \; \frac{1}{2}\overline{\chi}\gamma^\mu \mathcal{D}_\mu \chi \\ & - \; \frac{1}{2}\,g^2 W^+_\mu W^{-\mu} \; - \; \frac{1}{8}\,g^2 \; - \; i\,g\,\overline{\chi}\gamma_5\psi_2 \; + \; \frac{1}{4}\,(\overline{\chi}\chi)(\overline{\chi}\chi) \; , \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} W^{\pm}_\mu \; &= \; \frac{1}{2}\sqrt{2}\left({A_\mu}^{(1)} \; \pm \; i\,{A_\mu}^{(2)}\right) \\ \psi_{1,2} \; &= \; \frac{1}{2}\sqrt{2}\left(\lambda^{(1)} \; \pm \; i\,\gamma_5\lambda^{(2)}\right) \\ \mathcal{D}_\mu \chi \; &= \; \partial_\mu \chi \; + \; i\,g\,{A_\mu}^{(3)}\gamma_5 \chi \; . \end{aligned} \tag{4.31}$$

The $SU(2)$ symmetry implies that $D^{(a)2}$ is a constant. The constant is positive, so supersymmetry is spontaneously broken. The mass spectrum is as follows. The charged vector mesons $W^{\pm}_\mu$ are massive; they have eaten the scalars z and z^*. The massless vector meson A_μ is the gauge field corresponding to the unbroken $U(1)$ symmetry. Its supersymmetry partner is the massless Goldstone spinor λ^3. The Majorana spinors χ and ψ_2 are massive; they have combined to form one massive Dirac spinor, of mass proportional to the inverse radius R^{-1}. Finally, ψ_1 is both massless and charged. Note that this is just what one wants for grand unified theories. The CP^1 model has spontaneously broken supersymmetry, no leftover Higgs, and massless Weyl spinors in complex representations of the unbroken gauge group.

Properties like these hold for other homogeneous spaces $\mathcal{G}/\mathcal{H}$ which form Kähler manifolds. If $\mathcal{G}$ is gauged, one finds $\mathcal{G}$ spontaneously broken to $\mathcal{H}$. One also finds that supersymmetry is spontaneously broken and that all scalars are eaten. Furthermore, at the tree level, there are charged massless spinors in complex representations of $\mathcal{H}$. For example, $SU(5)/SU(3) \times SU(2) \times U(1)$ yields a massless $(\underline{3}, \underline{2})$ of $SU(3) \times SU(2)$ Weyl fermions. Other examples are listed in Table 2 [29].

Table 2
Complex Fermions on Kähler Manifolds $\mathcal{G}/\mathcal{H}$

$\mathcal{G}$	$\mathcal{H}$	Complex Fermions
$SU(r+s)$	$SU(r) \times SU(s) \times U(1)$	$(\underline{r}, \underline{s})$ of $SU(r) \times SU(s)$
$SO(10)$	$SU(5) \times U(1)$	$\underline{10}$ of $SU(5)$
E_6	$SO(10) \times U(1)$	$\underline{16}$ of $SO(10)$
E_7	$E_6 \times U(1)$	$\underline{27}$ of E_6

Despite these miraculous features, one must not take these models too seriously. The models are nonrenormalizable, and there is no clear way to break $\mathcal{H}$ down to $SU(3) \times SU(2) \times U(1)$. In the full quantum theory, one often cannot gauge $\mathcal{H}$—much less $\mathcal{G}$—because of anomalies. And what is even worse, in many cases a coordinate anomaly on $\mathcal{M}$ implies that the quantum effective action does not respect the symmetries of the classical theory [30]. Still, Table 2 is intriguing, and these models are remarkable because the particle spins (as well as their masses) violate supersymmetry. No model with unbroken supersymmetry has the same spin spectrum.

COUPLING THE GAUGE INVARIANT SUPERSYMMETRIC SIGMA MODEL TO SUPERGRAVITY

Having constructed the gauge invariant supersymmetric nonlinear sigma model, we would now like to couple it to supergravity [10,13]. As before, we shall first use superspace techniques to gauge the linear subgroup $\mathcal{H}$ of the isometry group $\mathcal{G}$. We will then expand the Lagrangian in components, and guess its extension to the full group $\mathcal{G}$. Of course, we must then verify by hand that the resulting Lagrangian is still supersymmetric.

In Section 3 we found that supersymmetric sigma models may be coupled to supergravity if the scalar fields ϕ^i lie on a Hodge manifold $\mathcal{M}$. In the remainder of this section, we shall see that the gauge interactions lead to no new restrictions on $\mathcal{M}$. The final Lagrangian will give a geometrical interpretation to the most general gauge invariant coupling of chiral multiplets in supergravity. It clarifies the deep relation between chiral invariance, the Fayet-Iliopoulos D-term and supergravity. It also leads to formulae of interest to model builders.

In parallel with the previous case, we start our construction by gauging that subgroup $\mathcal{H}$ of $\mathcal{G}$ which leaves invariant the Kähler potential $K(\phi^i, \phi^{*j})$. Comparing (3.21) with (4.19), we see that

$$\mathcal{L} \;=\; -\,3 \int d^4\theta \; E \;\exp\left\{ -\frac{1}{3}\, K\left[\Phi^i, (\Phi^+ e^{2gV})^{*j} \right] \right\} \tag{4.32}$$

is the $\mathcal{H}$-invariant generalization of the sigma model Lagrangian in local superspace. To gauge the full group $\mathcal{G}$, we decompose (4.32) in component fields. We then add the kinetic terms for the gauge fields, eliminate the auxiliary fields, and guess the extension to the full group $\mathcal{G}$. After a long calculation, we find the coupling to supergravity of the full gauge invariant supersymmetric nonlinear sigma model [10,13]:[♮4]

♮4 The authors of Ref. [10] only gauge the linear subgroup $\mathcal{H}$ of $\mathcal{G}$. However, they consider a more general action for the gauge field multiplet.

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{1}{2}eR \;-\; e\,g_{ij^*}\mathcal{D}_\mu\phi^i\mathcal{D}^\mu\phi^{*j} \\
& -\frac{1}{2}e\,g^2 D^{(a)2} \;-\; \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_5\gamma_\nu\mathcal{D}_\rho\psi_\sigma \\
& -\frac{1}{2}e\,\overline{\chi}^i\,\widehat{g_{ij^*}}\,\gamma^\mu\mathcal{D}_\mu\chi^j \;-\; \frac{1}{4}e\,F_{\mu\nu}{}^{(a)}F^{\mu\nu(a)} \\
& -\frac{1}{2}e\,\overline{\lambda}^{(a)}\gamma^\mu\mathcal{D}_\mu\lambda^{(a)} \;+\; \frac{1}{2}\sqrt{2}\,e\,\overline{\chi}^i\,\widehat{g_{ij^*}\mathcal{D}_\nu\phi^{*j}}\gamma^\mu\gamma^\nu\psi_\mu \\
& -\sqrt{2}\,eg\,\overline{\chi}^i\,\widehat{g_{ij^*}V^{(a)*j}}\,\lambda^{(a)} \;+\; \frac{i}{2}eg\,D^{(a)}\overline{\psi}_\mu\gamma^\mu\gamma_5\lambda^{(a)} \\
& +\frac{1}{4}e\,\overline{\psi}_\mu\sigma^{\alpha\beta}\gamma^\mu\lambda^{(a)}\left[2F_{\alpha\beta}{}^{(a)} \;+\; \overline{\psi}_\alpha\gamma_\beta\lambda^{(a)}\right] \\
& +\frac{1}{16}e\,g_{ij^*}\overline{\chi}^i(1+\gamma_5)\gamma_\sigma\chi^j\left[\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_\mu\gamma_\nu\psi_\rho \;-\; \overline{\psi}_\rho\gamma_5\gamma^\sigma\psi^\rho\right] \\
& -\frac{1}{64}e\left[g_{ij^*}g_{k\ell^*} \;-\; 2R_{ij^*k\ell^*}\right]\overline{\chi}^i(1+\gamma_5)\gamma_\mu\chi^j\overline{\chi}^k(1+\gamma_5)\gamma^\mu\chi^\ell \\
& +\frac{3}{64}e\,\overline{\lambda}^{(a)}(1+\gamma_5)\gamma_\mu\lambda^{(a)}\overline{\lambda}^{(b)}(1+\gamma_5)\gamma^\mu\lambda^{(b)} \\
& -\frac{1}{32}e\,g_{ij^*}\overline{\lambda}^{(a)}(1+\gamma_5)\gamma_\mu\lambda^{(a)}\overline{\chi}^i(1+\gamma_5)\gamma^\mu\chi^j\,.
\end{aligned}
\tag{4.33}
$$

The covariant derivatives are as follows,

$$
\begin{aligned}
\mathcal{D}_\mu\phi^i \;&=\; \partial_\mu\phi^i \;-\; gA_\mu{}^{(a)}V^{(a)i} \\
\mathcal{D}_\mu\chi^i \;&=\; \partial_\mu\chi^i \;+\; \frac{1}{2}\omega_{\mu\alpha\beta}\sigma^{\alpha\beta}\chi^i \;+\; \widehat{\Gamma^i{}_{jk}\mathcal{D}_\mu\phi^j}\,\chi^k \\
&\quad -\; gA_\mu{}^{(a)}\,\widehat{\frac{\partial V^{(a)i}}{\partial\phi^j}}\,\chi^j \;-\; \frac{i}{2}gA_\mu{}^{(a)}\mathrm{Im}\,F^{(a)}\,\gamma_5\,\chi^i \\
\mathcal{D}_\mu\lambda^{(a)} \;&=\; \partial_\mu\lambda^{(a)} \;+\; \frac{1}{2}\omega_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\lambda^{(a)} \;-\; g\,f^{abc}A_\mu{}^{(b)}\lambda^{(c)} \\
&\quad +\; \frac{i}{2}g\,A_\mu{}^{(b)}\mathrm{Im}\,F^{(b)}\,\gamma_5\,\lambda^{(a)} \\
\mathcal{D}_\mu\psi_\nu \;&=\; \partial_\mu\psi_\nu \;+\; \frac{1}{2}\omega_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\psi_\nu \;+\; \frac{i}{2}gA_\mu{}^{(a)}\mathrm{Im}\,F^{(a)}\,\gamma_5\,\lambda^{(a)}\,.
\end{aligned}
\tag{4.34}
$$

The Lagrangian (4.33) together with the covariant derivatives (4.34) is invariant under the isometry group $\mathcal{G}$. It is also invariant under the following supergravity transformations,

$$
\begin{aligned}
\delta e_{\alpha\mu} &= \bar{\epsilon}\gamma_\alpha\psi_\mu \\
\delta\phi^i &= \frac{1}{2}\sqrt{2}\,\bar{\epsilon}(1+\gamma_5)\chi^i \\
\delta\chi^i &= \sqrt{2}\,\gamma^\mu\widehat{\mathcal{D}_\mu\phi^{*i}}\epsilon \;+\; \frac{1}{4}\sqrt{2}\,(\gamma_5\chi^i)\left(\overline{\chi}^j\,\frac{\widehat{\partial K}}{\partial\phi^j}\,\gamma_5\epsilon\right) \\
&\quad -\; \frac{1}{2}\,(\gamma_\mu\epsilon)(\overline{\psi}^\mu\chi^i) \;-\; \frac{1}{2}\,(\gamma_5\gamma_\mu\epsilon)\,(\overline{\psi}^\mu\gamma_5\chi^i) \\
&\quad -\; \frac{1}{2}\sqrt{2}\,(\gamma_5\chi^k)(\overline{\chi}^j\,\widehat{\Gamma^i_{jk}}\,\gamma_5\epsilon) \;-\; \frac{1}{2}\sqrt{2}\,\chi^k\,(\overline{\chi}^j\,\widehat{\Gamma^i_{jk}}\,\epsilon) \\
\delta\psi_\mu &= 2\mathcal{D}_\mu\epsilon \;+\; \frac{1}{4}\,g_{ij^*}(\sigma_{\mu\nu}\gamma_5\epsilon)\left[\overline{\chi}^i(1+\gamma_5)\gamma^\nu\chi^j\right] \\
&\quad -\; \frac{1}{4}\sqrt{2}\,(\gamma_5\psi_\mu)\left(\overline{\chi}^i\,\frac{\widehat{\partial K}}{\partial\phi^i}\gamma_5\epsilon\right) \\
&\quad -\; \frac{1}{4}\,[g_{\mu\nu} \;-\; \sigma_{\mu\nu}]\,\gamma_5\epsilon\,(\overline{\lambda}^{(a)}\gamma^\nu\gamma_5\lambda^{(a)}) \\
\delta A_\mu{}^{(a)} &= -\,\bar{\epsilon}\gamma_\mu\lambda^{(a)} \\
\delta\lambda^{(a)} &= \left[F_{\alpha\beta}{}^{(a)} \;+\; \overline{\psi}_\alpha\gamma_\beta\lambda^{(a)}\right]\sigma^{\alpha\beta}\epsilon \;+\; i\,g\,D^{(a)}\gamma_5\epsilon\;,
\end{aligned}
\tag{4.35}
$$

as may be checked by hand.

Form the superspace Lagrangian (4.32), we see that $\mathcal{L}$ is not invariant under Kähler transformations, $K \to K + F + F^*$. As before, $\mathcal{L}$ is invariant if and only if the Kähler transformations are accompanied by chiral rotations of the Fermi fields,

$$
\begin{aligned}
\chi^i &\to \exp\left[\frac{i}{2}\,\mathrm{Im}\,F\,\gamma_5\right]\chi^i \\
\lambda^{(a)} &\to \exp\left[-\frac{i}{2}\,\mathrm{Im}\,F\,\gamma_5\right]\lambda^{(a)} \\
\psi_\mu &\to \exp\left[-\frac{i}{2}\,\mathrm{Im}\,F\,\gamma_5\right]\psi_\mu\,.
\end{aligned}
\tag{4.36}
$$

These chiral rotations lead to the same quantization condition as in Section 2. They imply that only Kähler manifolds of restricted type may be coupled to supergravity.

The chiral rotations (4.36) give rise to terms proportional to $2i\,\mathrm{Im}\,F^{(a)} = V^{(a)i}(\partial K/\partial\phi^i) - V^{(a)*i}(\partial K/\partial\phi^{*i}) + 2iD^{(a)}$ in the covariant derivatives (4.34). This is as expected, for isometries in $\mathcal{G}/\mathcal{H}$ give rise to shifts in the Kähler potential, $\delta K = F + F^*$, where $F = \epsilon^{(a)}[V^{(a)i}(\partial K/\partial\phi^i) + iD^{(a)}]$. Invariance of the Lagrangian $\mathcal{L}$ requires that these shifts be compensated by the chiral transformations (4.36). It is no surprise that gauging the isometries of $\mathcal{M}$ requires the gauging of the associated chiral rotations.

It is important to note that the Lagrangian (4.33) contains explicitly the functions $D^{(a)}$. Their existence is both necessary and sufficient to gauge the group $\mathcal{G}$. If the group $\mathcal{G}$ contains a $U(1)$ factor, equations (4.12) and (4.15) do not determine the $D^{(a)}$ uniquely. There is an arbitrary integration constant associated with each $U(1)$ factor,

$$D \rightarrow D + C . \tag{4.37}$$

In the globally supersymmetric case, shifts of these constants were shown to give rise to Fayet-Iliopoulos D-terms. The same is true in supergravity. By shifting the functions D, it is easy to recover the gauge invariant supergravity version of $\delta\mathcal{L}_{\mathrm{FI}}$ [31]:

$$\delta\mathcal{L}_{\mathrm{FI}} = -\frac{1}{2}eg^2\,\xi^2 - eg^2\xi\,D + \frac{i}{2}eg\,\xi\,(\overline{\psi}_\mu\gamma^\mu\gamma_5\lambda) . \tag{4.38}$$

Note that all the spinor covariant derivatives now contain chiral pieces proportional to the Fayet-Iliopoulos parameter ξ.[♮5]

The Lagrangian (4.33) contains the kinetic pieces necessary to couple the gauge invariant nonlinear sigma model to supergravity. As written, however, it completely ignores the existence of the superpotential. As explained in Section 3, the superpotential is an additional interaction term which is of crucial importance to realistic models. In superspace, it is given by

$$\Delta\mathcal{L} = \int d^2\theta\;\widetilde{\mathcal{E}}\,W(\Phi^i) + \text{h.c.}, \tag{4.39}$$

where W is an analytic function of the Φ^i, invariant under $\mathcal{H}$.

In the previous section, the superpotential was shown to be a section W of a holomorphic line bundle $\mathcal{F}$ over $\mathcal{M}$. The hermitian structure was shown to be

$$\|\,W\,\|^2 = e^K\,WW^* . \tag{4.40}$$

This is invariant under Kähler transformations provided

$$W \rightarrow W\,e^{-F} . \tag{4.41}$$

From this we see that a given superpotential W is gauge covariant if and

♮5 In the language of the trade, these are known as gauged R-transformations.

only if

$$W \;\rightarrow\; W \exp\left[\, - \,\epsilon^{(a)} (V^{(a)i} \frac{\partial K}{\partial \phi^i} \;+\; i\, D^{(a)})\right] \tag{4.42}$$

under the isometry generated by $\epsilon^{(a)}V^{(a)i}$. Potentials not obeying this law explicitly break the gauge symmetry of $\mathcal{M}$.

The Lagrangian (4.39) may be readily decomposed in terms of component fields.

$$\begin{aligned} e^{-1}\Delta\mathcal{L} \;&=\; \exp(K/2)\Big\{\overline{\psi}_\mu \sigma^{\mu\nu}\widehat{W}^*\psi_\nu \;+\; \frac{1}{2}\sqrt{2}\,\overline{\psi}\cdot\gamma\,\widehat{D_iW}\chi^i \\ &\quad -\; \frac{1}{2}\overline{\chi}^i\,\widehat{\left[\mathcal{D}_j(D_iW)\right]}\,\chi^j\Big\} \\ &\quad +\; \exp(K)\left[3\,|W|^2 \;-\; (D_iW)(D_jW)^* g^{ij^*}\right] \;. \end{aligned} \tag{4.43}$$

The covariant derivatives D_iW and $\mathcal{D}_j(D_iW)$ are given below (4.4). It is easy to verify that (4.43) is invariant under the transformations (3.10), (4.36) and (4.41). The combined Lagrangian (4.33) plus (4.43) is also invariant under the following supergravity transformations:

$$\begin{aligned} \delta e_{\alpha\mu} \;&=\; \overline{\epsilon}\gamma_\alpha\psi_\mu \\ \delta\phi^i \;&=\; \frac{1}{2}\sqrt{2}\,\overline{\epsilon}(1+\gamma_5)\chi^i \\ \delta\chi^i \;&=\; \sqrt{2}\,\gamma^\mu\,\widehat{D_\mu\phi^{*i}}\epsilon \;+\; \frac{1}{4}\sqrt{2}(\gamma_5\chi^i)\left(\overline{\chi}^j\,\widehat{\frac{\partial K}{\partial\phi^j}}\gamma_5\epsilon\right) \\ &\quad -\; \frac{1}{2}(\gamma_\mu\epsilon)(\overline{\psi}^\mu\chi^i) \;-\; \frac{1}{2}(\gamma_5\gamma_\mu\epsilon)(\overline{\psi}^\mu\gamma_5\chi^i) \\ &\quad -\; \frac{1}{2}\sqrt{2}(\gamma_5\chi^k)(\overline{\chi}^j\,\widehat{\Gamma^i{}_{jk}}\,\gamma_5\epsilon) \\ &\quad -\; \frac{1}{2}\sqrt{2}\,\chi^k(\overline{\chi}^j\,\widehat{\Gamma^i{}_{jk}}\,\epsilon) \;-\; \sqrt{2}\,\widehat{\exp(K/2)g^{ij^*}(D_jW)^*}\epsilon \end{aligned}$$

$$
\begin{aligned}
\delta\psi_\mu &= 2\mathcal{D}_\mu\epsilon + \frac{1}{4}g_{ij^*}(\sigma_{\mu\nu}\gamma_5\epsilon)\left[\overline{\chi}^i(1+\gamma_5)\gamma^\nu\chi^j\right] \\
&- \frac{1}{4}\sqrt{2}(\gamma_5\psi_\mu)\left(\overline{\chi}^i\frac{\widehat{\partial K}}{\partial\phi^i}\gamma_5\epsilon\right) \\
&- \frac{1}{4}\left[g_{\mu\nu} - \sigma_{\mu\nu}\right]\gamma_5\epsilon\left(\overline{\lambda}^{(a)}\gamma^\nu\gamma_5\lambda^{(a)}\right) + \frac{1}{2}\exp(K/2)\widehat{W}\gamma_\mu\epsilon \\
\delta A_\mu{}^{(a)} &= -\,\overline{\epsilon}\gamma_\mu\lambda^{(a)} \\
\delta\lambda^{(a)} &= \left[F_{\alpha\beta}{}^{(a)} + \overline{\psi}_\alpha\gamma_\beta\lambda^{(a)}\right]\sigma^{\alpha\beta}\epsilon + i\,g\,D^{(a)}\gamma_5\epsilon\,.
\end{aligned}
\tag{4.44}
$$

The Lagrangian (4.33) plus (4.43) is rather complicated. The potential, however, takes a relatively simple form:

$$
\mathcal{V} = \frac{1}{2}g^2 D^{(a)2} + e^K\left[(D_iW)(D_jW)^*g^{ij^*} - 3|W|^2\right]\,. \tag{4.45}
$$

In contrast to the globally supersymmetric case, it is no longer positive semidefinite. This gives us hope that we might construct realistic theories with spontaneously broken supersymmetry and no cosmological constant. Form the transformations (4.44), we see that χ^i and $\lambda^{(a)}$ transform nonlinearly whenever

$$
\left\langle e^{K/2}g^{ij^*}(D_jW)^*\right\rangle \neq 0\,, \qquad \left\langle D^{(a)}\right\rangle \neq 0\,. \tag{4.46}
$$

These conditions identify a linear combination of χ^i and $\lambda^{(a)}$ as the Goldstone spinor, and signal spontaneous supersymmetry breaking. From equation (4.45), we see that spontaneous supersymmetry breaking and no cosmological constant are consistent if and only if

$$
\langle W\rangle \neq 0\,. \tag{4.47}
$$

Because of (4.41), equation (4.47) breaks all the symmetries in $\mathcal{G}$ not in $\mathcal{H}$. Only symmetries for which $F = 0$ are left unbroken.

5. $N = 2$ SUPERSYMMETRY IN FOUR DIMENSIONS

$N = 2$ Supersymmetry and Hyperkähler Geometry

In the previous sections, we derived the most general $N = 1$ supersymmetric Lagrangian in $3 + 1$ dimensions. We found that the matter couplings had a natural description in the language of the nonlinear sigma model. In this section we shall extend these results to $N = 2$. We will begin to construct the most general Lagrangian for the massless spin $(0, \frac{1}{2})$ multiplet [14,32]. As before, we shall see that our results have a geometrical interpretation in the language of the sigma model.

In what follows, we will not attempt to include mass terms or potentials, nor will we discuss the spin $(0, \frac{1}{2}, 1)$ gauge field multiplet. Including mass terms and potentials is more difficult in $N = 2$ supersymmetry than in $N = 1$. This is because the $N = 2$ multiplets contain different sets of component fields. As shown in Table 3, the $N = 1$ algebra admits both massless and massive spin $(0, \frac{1}{2})$ representations. The $N = 2$ algebra, on the other hand, has only massless spin $(0, \frac{1}{2})$ representations. The equivalent massive representations contain spins 0, $\frac{1}{2}$ and 1. Thus in $N = 2$ supersymmetry, one cannot pass from massless to massive representations simply by adding a superpotential to the Lagrangian $\mathcal{L}$. One must change the field content of the theory as well.

The question of gauge fields is also more subtle in $N = 2$ supersymmetry than in $N = 1$. Again this is because of the different field content of the $N = 2$ multiplets. In $N = 1$ supersymmetry, gauge field multiplets contain spins $\frac{1}{2}$ and 1. In contrast, $N = 2$ gauge multiplets contain spins 0, $\frac{1}{2}$ and 1. If the $N = 2$ Lagrangian is to be described in geometrical language, the extra scalar fields in the $N = 2$ gauge multiplets must have a sigma model interpretation. They too must lie on a sigma model manifold, whose dimension should be equal to the dimension of the isometry group $\mathcal{G}$.

Despite these apparent difficulties, much progress has been made on these issues. One elegant approach is to work in five or six dimensions, where the $N = 2$ supersymmetry becomes $N = 1$. In the higher dimensions, it is easier to add mass terms and gauge fields. One can then recover a $3 + 1$ dimensional model by dimensional reduction. Of course, there is no guarantee that this procedure gives the most general $3 + 1$ dimensional model, but it does give insight into the geometrical structure of the lower-dimensional theory. Recent developments along these and other lines are discussed in Ref. [33].

In the remainder of this section, we shall restrict our attention to massless spin $(0, \frac{1}{2})$ multiplets. We will work out the most general Lagrangian containing these fields, and we will describe the matter couplings in the language of the nonlinear sigma model. As before, we shall see that different $N = 2$ couplings correspond to different manifolds, and constraints on the matter couplings arise as restrictions on the manifolds $\mathcal{M}$.

Table 3
Supersymmetry Representations in $3+1$ Dimensions

N	Representation	Spin	Multiplicity	Type
1	massless matter	0	1	complex
		1/2	1	Majorana
1	massive matter	0	1	complex
		1/2	1	Majorana
1	massless gauge field	1/2	1	Majorana
		1	1	real
2	massless matter	0	2	real
		1/2	1	Majorana
2	massive matter	0	5	real
		1/2	4	Majorana
		1	1	real
2	massless gauge field	0	2	real
		1/2	2	Majorana
		1	1	real

In $N = 2$ rigid supersymmetry, we shall find that sigma models exist for all hyperkähler manifolds $\mathcal{M}$ [4]. This is, of course, expected, since $N = 4$ supersymmetry in $1+1$ dimensions is related to $N = 2$ supersymmetry in $d = 3+1$. However, we will see that hyperkähler manifolds arise in a different way than in the $1+1$ dimensional models of Section 2. After discussing rigid supersymmetry, we shall move on to consider $N = 2$ local supersymmetry. We will see that $N = 2$ local supersymmetry requires the scalar fields to be the coordinates—not of a hyperkähler manifold—but rather of a quaternionic manifold [14]. Quaternionic and hyperkähler manifolds are related to each other,

Thus one of our results is that matter couplings allowed in $N = 2$ supersymmetry are forbidden in $N = 2$ supergravity, and vice versa.[♯6] A second surprising result is that in $N = 2$ supergravity, the matter couplings cannot be trivially reduced to $N = 1$. The $N = 1$ and $N = 2$ rigid and local matter couplings are summarized in Table 4.

In $3+1$ dimensions, we will find that $N = 2$ sigma models are best described according to their holonomy groups G. As discussed in Section 2, the holonomy group of a connected n-dimensional Riemannian manifold is

♯6 This result was partly anticipated in Ref. [35], in which an $N = 2$ supersymmetric sigma model was found that could not be coupled to supergravity.

Table 4
Matter Couplings in $3+1$ Dimensions

	rigid supersymmetry	local supersymmetry
$N=1$	Kähler	Hodge
$N=2$	hyperkähler	quaternionic

the group of transformations generated by parallel transporting all vectors around all possible closed curves in $\mathcal{M}$. If the parallel transport is done with respect to the Riemann connection, then the holonomy group G is contained in $O(n)$.

In Section 2, we defined a hyperkähler manifold as a $4n$-dimensional real Riemannian manifold endowed with three parallel complex structures, obeying the relations (2.20) and (2.21) [35]. In this section we take a slightly different point of view. We now *define* a hyperkähler manifold to be a $4n$-dimensional real Riemannian manifold whose holonomy group is contained in $Sp(n) \subseteq O(4n)$. It is easy to show that this definition is equivalent to that given in terms of the parallel complex structures.

As shown in Table 3, the $N=2$ supersymmetry algebra has representations with 2 massless spin 0 states, and 2 massless spin $\pm\frac{1}{2}$ states. However, interacting field theories seem to exist only if the number of real scalar fields is divisible by four. This is related to the fact that $\mathcal{M}$ must be hyperkähler. Therefore, we consider theories with $4n$ real scalars ϕ^i, $2n$ Majorana spinors χ^Z, and 2 Majorana spinors ϵ^A. The spinors χ^Z are the supersymmetry partners of the scalars ϕ^i, and the spinors ϵ^A are the two supersymmetry parameters.

On dimensional grounds, the supersymmetry transformation of the scalar fields ϕ^i must take the following form

$$\delta\phi^i \;=\; \gamma^i_{AZ}(\bar{\epsilon}_R{}^A\chi_L{}^Z \;+\; \bar{\epsilon}_L{}^A\chi_R{}^Z)\ , \tag{5.1}$$

where γ^i_{AZ} is a nonsingular function of the ϕ^j. Since the variation of a coordinate is a vector, $\delta\phi^i$ takes its values in the $4n$-dimensional tangent bundle T of $\mathcal{M}$. The spinors ϵ^A and χ^Z take their values in 2 and $2n$-dimensional bundles H and P. The supersymmetry transformation (5.1) says that $\mathcal{M}$ must admit structures γ^i_{AZ} that split the tangent space in two, $T = H \otimes P$. This immediately gives a strong restriction on the allowed manifolds $\mathcal{M}$.

When we actually construct the sigma model, we shall see that supersymmetry demands an even stronger condition. Cancellation of the $\bar{\epsilon}\chi$ terms requires $\nabla_j \gamma^i_{AZ} = 0$. That is to say, there must exist suitable connections in T, H and P such that the γ^i_{AZ} are covariantly constant.

To make the γ^i_{AZ} more familiar, let us specialize for the moment to $n = 1$. Then $\mathcal{M}$ and T are four-dimensional, and H and P are both two-dimensional. In this case, H and P are bundles of left- and right-handed spinors. In four dimensions, T is indeed the product of two spinor bundles, and the γ^i_{AZ} are simply the Dirac γ-matrices. And in Riemannian geometry, the Dirac γ-matrices are indeed covariantly constant.

There is one more piece of information we have not yet used. That is that the bundle T is real. The fact that T is real implies that H and P are both real or both pseudoreal. As far as is known, H and P real does not lead to a sigma model. Therefore we take H and P to be pseudoreal. In down-to-earth terms, this just says that A is an $Sp(1)$ index, and Z is an $Sp(n)$ index.

In rigid supersymmetry, the supersymmetry parameters ϵ^A are constants, so the bundle H is trivial, and the $Sp(1)$ connection is flat. In local supersymmetry, we shall see that H cannot be trivial, so the holonomy group $G \subseteq Sp(1) \times K$, where $K \subseteq Sp(n)$.

To actually prove these assertions, one must do a little work. The fact that the γ^i_{AZ} are covariantly constant implies that they satisfy several relations similar to the Dirac algebra:[♯7]

$$
\begin{aligned}
\gamma^i_{AY}\,\gamma^j_{BZ}\,g_{ij} &= \epsilon_{AB}\,\epsilon_{YZ} \\
\gamma^i_{AZ}\,\gamma^{jBZ} + \gamma^j_{AZ}\,\gamma^{iBZ} &= g^{ij}\delta_A{}^B \\
\gamma^i_{AY}\,\gamma^{jAZ} + \gamma^j_{AY}\,\gamma^{iAZ} &= n^{-1}g^{ij}\delta_Y{}^Z\ .
\end{aligned}
\tag{5.2}
$$

Here ϵ_{AB} and ϵ_{YZ} are the totally antisymmetric $Sp(1)$ and $Sp(n)$ invariant tensors. Furthermore, the fact that $[\nabla_i, \nabla_j]\gamma^k_{AZ} = 0$ implies

$$
R_{ijk\ell}\,\gamma^\ell_{AY}\gamma^k_{BZ} = \epsilon_{AB}\,R_{ijYZ} + \epsilon_{YZ}\,R_{ijAB}\ , \tag{5.3}
$$

where R_{ijAB} and R_{ijYZ} are the $Sp(1)$ and $Sp(n)$ curvatures, formed form the appropriate connections. In rigid supersymmetry, the ϵ^A are constants,

♯7 Our raising and lowering conventions are as follows: $\chi^A = \epsilon^{AB}\chi_B$, $\chi_A = \epsilon_{BA}\chi^B$ and $\epsilon_{AB}\,\epsilon^{BC} = -\delta_A{}^C$ (and likewise for ϵ_{XY}).

so $R_{ijAB} = 0$. In this case

$$R_{ijk\ell} = \gamma_k{}^{AZ} \gamma_{\ell A}{}^{Y} R_{ijYZ} \tag{5.4}$$

and we see that the holonomy group G is not all of $O(4n)$, but rather is contained in $Sp(n)$. That is why $N = 2$ rigid supersymmetry in $3+1$ dimensions demands that $\mathcal{M}$ be hyperkähler.

Because of the antisymmetry of ϵ_{AB}, it is easy to check that (5.4) implies

$$R_{ijk\ell} = - R_{ij\ell k} \,, \tag{5.5}$$

as required. Furthermore, the cyclic identity on the curvature requires

$$R_{ijYZ} = \gamma_i{}^{AW} \gamma_{jA}{}^{X} \Omega_{XYZW} \,. \tag{5.6}$$

The object Ω_{XYZW} is a totally symmetric $Sp(n)$ tensor; it is known as the hyperkähler curvature. Note that (5.4) and (5.6) are sufficient to prove that hyperkähler manifolds are Ricci flat.

The relations (5.2)-(5.6) are all that are needed to prove the invariance of the $N = 2$ supersymmetric nonlinear sigma model. The Lagrangian is as follows [4,14],[♯8]

$$\begin{aligned} \mathcal{L}_{SS} &= - g_{ij}\, \partial_\mu \phi^i \partial^\mu \phi^j - \frac{1}{2} \overline{\chi}_Z \gamma^\mu \mathcal{D}_\mu \chi^Z \\ &\quad + \frac{1}{16} \Omega_{XYZW} \left(\overline{\chi}_L{}^X \gamma_\mu \chi_L{}^Y \right) \left(\overline{\chi}_L{}^Z \gamma^\mu \chi_L{}^W \right) \,, \end{aligned} \tag{5.7}$$

where $\chi^Z_{L,R} = (1 \pm \gamma_5)\chi^Z$. The covariant derivative $\mathcal{D}_\mu \chi^Z$ contains the $Sp(n)$ connection coefficients $\Gamma^{iZ}{}_Y$,

$$\mathcal{D}_\mu \chi^Z = \partial_\mu \chi^Z + \Gamma_i{}^Z{}_Y \, \partial_\mu \phi^i \chi^Y \,, \tag{5.8}$$

and the transformation laws are given by

$$\begin{aligned} \delta \phi^i &= \gamma^i_{AZ} \left(\overline{\epsilon}_R{}^A \chi_L{}^Z + \overline{\epsilon}_L{}^A \chi_R{}^Z \right) \\ \delta \chi_L{}^Z &= 2\, \partial_\mu \phi^i \, \gamma_i{}^{AZ} \gamma^\mu \epsilon_{RA} - \Gamma_i{}^Z{}_Y \, \delta\phi^i \, \chi_L{}^Y \,. \end{aligned} \tag{5.9}$$

Cancellation of the $\overline{\epsilon}\chi$ terms implies that $R_{ijAB} = 0$.

♯8 When no helicity is specified, we adopt the following conventions: $\overline{\chi}_A \lambda^A = \overline{\chi}_{RA} \lambda_L{}^A - \overline{\chi}_L{}^A \lambda_{RA}$, $\overline{\chi}_A \gamma^\mu \lambda^A = \overline{\chi}_{LA} \gamma^\mu \lambda_L{}^A + \overline{\chi}_R{}^A \gamma^\mu \lambda_{RA}$ and $\overline{\chi}_A \sigma^{\mu\nu} \lambda^A = \overline{\chi}_{RA} \sigma^{\mu\nu} \lambda_L{}^A - \overline{\chi}_L{}^A \sigma^{\mu\nu} \lambda_{RA}$.

The Lagrangian (5.7) is invariant under rigid $N = 2$ supersymmetry transformations. To gauge them, we shall use the well-known Noether procedure, adding terms proportional to $\kappa^2 = 8\pi G_N$ to the Lagrangian and transformation laws. We begin by adding the pure $N = 2$ supergravity Lagrangian $\mathcal{L}_{SG}$ [36] to the globally supersymmetric matter Lagrangian $\mathcal{L}_{SS}$:

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{SG} + \mathcal{L}_{SS} \\
\mathcal{L}_{SG} &= -\frac{1}{2\kappa^2}\, e\, R - \frac{1}{2}\,\epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}_{\mu A}\gamma_5\gamma_\nu \mathcal{D}_\rho \psi_\sigma{}^A - \frac{1}{4}\, e\, F_{\mu\nu}F^{\mu\nu} \\
&\quad + \frac{1}{4}\sqrt{2}\, e\,\kappa\,\overline{\psi}_{\mu A}\left(F^{\mu\nu} + \frac{1}{2}\, e^{-1}\widetilde{F}^{\mu\nu}\gamma_5\right)\psi_\nu{}^A \\
&\quad - \frac{1}{8}\, e\,\kappa^2\,\overline{\psi}_{\mu A}\left(\overline{\psi}^\mu{}_B\psi^{\nu B} + \frac{1}{2}\, e^{-1}\,\epsilon^{\mu\nu\rho\sigma}\,\overline{\psi}_{\rho B}\psi_\sigma{}^B\gamma_5\right)\psi_\nu{}^A\,,
\end{aligned}
\tag{5.10}
$$

where $\widetilde{F}_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}\, F^{\rho\sigma}$. The pure supergravity action contains two gravitinos $\psi_\mu{}^A$, and one $U(1)$ gauge boson A^μ. It is invariant under the following supergravity transformations:

$$
\begin{aligned}
\delta e_{\mu\alpha} &= \kappa\,\overline{\epsilon}_A\gamma_\alpha\psi_\mu{}^A \\
\delta A_\mu &= \sqrt{2}\;\overline{\epsilon}_A\psi_\mu{}^A \\
\delta\psi_{\mu L}{}^A &= \frac{2}{\kappa}\,\mathcal{D}_\mu\epsilon_L{}^A + \frac{1}{2}\sqrt{2}\left(F_{\mu\nu} - \frac{1}{2}\, e\,\widetilde{F}_{\mu\nu}\gamma_5\right)\gamma^\nu\epsilon_R{}^A \\
&\quad - \frac{1}{2}\,\kappa\left(\overline{\psi}_{\mu A}\psi_\nu{}^A - \frac{1}{2}\, e\,\epsilon_{\mu\nu\rho\sigma}\,\overline{\psi}^\rho{}_A\psi^{\sigma A}\gamma_5\right)\gamma^\nu\epsilon_R{}^A \\
&\quad - \Gamma_i{}^A{}_B\,\delta\phi^i\,\psi_{\mu L}{}^B\,.
\end{aligned}
\tag{5.11}
$$

The covariant derivatives in $\mathcal{L}_{SS}$ and $\mathcal{L}_{SG}$ now contain the spin connection $\omega_{\mu\alpha\beta}$:

$$
\begin{aligned}
\mathcal{D}_\mu\psi_\nu{}^A &= \partial_\mu\psi_\nu{}^A + \frac{1}{2}\,\omega_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\psi_\nu{}^A + \Gamma_i{}^A{}_B\,\partial_\mu\phi^i\,\psi_\nu{}^B \\
\mathcal{D}_\mu\chi^Z &= \partial_\mu\chi^Z + \frac{1}{2}\,\omega_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\chi^Z + \Gamma_i{}^Z{}_Y\,\partial_\mu\phi^i\,\chi^Y \\
\mathcal{D}_\mu\epsilon^A &= \partial_\mu\epsilon^A + \frac{1}{2}\,\omega_{\mu\alpha\beta}\,\sigma^{\alpha\beta}\epsilon^A + \Gamma_i{}^A{}_B\,\partial_\mu\phi^i\,\epsilon^B\,.
\end{aligned}
\tag{5.12}
$$

Note that the covariant derivatives for ϵ^A and $\psi_\mu{}^A$ also contain the $Sp(1)$ connection coefficients $\Gamma^{iA}{}_B$.

We shall use the 1.5-order formalism [16,37], which says that $\omega_{\mu\alpha\beta}$ obeys its own equation of motion and need not be explicitly varied under supergravity transformations. To lowest order, we couple $\mathcal{L}_{SG}$ and $\mathcal{L}_{SS}$ through the Neother current $J_\mu{}^A$:

$$\mathcal{L}_N \;=\; -\,\frac{1}{2}\,\overline{\psi}_{\mu A} J^{\mu A} \;=\; e\,\kappa\,\gamma_{iAZ}\left(\overline{\chi}_R{}^Z\gamma^\mu\gamma^\nu\psi_{\mu L}{}^A + \overline{\chi}_L{}^Z\gamma^\mu\gamma^\nu\psi_{\mu R}{}^A\right)\,\partial_\nu\phi^i\;. \tag{5.13}$$

These terms ensure that $\mathcal{L} = \mathcal{L}_{SG} + \mathcal{L}_{SS} + \mathcal{L}_N$ is invariant to order κ^0 under local supersymmetry transformations. The $\bar{\epsilon}\chi$ terms cancel provided $\nabla_i\gamma^j_{AZ} = 0$. The $\bar{\epsilon}\psi$ terms require

$$R_{ijAB} \;=\; \kappa^2\left(\gamma_{iAZ}\,\gamma_{jB}{}^Z \;-\; \gamma_{jAZ}\,\gamma_{iB}{}^Z\right)\;. \tag{5.14}$$

Substituting (5.14) into (5.3), and using the cyclic identity on the curvature, we find

$$R_{ijXY} \;=\; \kappa^2\left(\gamma_{iAX}\,\gamma_j{}^A{}_Y \;-\; \gamma_{jAX}\,\gamma_i{}^A{}_Y\right) \;+\; \gamma_i{}^{AW}\gamma_{jA}{}^Z\,\Omega_{XYZW}\;. \tag{5.15}$$

Equation (5.14) tells us, as mentioned before, that the $Sp(1)$ curvature is nonzero. A manifold with holonomy contained in $Sp(1)\times SU(n)$ and nonzero $Sp(1)$ curvature is called a *quaternionic* manifold [38,39]. When $R_{ijAB} \neq 0$, the supersymmetry parameter ϵ^A cannot be chosen to be covariantly constant.

Note that equations (5.3), (5.14) and (5.15) fix the scalar curvature in terms of Newton's constant:

$$R \;=\; -\,8\kappa^2\,(n^2 + 2n)\;. \tag{5.16}$$

This is the analogue of the quantization condition found previously for $N = 1$. Here, however, we find that only one value of the scalar self-coupling is consistent with supergravity. These results imply that the order κ^0 variations in $\mathcal{L}$ restrict $\mathcal{M}$ to be a quaternionic manifold of negative scalar curvature. They lead to the surprising conclusion that rigidly supersymmetric theories with $N = 2$ scalar multiplets may not be coupled to ($N = 2$) supergravity.[♯9]

♯9 Note that as $\kappa \to 0$, equation (5.15) approaches (5.6). This tells us that quaternionic manifolds of zero scalar curvature are precisely the hyperkähler manifolds discussed before.

It is remarkable that all this information can be gleaned from the order κ^0 variations of the supergravity Lagrangian. The higher order variations simply confirm the above results. It is a long and tedious calculation to compute the Lagrangian and transformation laws. The calculation is aided by using the 1.5-order formalism and by collecting various terms into the following "supercovariant" expressions:

$$
\begin{aligned}
\widehat{F}_{\mu\nu} &= F_{\mu\nu} - \frac{1}{2}\kappa\sqrt{2}\,\overline{\psi}_{\mu A}\psi_\nu{}^A \\
\widehat{D}_\mu\phi^i &= \partial_\mu\phi^i - \frac{1}{2}\kappa\gamma^i_{AZ}\left(\overline{\psi}_{\mu R}{}^A\chi_L{}^Z + \overline{\psi}_{\mu L}{}^A\chi_R{}^Z\right) .
\end{aligned}
\tag{5.17}
$$

These expressions are supercovariant because their supersymmetry variations contain no $\partial_\mu\epsilon^A$ pieces.

Using (5.2), (5.3), (5.14), (5.15) and (5.16), one can verify that

$$
\begin{aligned}
\mathcal{L} = &- \frac{1}{2\kappa^2}eR - e g_{ij}\widehat{D}_\mu\phi^i\widehat{D}^\mu\phi^j - \frac{1}{4}e\widehat{F}_{\mu\nu}\widehat{F}^{\mu\nu} \\
&- \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\overline{\psi}_{\mu A}\gamma_5\gamma_\nu D_\rho\psi_\sigma{}^A - \frac{1}{2}e\overline{\chi}_Z\gamma^\mu D_\mu\chi^Z \\
&+ \frac{1}{16}\kappa\sqrt{2}\,\overline{\psi}_{\mu A}\gamma_5\psi_\nu{}^A\left(e\widehat{F}^{\mu\nu} + \widehat{\widetilde{F}}^{\mu\nu}\right) \\
&+ e\kappa\gamma_{iAZ}\left(\overline{\chi}_R{}^Z\sigma^{\mu\nu}\psi_{\mu L}{}^A + \overline{\chi}_L{}^Z\sigma^{\mu\nu}\psi_{\mu R}{}^A\right)\left(\partial_\nu\phi^i + \widehat{D}_\nu\phi^i\right) \\
&+ \frac{1}{4}\kappa\sqrt{2}\,e\overline{\chi}_Z\sigma^{\mu\nu}\chi^Z\widehat{F}_{\mu\nu} - \frac{1}{32}e\kappa^2\left(\overline{\chi}_Y\gamma_\mu\gamma_5\chi^Y\right)\left(\overline{\chi}_Z\gamma^\mu\gamma_5\chi^Z\right) \\
&+ \frac{1}{16}e\kappa^2\left(\overline{\chi}_Y\sigma_{\mu\nu}\chi^Y\right)\left(\overline{\chi}_Z\sigma^{\mu\nu}\chi^Z\right) \\
&+ \frac{1}{16}e\,\Omega_{XYZW}\left(\overline{\chi}_L{}^X\gamma_\mu\chi_L{}^Y\right)\left(\overline{\chi}_L{}^Z\gamma^\mu\chi_L{}^W\right)
\end{aligned}
\tag{5.18}
$$

is invariant under the following supergravity transformations:

$$
\begin{aligned}
\delta e_{\mu\alpha} &= \kappa\overline{\epsilon}_A\gamma_\alpha\psi_\mu{}^A \\
\delta A_\mu &= \sqrt{2}\,\overline{\epsilon}_A\psi_\mu{}^A \\
\delta\phi^i &= \gamma^i_{AZ}\left(\overline{\epsilon}_R{}^A\chi_L{}^Z + \overline{\epsilon}_L{}^A\chi_R{}^Z\right) \\
\delta\chi_L{}^Z &= 2\,\partial_\mu\phi^i\gamma_i{}^{AZ}\gamma^\mu\epsilon_{RA} - \Gamma_i{}^Z{}_Y\,\delta\phi^i\chi_L{}^Y \\
\delta\psi_{\mu L}{}^A &= \frac{2}{\kappa}D_\mu\epsilon_L{}^A + \frac{1}{2}\sqrt{2}\left(\widehat{F}_{\mu\nu} - \frac{1}{2}e\widehat{\widetilde{F}}_{\mu\nu}\gamma_5\right)\gamma^\nu\epsilon_R{}^A \\
&\quad - \Gamma_i{}^A{}_B\,\delta\phi^i\psi_{\mu L}{}^B + \frac{1}{2}\gamma^\nu\gamma_5\epsilon_R{}^A\left(\overline{\chi}_Z\sigma_{\mu\nu}\gamma_5\chi^Z\right) .
\end{aligned}
\tag{5.19}
$$

The spin connection ω contains both χ- and ψ-torsion, and the $4n$ real scalar fields are restricted to lie on a negatively curved quaternionic manifold with holonomy group G contained in $Sp(n) \times Sp(1)$.

Even if the holonomy group is contained in $Sp(n) \times Sp(1)$, the γ^i_{AZ} may not exist globally. [In the case $n = 1$, the condition that the γ^i_{AZ} exist globally is precisely the condition that the manifold admit a spin structure.] Because they appear explicitly in (5.18) and (5.19), the γ^i_{AZ} must exist globally for the Lagrangian and transformation laws to make sense. For quaternionic manifolds of positive scalar curvature, this imposes a severe restriction—the only allowed manifolds are the quaternionic projective spaces $HP(n)$ [39]. However equation (5.16) limits us to negatively curved (and typically noncompact) manifolds. It is not known if there are negatively curved quaternionic manifolds which do not admit globally defined γ's.

For $n > 1$, all quaternionic manifolds are Einstein space of constant (nonzero) scalar curvature. The only known compact cases are the symmetric spaces discussed by Wolf [40,41]. These consist of the three families

$$HP(n) = \frac{Sp(n+1)}{Sp(n) \times Sp(1)} \qquad X(n) = \frac{SU(n+2)}{SU(n) \times SU(2) \times U(1)}$$

$$Y(n) = \frac{SO(n+4)}{SO(n) \times SO(4)}\,, \tag{5.20}$$

where $n > 1$, as well as

$$\frac{G_2}{SO(4)} \qquad \frac{F_4}{Sp(3) \times Sp(1)} \qquad \frac{E_6}{SU(6) \times Sp(1)}$$

$$\frac{E_7}{SO(12) \times Sp(1)} \qquad \frac{E_8}{E_7 \times Sp(1)}\,. \tag{5.21}$$

Note that $X(2) \simeq Y(2)$. The only known noncompact examples are the noncompact analogues of (5.20) and (5.21), as well as the homogeneous but not symmetric spaces found by Alekseevskii [41].

By way of conclusion, let us now relate the Lagrangian (5.18) to the $N = 1$ results derived before. In rigid supersymmetry, the reduction from $N = 2$ to $N = 1$ is trivial. The reduction is trivial because any hyperkähler manifold is Kähler with respect to each of its three parallel complex structures. In local supersymmetry, however, the story is more complicated, and the reduction from $N = 2$ to $N = 1$ is *not* trivial. The simple truncation $\epsilon^1 = \psi^1_\mu = A_\mu = 0$ is not preserved by the supergravity transformations (5.19). This is easy to understand in mathematical terms. On a quaternionic manifold, three almost complex structures $I^{(A)i}{}_j$ are defined *locally*, and they locally obey the Clifford algebra relation (2.20). However, the three almost complex structure are *not* defined globally. As one moves from one coordinate patch to another, the almost complex structures rotate into each other via $Sp(1)$ transformations [38]. Since $N = 1$ supergravity requires that $\mathcal{M}$ be Hodge, and Hodge manifolds have globally defined complex structures, we see that the nonzero $Sp(1)$ curvature of a quaternionic manifold not only obstructs the global definition of the $I^{(A)i}{}_j$, but also prevents a trivial reduction from $N = 2$ to $N = 1$.

ACKNOWLEDGEMENTS

Much of the material discussed here was developed in collaboration with Edward Witten. My debt to him is substantial. I am also grateful to Luis Alvarez-Gaumé, Martin Roček, Stuart Samuel, and especially, Julius Wess, for many discussions on supersymmetry and sigma models. Finally, I would like to thank Vladimir Rittenberg and the Organizing Committee for a very pleasant and fruitful stay in Bonn.

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