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SUPERSYMMETRIC MODELS FOR QUARKS AND LEPTONS WITH NONLINEARLY REALIZED E_8 SYMMETRY*

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Abstract

We propose three supersymmetric nonlinear sigma models with global symmetry E_8. The models can accomodate three left-handed families of quarks and leptons without incurring Adler–Bell–Jackiw anomaly with respect to either the standard $SU(3) \times SU(2) \times U(1)$ gauge group, or the $SU(5)$, or $SO(10)$ grand unifying gauge group. They also predict unambiguously a right-handed, fourth family of quarks and leptons. In order to explore the structure of the models, we develop a differential form formulation of the Kahler manifolds, resulting in general expressions for the curvature tensors and other geometrical objects in terms of the structure constants of the algebra, and the squashing parameters. These results, in turn, facilitate a general method for determining the Lagrangian to quartic order, and so the structure of the inherent four-fermion interactions of the models. We observe that the Kahlerian condition $d\omega = 0$ on the fundamental 2-form ω greatly reduces the number of the independent squashing parameters. We also point out two plausible mechanisms for symmetry breaking, involving gravity.

1. Introduction and Summary

Among the field theories that are useful in physics, pure gauge field theories, the general theory of relativity, and nonlinear sigma models distinguish themselves by their elegant embodiment of symmetry. Both the particle content and the form of interactions in these theories are uniquely determined by symmetry, leaving only the magnitude of the coupling constants free. The coupling constants of gauge field theory are dimensionless while their counterparts in general theory of relativity and nonlinear sigma model are dimensionful. Consequently only gauge field theories are renormalizable. For the other two, the dimensional coupling constants define the critical mass scales beyond which the theories fail; in other words, the underlying physical systems enter a new phase at these critical mass scales.

In the nonlinear sigma model for pions,[1] the dimensional coupling constant F_π denotes the mass scale characterizing the dynamical spontaneous breakdown of global chiral symmetry $SU(2)_L \times SU(2)_R$ to $SU(2)_V$. The origin of the chiral symmetry is easily understood in terms of the quark model of hadronic matter. Analogously, in general relativity, the Planck mass, M_P, may be taken as a critical mass scale beyond which space–time enters a different phase.

The standard $SU(3) \times SU(2) \times U(1)$ gauge theory[2] for the strong, weak, and electromagnetic interactions is renormalizable. When it is extended to incorporate the quarks, leptons, and the Higgs scalar fields, care is taken to preserve renormalizability. The only dimensional parameters appearing in the Lagrangian are the mass terms of the scalar fields. Extension to grand unified gauge theories[3] with asymptotic freedom further tames the running coupling constant of the original $U(1)$ in the ultraviolet region. The characteristic mass scales brought out by renormalization procedure, and at which the gauge couplings diverge, appear at the infrared region. We would like to retain, as far as possible, this picture of gauge interactions, grand unified or not, in the present paper. The standard formulation carries with it an implicit assumption that quarks and leptons are elementary, or equivalently, of the absence of a critical mass scale (Λ_σ) in the ultraviolet region, around and beyond which the quarks and leptons will not be the proper dynamical degrees of freedom. So far there is no experimental evidence in direct conflict with the assumption of elementary quarks and leptons.

If Λ_σ does exist in nature, then from the $(g-2)$ factor of electron and muon one estimates $\Lambda_\sigma > 10^3$ TeV,[4] and from e^+e^- Bhabha scattering $\Lambda_\sigma > 750$ GeV.[5] In either case Λ_σ is much greater than the known masses of quarks and leptons.

In this paper we explore the implications of the plausible existence of Λ_σ in terms of supersymmetric nonlinear sigma model.[6] What we hope to derive ultimately is a natural explanation of some features of elementary particle physics not accounted for by standard renormalizable field theory.[2,3] These features include the three–family structure of the observed quark–lepton spectrum, and the mass matrix of quarks and leptons. The mass matrix would involve inevitably the scale characterizing the breakdown of the $SU(2)_L \times U(1)$ symmetry to $U(1)_{e.m.}$, therefore the physics flowing from Λ_σ, though perhaps necessary, is certainly not sufficient for this purpose. The three–family structure, in contrast, is independent of the mechanism for the symmetry breakdown and so might be determined completely by physics at Λ_σ.

Supersymmetric nonlinear sigma model provides a field-theoretic setting that accommodates massless fermions, henceforth to be referred to as σ–fermions; massless spin-0 bosons; and a critical mass scale Λ_σ in the form of a dimensionful coupling constant. The model assumes that the phase beyond Λ_σ, which we shall call the preonic phase, possesses supersymmetry. It makes no assumption about the proper dynamical degrees of freedom in the preonic phase of matter.

Each nonlinear sigma model is characterized by an abstract manifold on which the spin-0 Bose fields take values.[7] The collection of general coordinate transformations on the manifold which leave the length of infinitesimal line element on the manifold invariant forms a group called the isometry group. A subgroup of the isometry group which leaves a point p of the manifold fixed is called the isotropy group at the point p. Obviously the tangent space at the point p forms a linear representation of the isotropy group, called the isotropy representation. For the manifolds that interest us, both the isotropy group and the isotropy representation are the same for every point of the manifold. The action of a nonlinear sigma model is invariant under the isometry group of transformations of the abstract manifold.

The abstract manifold for $N = 1$ supersymmetric nonlinear sigma model[6] in $(3+1)-$dimensional space–time must perforce be a Kahlerian complex manifold.[8] The spin-0 bosons are represented by complex scalar fields, denoted by ϕ^i, and the σ–fermions by two–component Weyl fermions, denoted by χ^i. Together with auxiliary complex scalar field F^i, ϕ^i and χ^i form a chiral superfield Φ^i. The superspace action takes the form

$$I = \int d^4x \; d^2\theta \; d^2\bar{\theta} \; K(\Phi^i, \; \Phi^{*j}) \tag{1}$$

where $K(\Phi^i, \; \Phi^{*j})$ is a real function of the chiral superfields Φ^i, and antichiral superfields Φ^{*j}, which is obtained from the Kahler potential $K(\phi^i, \; \phi^{*j})$ by simply substituting Φ^i for ϕ^i , and Φ^{*j} for ϕ^{*j}. Obviously the action is invariant under the transformation

$$K(\phi^i, \; \phi^{*j}) \rightarrow K(\phi^i, \; \phi^{*j}) + F(\phi^i) + F^*(\phi^{*j}) \tag{2}$$

where $F(\phi^i)$ is any holomorphic function of the ϕ^i. After integration over θ and $\bar{\theta}$, and elimination of the auxiliary fields F^i, Eq. (1) yields the following Lagrangian density

$$
\begin{aligned}
\mathcal{L} = & -g_{ij*}\partial_\mu\phi^i\partial^\mu\phi^{*j} - \frac{i}{2}g_{ij*}\chi^i\ \sigma^\mu D_\mu\bar{\chi}^j \\
& - \frac{i}{2}g_{ij*}\ \bar{\chi}^j\ \bar{\sigma}^\mu D_\mu\chi^i + \frac{1}{4}R_{ik^*j\ell^*}(\chi^i\chi^j)(\bar{\chi}^k\bar{\chi}^\ell)
\end{aligned}
\tag{3}
$$

Here g_{ij*}, $R_{ik^*j\ell^*}$, and $D\chi^i$ are respectively the metric tensor, curvature tensor, and covariant derivatives defined on the Kahler manifold. We have explicitly

$$g_{ij*} = \partial_i\partial_{j*}K(\phi,\phi^*) \tag{4}$$

$$R_{ik^*j\ell^*} = \partial_j\partial_{\ell^*}g_{ik^*} - g^{mn*}\partial_j g_{in^*}\partial_{\ell^*}g_{mk^*} \tag{5}$$

and

$$D\chi^i = \partial\chi^i + g^{i\ell^*}\partial_k g_{j\ell^*}\partial\phi^j\chi^k \tag{6}$$

where $g^{i\ell^*}$ is the inverse of the metric tensor g_{ij^*}.

A beautiful class of Kahler manifolds exist, where each can be expressed as a coset space G/H. Here G is a compact, connected, simple Lie group and H a closed subgroup which is the centralizer of a torus in G. G/H is a homogeneous space with isometry group G and isotropy group H. The scalar fields ϕ^i, being a local coordinate system of the manifold, form a nonlinear realization of G. The $\sigma-$fermions χ^i, transforming in the same way as $d\phi^i$ under the action of G, transform like an isotropy representation under the action of H. Most of the $\sigma-$fermions of a desirable model will be identified as the known quarks and leptons.[10,11] The $SU(3)_C \times SU(2)_L \times U(1)$ gauge interactions, and similarly the grand unifying gauge groups $SU(5)$ or $SO(10)$, will be obtained by gauging a part of H. The isotropy representation of a manifold is thus the chief means by which we identify the promising models. The content of an isotropy representation is completely determined once the G, H, and an invariant complex structure[12] are chosen.

In Ref. 10 we showed that, within the class of models mentioned above, only those with $G = E_7$, E_8 can have an isotropy representation capable of accommodating three families of quarks and leptons. In the case of $G = E_7$, the grand unifying gauge group can be $SU(5)$, and there are three possible choices of H, namely $H = SU(5) \times U(1)^3$, $SU(5) \times SU(2) \times U(1)^2$, and $SU(5) \times SU(3) \times U(1)$.[13] But they suffer from Adler-Bell-Jackiw–anomaly.[14] It is not possible to accommodate both an $H \supset SO(10)$ and three families of quarks and leptons simultaneously when $G = E_7$. In the present paper we will show that when $G = E_8$ and $H = SO(10) \times U(1)^3$, $SO(10) \times SU(2) \times U(1)^2$, and $SO(10) \times SU(3) \times U(1)$, the corresponding models can accommodate the three left handed families of quarks and leptons without incurring the $ABJ-$anomaly with respect to either $SU(3)_C \times SU(2)_L \times U(1)$, $SU(5)$, or $SO(10)$–gauge group. An additional surprising prediction of the E_8 models is that there is a right–handed, fourth family of quarks and leptons. The fourth family differs from the first three also in the $U(1)^3-$, $SU(2) \times U(1)^2-$, and $SU(3) \times U(1)-$representation content for the cases where $H = SO(10) \times U(1)^3$, $SO(10) \times SU(2) \times U(1)^2$, and $SO(10) \times SU(3) \times U(1)$ respectively. It is not possible to find an invariant complex structure such that all four families are left–handed simultaneously.

The E_8-, and E_7-models are highly interesting in yet another respect which we shall mention now. The isotropy representations of the associated abstract manifolds are reducible. Generally, a homogeneous manifold G/H, which need not be Kahlerian or even complex, with a reducible isotropy representation would allow independent rescalings[15] for coframes (vielbeins) corresponding to the different irreducible components of the isotropy representation without affecting the isometry group. The rescaling act is often referred to as squashing. Thus, an ordinary (i.e., without supersymmetry) nonlinear sigma model based on such a manifold would carry as many independent squashing parameters as the number of irreducible components in the isotropy representation. The Kahler manifolds for the E_8-, and E_7-models are indeed squashed manifolds. But we will show that the Kahlerian condition on the metric tensors of the manifolds greatly reduces the degrees of independent rescalings. Furthermore, we will show that there exists a unique choice of the ratio of rescalings for which a Kahlerian manifold is Einsteinian.

Let us now outline the order of presentation in the present paper. In Section 2 we determine the invariant complex structures for the E_8/H manifolds by analyzing the root space of the Lie algebra E_8. We then find a proper basis for the algebra, and construct the commutators in terms of this basis. The isotropy representations, and so the field content of the E_8 models, are determined in this section.

In Section 3 we develop a differential form approach to Kahler manifold. We derive the general expressions for the various geometrical objects such as connection 1–form, curvature 2–form, and Ricci 2–form, which are not available elsewhere in the literature. It is here that we find the results relating to squashing that we mentioned above.

In Section 4 we propose a general method for determination of the Kahler potential to the quartic order. We apply the method to one of the E_8 models, namely that with $H = SO(10) \times SU(3) \times U(1)$.

In Section 5 we point out some general features of the four–fermion interactions inherent in supersymmetric nonlinear sigma model. For the E_8-models, the inherent four–fermion interactions can induce proton instability even in the absence of any grand unifying gauge interactions. Thus the critical mass scale Λ_σ of the models should be around or beyond 10^{15} GeV.

In Section 6 we propose two plausible mechanisms for the explicit breakdown of the global symmetry G when a supersymmetric nonlinear sigma model is coupled to supergravity.

In Section 7 we conclude the paper with some further discussions. These include the value of the mass scale Λ_σ, and the implications of the E_8−models.

Appendix I provides the commutators of E_8 algebra in terms of a basis adapted to the structure of the E_8−models.

2. The Field Content of the E_8 Models–Invariant Complex Structure and E_8 Algebra

Besides being a nonlinear realization of the global symmetry G, the complex scalar fields ϕ^i also take part in the linear representations, namely, the chiral supermultiplets Φ^i, of supersymmetry. In order to ensure that the algebra of G commutes with the supersymmetry algebra, it is necessary that G-transformations do not mix chiral supermultiplets with anti–chiral supermultiplets. Accordingly the action of G on ϕ^i which are complex coordinates on the manifold G/H, is characterized by a set of holomorphic Killing vectors on the manifold. In other words, the manifold is endowed with a G-invariant complex structure.

The criterion[12] for an invariant complex structure on Kahler manifold G/H can be stated in terms of some positivity and closure properties defined on a system of roots of the algebra G. Let $\Omega = \{\omega_1, \omega_2, ..., \omega_m, \omega_{m+1}, ..., \omega_{m+h}\}$ be a system of positive roots of the algebra G such that a subset of it, say $\Theta = \{\omega_{m+1}, \omega_{m+2}, ..., \omega_{m+h}\}$ form a system of positive roots for H. Then the subset $\Phi = \{ \omega_1, \omega_2, ..., \omega_m \}$ defines an invariant complex structure if it is a closed system of roots. A set of roots of G is said to be closed if it contains the sum of any two of its elements whenever this sum is a root of G. The system of roots of an invariant structure, Ψ, can be split into further subsets each of which can be identified with the weights of an irreducible linear representation of the group H. This provides an algorithm for determining the irreducible pieces of a reducible isotropy representation, and thus the corresponding invariant complex structure.

The root space of E_8 is an eight–dimensional Euclidean space.[16] In terms of an orthogonal basis e_i, $i = 1, ..., 8$, the roots of E_8 can be expressed as

$$\pm e_i \pm e_j \quad , \quad 1 \le i \ne j \le 8 \ ,$$

and

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8)$$

with an even number of plus signs in the bracket (...). We chose the following system of simple roots for E_8, namely $\{\alpha_i, \ i = 1, \ 2, \ ..., \ 8\}$, where

$$\begin{aligned}
&\alpha_1 = e_1 - e_2 \ , \quad \alpha_2 = e_2 - e_3 \ , \quad \alpha_3 = e_3 - e_4 \ , \\
&\alpha_4 = e_4 - e_5 \ , \quad \alpha_5 = \frac{1}{2}(-e_1 - e_2 - e_3 - e_4 + e_5 - e_6 + e_7 - e_8) \ , \\
&\alpha_6 = e_6 - e_7 \ , \quad \alpha_7 = e_7 + e_8 \ , \quad \alpha_8 = e_4 + e_5 \ .
\end{aligned}$$

The resulting system of positive roots of E_8 is

$$
\begin{aligned}
\Omega = \{ \; & e_i \pm e_j \; , \; 1 \leq i < j \leq 5 \; , \\
& e_6 \pm e_7 \; , \\
& e_7 \pm e_8 \; , \\
& e_6 \pm e_8 \; , \\
& \pm e_i + e_6 \; , \; 1 \leq i \leq 5 \; , \\
& \pm e_i + e_7 \; , \; 1 \leq i \leq 5 \; , \\
& \pm e_i - e_8 \; , \; 1 \leq i \leq 5 \; , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 - e_7 - e_8) \; , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 - e_6 + e_7 - e_8) \; , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 + e_7 + e_8) \; , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 + e_7 - e_8) \; . \}
\end{aligned}
$$

Again the total number of plus signs in each bracket (...) above should be even. For the case $G/H = E_8/SO(10) \times SU(3) \times U(1)$, one can verify that the system of positive roots of H is

$$
\begin{aligned}
\Theta = \{ \; & e_i \pm e_j \; , \; 1 \leq i < j \leq 5 \; , \\
& e_6 - e_7 \; , \\
& e_6 + e_8 \; , \\
& e_7 + e_8 \; . \}
\end{aligned}
$$

And the system of roots of invariant complex structure is

$$
\begin{aligned}
\Psi = \{\ & e_6 + e_7\ ,\ e_6 - e_8\ ,\ e_7 - e_8\ , \\
& \pm e_i + e_6\ ,\ 1 \le i \le 5\ , \\
& \pm e_i + e_7\ ,\ 1 \le i \le 5\ , \\
& \pm e_i - e_8\ ,\ 1 \le i \le 5\ , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 - e_7 - e_8)\ , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 - e_6 + e_7 - e_8)\ , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 + e_7 + e_8)\ , \\
& \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 + e_6 + e_7 - e_8)\ .\}
\end{aligned}
$$

Thus the isotropy representation can be taken to be

$$
\begin{aligned}
\Gamma = & (\underline{16},\ \underline{3},\ 1/3) + (\underline{16}^*,\ \underline{1},\ 1) \\
& + (\underline{10},\ \underline{3}^*,\ 2/3) + (\underline{1},\ \underline{3},\ 4/3)
\end{aligned}
\tag{7}
$$

Next one can check that $\Psi' = \Psi\ \cup \{e_6 + e_8,\ e_7 + e_8\}$ and $\Psi'' = \Psi' \cup \{e_6 - e_7\}$ are the systems of roots of invariant complex structure for the Kahler manifolds $E_8/SO(10) \times SU(2) \times U(1)^2$, and $E_8/SO(10) \times U(1)^3$ respectively. We have the regular embedding $U(1)^3 \subset SU(2) \times U(1)^2 \subset SU(3) \times U(1)$. The isotropy representation of the Kahler manifold $E_8/SO(10) \times SU(2) \times U(1)^2$ with the structure Ψ' is

$$
\begin{aligned}
\Gamma' = & (\underline{16},\ \underline{2},\ 1/2,\ 1/3) + (\underline{16},\ \underline{1},\ -1,\ 1/3) \\
& + (\underline{16}^*,\ \underline{1},\ 0,\ 1) + (\underline{10},\ \underline{2},\ -1/2,\ 2/3) \\
& + (\underline{10},\ \underline{1},\ 1,\ 2/3) + (\underline{1},\ \underline{2},\ 1/2,\ 4/3) \\
& + (\underline{1},\ \underline{1},\ -1,\ 4/3) + (\underline{1},\ \underline{2},\ 3/2,\ 0)
\end{aligned}
\tag{8}
$$

with respect to $SO(10) \times SU(2) \times U(1)' \times U(1)$. The isotropy representation of the Kahler manifold $E_8/SO(10) \times U(1)^3$ with the structure Ψ'' is

$$
\begin{aligned}
\Gamma'' = & (\underline{16},\ 1,\ 1/2,\ 1/3) + (\underline{16},\ -1,\ 1/2,\ 1/3) \\
& + (\underline{16},\ 0,\ -1,\ 1/3) + (\underline{16}^*,\ 0,\ 0,\ 1) \\
& + (\underline{10},\ 1,\ -1/2,\ 2/3) + (\underline{10},\ -1,\ -1/2,\ 2/3) \\
& + (\underline{10},\ 0,\ 1,\ 2/3) + (\underline{1},\ 1,\ 1/2,\ 4/3) \\
& + (\underline{1},\ -1,\ 1/2,\ 4/3) + (\underline{1},\ 0,\ -1,\ 4/3) \\
& + (\underline{1},\ 1,\ 3/2,\ 0) + (\underline{1},\ -1,\ 3/2,\ 0) \\
& + (\underline{1},\ 2,\ 0,\ 0)\ .
\end{aligned}
\tag{9}
$$

with respect to $SO(10) \times U(1)'' \times U(1)' \times U(1)$.

The basis of the E_8 algebra can always be chosen to reflect a given isotropy representation. For the isotropy representation Γ of Eq. (7), we choose the basis

$$
\begin{aligned}
\{\ & L_{AB},\ T^I_J,\ T,\ \bar{X}_I,\ \bar{Y}^I_A,\ \bar{W}_{I\dot{\alpha}},\ \bar{Z}_\alpha, \\
& X^I,\ Y_{IA},\ W^I_\alpha,\ Z_{\dot{\alpha}}\ \}
\end{aligned}
\tag{10}
$$

Here the indices $I,\ J = 1,\ 2,\ 3$ are the $SU(3)$ indices; $A,\ B = 1,\ 2,\ ...,\ 10$ are $SO(10)$ indices; $\alpha = 1,\ 2,\ ...,\ 16$ is the $SO(10)$ spinor $\underline{16}$–index; and $\dot{\alpha}$ the $\underline{16}^*$–index. $L_{A,B}$, T^I_J, and T are anti hermitian, and $\sum_I\ T^I_I = 0$. They generate the $SO(10)$, $SU(3)$ and $U(1)$. The barred generators are antihermitian conjugate of the corresponding unbarred generators. The collection $\{\ X^I,\ Y_{IA},\ W^I_\alpha,\ Z_{\dot{\alpha}}\ \}$ has the identical representation content as the Γ. The explicit expressions for the commutators of E_8 algebra in terms of the basis given by Eq. (10) are collected in Appendix I.

A basis of the E_8 algebra which is adapted to the isotropy representation Γ' of Eq. (8) can be obtained from the basis of Eq. (10) by splitting the set of $SU(3)$ generators as follows

$$
\begin{aligned}
\{\, T^I_J,\ I,\ J = 1,\ 2,\ 3\} = \{\, & S^i_j = T^i_j - \frac{1}{2}\delta^i_j \sum_1^2 T^\ell_\ell,\ i,\ j = 1,\ 2\} \\
& \cup \{\, S = \frac{1}{2}\sum_1^2 T^\ell_\ell - T^3_3\} \\
& \cup \{\, V^j = T^j_3,\ \bar{V}_j = T^3_J,\ j = 1,\ 2.\}
\end{aligned}
$$

Similarly we obtain a basis of E_8 adapted to the isotropy representation Γ'' from Eq. (10) by further splitting of the set of $SU(2)$ generators:

$$
\begin{aligned}
\{\, S^i_j,\ i,\ j = 1,\ 2\} = \ & \{\, N = T^1_1 - T^2_2\} \\
& \cup\ \{\, U = T^1_2,\ \bar{U} = T^2_1\}
\end{aligned}
$$

The isotropy representations, Γ, Γ', and Γ'' establish our claim that the E_8 models allow three left–handed, and one right–handed family of quarks and leptons. They are obviously free from the $ABJ-$anomaly with respect to the group $SO(10)$, and so are with respect to the regular subgroups $SU(5)\times U(1)\subset SO(10)$, and $SU(3)\times SU(2)\times U(1)^2\subset SO(10)$. We note that the isotropy representation Γ of $E_8/SO(10)\times SU(3)\times U(1)$ is not $ABJ-$anomaly free with respect to the $SU(3)$ factor of the isotropy group, nor with respect to the $U(1)$ factor. We also observe that both Γ' and Γ'' incur $ABJ-$anomaly with respect to at least one $U(1)$ factor of the respective isotropy groups. One cannot gauge these anomalous symmetries. The anomalous $U(1)('s)$ provides naturally a Peccei-Quinn symmetry because of the $U(1) - SO(10) - SO(10)$ triangle anomaly.

3. Geometrical Objects on Kahler Manifold

We regard Kahler manifold G/H as a collection of right cosets $\{gH\}$ parametrized by complex numbers $\{z^i\}$. The left action of $g \in G$ on a coset representative $L(z, z^*)$ is given by

$$gL(z,\ z^*) = L(z',\ z'^*)\ h \tag{11}$$

where h, an element of H, and $L(z', z^{*\prime})$ are functions of g and $L(z, z^*)$. The explicit form of $L(z, z^*)$ depends on the specific embedding of G/H in G.

A left invariant 1–form can be defined in terms of $L(z, z^*)$:

$$e(z,\ z^*) \equiv L^{-1}(z,\ z^*)\ dL(z,\ z^*) \tag{12}$$

It takes value on Lie algebra of G. We shall denote by $\{T_\alpha\}$ a basis for the algebra, adapted to an invariant complex structure. We use the following index convention: $\alpha,\ \beta,\ ...$ for generic G indices; $a,\ b,\ ...$ for H indices; $I,\ J,\ ...$ for holomorphic flat coset indices; and $i,\ j,\ ...$ for holomorphic curved coset indices. Antiholomorphic indices are obtained by putting $*$ on top of the corresponding holomorphic indices. We can write the 1–form $e(z, z^*)$ as

$$\begin{aligned} e(z,\ z^*) &= T_I e^I + T_{I^*} e^{I^*} + T_a e^a \\ &= T_I\ e^I_i\ dz^i + T_{I^*} e^{I^*}_{i^*}\ dz^{i^*} + T_a e^a_i dz^i \\ &+ T_a e^a_{i^*} dz^{i^*} \end{aligned} \tag{13}$$

where e^I_i, e^{I*}_{i*}, e^a_i, and e^a_{i*} are functions of both z and z^*. In particular e^I_i and e^{I*}_{i*} will play the role of vielbein (coframe) on the manifold G/H. It follows from Eq. (12) that the 1–form satisfies the Maurer–Cartan equation:

$$de = -e \wedge e \tag{14}$$

And from the antihermiticity of e and T_α, i.e. $e^+ = -e$, $T^+_I = -T_{I*}$, and $T^+_a = -T_a$, we have $(e^I_i)^* = e^{I*}_{i*}$ and $(e^a_i)^* = e^a_{i*}$.

The action of $g \in G$ on the 1–form is given by the transformation law

$$\begin{aligned} e(z', z'^*) &= h(g, z, z^*)\ e(z, z^*)\ h^{-1}(g, z, z^*) \\ &+ h(g, z, z^*)\ dh^{-1}(g, z, z^*) \end{aligned} \tag{15}$$

Or, equivalently,

$$e^I(z', z'^*) = D^I_J(h^{-1})\ e^J(z, z^*) \tag{16}$$

$$e^a(z', z'^*) = D^a_b(h^{-1})\ e^b(z, z^*) + (h\ d\ h^{-1})^a \tag{17}$$

and the complex conjugate of (16). Here $D^\beta_\alpha(g)$ denotes the adjoint representation of $g \in G$: $g^{-1}T_\alpha g = D^\beta_\alpha(g)T_\beta$. The matrices $D^I_J(h^{-1})$, $h \in H$, are block diagonal in our chosen basis. Each block acts on an irreducible representation of H. Therefore the transformation laws (16) and its complex conjugate allows the construction of a G–invariant hermitian metric on G/H:

$$ds^2 = g_{ij^*}\ dz^i\ dz^{*j} = c_I\ e^I\ e^{I*} \tag{18}$$

where C_I are positive definite real numbers, but otherwise could be different for different H-irreducible sectors. The metric can be brought back to the canonical form:

$$ds^2 = \tilde{e}^I \, \tilde{e}^{I^*} \tag{19}$$

by adopting the rescaled vielbein

$$\tilde{e}^I = \sqrt{c_I} \, e^I \tag{20}$$

For ordinary (i.e. without supersymmetry) nonlinear sigma model based on an abstract manifold with a reducible isotropy representation, the C_I associated with each irreducible sector is in principle an independent parameter. Thus the model is characterized by as many independent characteristic mass scales as the number of H–irreducible sectors in the isotropy representation.[15] The rescalings of vielbein, Eq. (20), is often referred to as the squashings of the manifold. In supersymmetric nonlinear sigma model the C_I's are constrained by additional conditions which will now be explained.

To the metric defined by Eq. (18) we associate a 2–form:

$$\omega = \frac{1}{2} i \, g_{jk^*} \, dz^j \wedge dz^{k^*} = \frac{1}{2} i \, C_I e^I \wedge e^{I^*} \tag{21}$$

A complex manifold is a Kahler manifold if and only if the 2–form is closed, i.e. $d\omega = 0$. By applying the Maurer–Cartan structure equation (14) and eq. (21), we find readily that the condition $d\omega = 0$ demands that

$$C_I + C_J = C_K \qquad \text{if} \qquad f_{IJ}^K \neq 0 \tag{22}$$

Here f_{IJ}^K is the structure constant: $[T_I, \, T_J] = f_{IJ}^K T_K$. If we pause now to examine the E_8 model with the isotropy group $H = SO(10) \times SU(3) \times U(1)$, the commutators of the E_8 algebra (see the Appendix) are such that the ratio among C_I is unambiguously determined. That is $C_X : C_Y : C_W : C_Z = 4 : 2 : 1 : 3$. And the model has only one independent characteristic mass scale. This property is shared by all supersymmetric nonlinear sigma model in which the center of the isotropy group H has only one $U(1)$ factor; the ratio among C_I is identical to the ratio among the eigenvalues of the $U(1)$ generator in the isotropy representation.

In the case when the center of the isotropy group H has two or more than two $U(1)$ factors, C_M can be any linear combination of $Q^a(M)$, the generators of the $U(1)$'s,

$$C_M \equiv \alpha_a \, Q^a(M) \tag{22a}$$

Here the coefficients α_a are such that C_M is positive definite for all values of M, but are otherwise arbitrary. Consequently, the corresponding model has as many independent characteristic mass scales as the number (n) of the $U(1)$ factors. (Of course the condition that $C_M > 0$ for all M puts some bounds on the mass scales.) Alternatively, one can parametrize with just one mass scale Λ_σ, corresponding to the overall squashing, plus the angles of the spherical polar coordinate system in n dimensions.

In terms of the rescaled vielbein, the connection 1–form Γ^I_J is given by

$$d\tilde{e}^I + \Gamma^I_J \wedge \tilde{e}^J = 0 \tag{23}$$

and the antihermiticity condition:

$$\left(\Gamma^I_J\right)^* = -\Gamma^J_I \tag{24}$$

By applying the Maurer–Cartan structure equation, we obtain from Eq. (23)

that

$$\Gamma^I_J = \sqrt{\frac{C_J}{C_I C_K}}\; f^I_{KJ}\; \tilde{e}^K + \sqrt{\frac{C_I}{C_J C_K}}\; f^I_{K^* J}\; \tilde{e}^{K*} + f^I_{aJ}\; \tilde{e}^a \tag{25}$$

The curvature 2–form R^I_J can now be evaluated. By definition

$$R^I_J = d\Gamma^I_J + \Gamma^I_K \wedge \Gamma^K_J \tag{26}$$

Substituting Eq. (25) in Eq. (26), we obtain

$$\begin{aligned} R^I_J = \Bigg\{ & -\frac{1}{\sqrt{C_M C_N}}\; f^I_{aJ}\; f^a_{MN^*} - \sqrt{\frac{C_J}{C_I C_M C_N}}\; f^I_{KJ}\; f^K_{MN^*} \\ & - \sqrt{\frac{C_I}{C_J C_M C_N}}\; f^I_{K^* J}\; f^{K^*}_{MN^*} - \frac{1}{C_K}\sqrt{\frac{C_I C_J}{C_M C_N}}\; f^I_{N^* K}\; f^K_{MJ} \\ & + \frac{C_K}{\sqrt{C_I C_J C_M C_N}}\; f^I_{MK}\; f^K_{N^* J} \Bigg\}\; \tilde{e}^M \wedge \tilde{e}^{N^*} \end{aligned} \tag{27}$$

Thus the nonvanishing components of the curvature tensor are of the form $R^I_{JMN^*}$ $(R^I_{JN^*M} = -R^I_{JMN^*})$. The Ricci tensor has the nonvanishing components

$$\begin{aligned} S_{MN^*} &= \sum_I R^I_{IMN^*} \\ &= \frac{-2}{C_M}\; f^I_{aI}\; f^a_{MN^*} \end{aligned} \tag{28}$$

The scalar curvature can also be evaluated to yield

$$R = \frac{-2}{C_M}\; f^I_{aI}\; f^a_{MM^*} \tag{29}$$

The nonvanishing contributions to the factor $\sum_I f^I_{aI}$ of Eqs. (28) and (29) comes only from those indices a that correspond to the generators of the center of H.

And for such indices a, the structure constant

$$f^a_{MN^*} \propto Q^a(M)\ \delta_{MN^*} \tag{30}$$

where $Q^a(M)$ denotes the ath toral $U(1)$ charge of the generator indexed by M. Therefore the Kahler manifold becomes Einsteinian, i.e., $S_{MN^*} \propto \delta_{MN^*}$, for the following value of C_M:

$$C_M \propto \sum_{a,I} Q^a(I)\ Q^a(M) \tag{31}$$

This choice of C_M obviously satisfies the condition (22). (The proportionality constant of Eq. (31) can be shown to be positive definite.)

4. Determination of Kahler Potential to the Quartic Order

The Kahler potential is of interest because it determines the Lagrangian density, Eq. (1), of the supersymmetric nonlinear sigma model. We have not yet found a practical and general method for computing the full expression for the Kahler potential. We have succeeded in inventing a general method for calculating the Kahler potential to quartic order, which will be explained in the following.

Given a connection 1–form for a Riemann manifold, the antisymmetric part of the connection is simply the torsion, which transforms as a tensor under general coordinate transformation on the manifold. The symmetric part is not a tensor, and its value at a given point may be created or annihilated by proper choice of a local coordinate system. A Kahler manifold has zero torsion; therefore, a coordinate system exists such that the connection vanishes at the origin of the coordinates. We denote by $\{\phi^i\}$ the coordinate system, and $K(\phi, \phi^*)$ the Kahler potential with respect to these coordinates, then

$$g_{ij^*}\Big|_{\phi=\phi^*=0} = \partial_i \partial_{j^*} \, K(\phi, \, \phi^*) \, \Big|_{\phi=\phi^*=0}$$

and

$$R_{ik^*j\ell^*}\Big|_{\phi=\phi^*=0} = \partial_i \partial_{k^*} \, \partial_j \partial_{\ell^*} \, K(\phi, \, \phi^*) \, \Big|_{\phi=\phi^*=0}$$

The first order, and the third order derivatives of $K(\phi, \phi^*)$ vanish at the origin. Therefore the values of the metric tensor and curvature tensor at the origin of the coordinates are sufficient to determine the Kahler potential to the quartic order. The metric tensor at the origin is simply δ_{ij*}, and the value of curvature tensor at the origin is determined by Eq. (27) in terms of the squashing ratios C_I and the structure constants $f^{\alpha}_{\beta\gamma}$.

Amongst the three E_8 models, we have calculated $K(\phi, \phi*)$ for the one with isotropy groups $H = SO(10) \times SU(3) \times U(1)$, using the method. Let $\{x^I, \, y_{IA}, \, w^I_{\alpha}, \, z_{\alpha}\}$ be complex coordinates, consistent with our method, on the Kahler manifold $E_8/SO(10) \times SU(3) \times U(1)$. Then, up to quartic order, the Kahler potential is

$$
\begin{aligned}
K(\phi,\ \phi^*) = {} & x^I\bar{x}_I + y_{IA}\bar{y}^I_A + w^I_\alpha\bar{w}_{I\dot{\alpha}} + z_{\dot{\alpha}}\bar{z}_\alpha \\
& + \lambda\Big\{\ \frac{1}{4}(x^I\bar{x}_I)^4 + \frac{1}{2}(y_{IA}\bar{y}^I_A)(x^K\bar{x}_K) \\
& - \frac{1}{2}(x^I\bar{x}_K)(y_{IA}\bar{y}^K_A) + \frac{1}{2}(x^I\bar{w}_{I\dot{\alpha}})(w^K_\alpha\bar{x}_K) \\
& + \frac{1}{2}(x^I\bar{x}_I)(z_{\dot{\alpha}}\bar{z}_\alpha) + \frac{1}{2}(y_{IA}\bar{y}^K_A)(y_{KB}\bar{y}^I_B) \\
& - \frac{1}{4}(y_{IA}y_{KA})(\bar{y}^I_B\bar{y}^K_B) - \frac{1}{3}(y_{IA}\bar{y}^K_B)(\bar{w}_K\Sigma_{AB}w^I) \\
& + (y_{IA}\bar{y}^I_B)(\bar{w}_K\Sigma_{AB}w^K) - \frac{5}{6}(y_{IA}\bar{y}^K_A)(w^I\bar{w}_K) \\
& + \frac{1}{2}(y_{IA}\bar{y}^I_A)(w^K\bar{w}_K) - \frac{2}{3}(y_{IA}\bar{y}^I_B)(z\Sigma_{AB}\bar{z}) \\
& + \frac{1}{3}(y_{IA}\bar{y}^I_A)(z\bar{z}) + (w^I\bar{w}_K)(w^K\bar{w}_I) \\
& - \frac{1}{4}(\bar{w}_I\Gamma_A\bar{w}_K)(w^I\Gamma^*_A w^K) - \frac{1}{2}(zw^I)(\bar{w}_I\bar{z}) \\
& + \frac{1}{3}(\bar{w}_I\Gamma_A z)(w^I\Gamma^*_A\bar{z}) + \frac{1}{3}(z\bar{z})^2 \\
& - \frac{1}{12}(z\Gamma_A z)(\bar{z}\Gamma^*_A\bar{z}) \\
& + \frac{1}{2\sqrt{3}}\epsilon_{IJK}x^I\bar{y}^J_A(\bar{z}\Gamma^*_A w^K) \\
& + \frac{1}{2\sqrt{3}}\epsilon^{IJK}\bar{x}_I y_{JA}(z\Gamma_A\bar{w}_K) \\
& + \frac{1}{3}x^I y_{IA}(\bar{z}\Gamma^*_A\bar{z}) + \frac{1}{3}\bar{x}_I\bar{y}^I_A(z\Gamma_A z) \\
& - \frac{1}{\sqrt{3}}\epsilon^{IJK}y_{IA}y_{JB}(\bar{w}_K\Sigma_{AB}\bar{z}) \\
& - \frac{1}{\sqrt{3}}\epsilon_{IJK}\bar{y}^I_A\bar{y}^J_B(w^K\Sigma^*_{AB}z)\Big\}
\end{aligned}
\qquad (32)
$$

where λ being related to the overall squashing is undetermined, and $\bar{x}_I \equiv (\chi^I)^*$, $\bar{y}^I_A \equiv (y_{IA})^*$, etc.

5. The Four–Fermion Interactions

A supersymmetric nonlinear sigma model has four–fermion interactions, Eq. (3). The structure of these interactions is determined by the curvature tensor of the Kahler manifold. The general formula derived in Section 3, namely Eq. (27), allows a few general conclusions to be drawn about the interactions.

For a Kahler manifold with irreducible isotropy representation, e.g. the Hermitian symmetric spaces,[17] the expression for curvature tensor, by Eq. (27), becomes

$$R^I_{JKMN^*} = \frac{-1}{C_M}\, f^I_{aJ}\, f^a_{MN^*} \tag{33}$$

We recall that the index a runs through all generators of the isotropy group H. Therefore the four–fermion interactions of the model based on such a Kahler manifold have the same group theoretical structure as the effective four–fermion interactions that would have resulted from exchange of one massive gauge boson of gauge group H.

For a Kahler manifold with reducible isotropy representation, not all components of curvature tensor can be reduced to the form given by Eq. (33).

The general expression for the curvature tensor is, from Eq. (27),

$$
\begin{aligned}
R^{I}_{JMN^*} = &-\frac{1}{\sqrt{C_M C_N}}\, f^{I}_{aJ}\, f^{a}_{MN^*} - \sqrt{\frac{C_J}{C_I C_M C_N}}\, f^{I}_{KJ}\, f^{K}_{MN^*} \\
&- \sqrt{\frac{C_I}{C_J C_M C_N}}\, f^{I}_{K^*J}\, f^{K^*}_{MN^*} - \frac{1}{C_K}\sqrt{\frac{C_I C_J}{C_M C_N}}\, f^{I}_{N^*K}\, f^{K}_{MJ} \qquad (34) \\
&+ \frac{C_K}{\sqrt{C_I C_J C_M C_N}}\, f^{I}_{MK}\, f^{K}_{N^*J}
\end{aligned}
$$

There are values of $(I,\ J,\ M,\ N)$–index for which the last four terms at the right hand side of Eq. (34) result in four–fermion interactions which cannot be induced by exchange of a gauge boson of the gauge group H. A glance at the last six terms of the Kahler potential given by Eq. (32) immediately verifies the point.

Specializing to the E_8 models, where $H = SO(10)\times SU(3)\times U(1)$, $SO(10)\times SU(2)\times U(1)^2$, and $SO(10)\times U(1)^3$, we conclude first that the critical mass scale Λ_σ may be around or beyond the usual scale for grand unification, namely $\Lambda_{GUT}\approx 10^{15}$ GeV. Otherwise, the four–fermion interactions would likely induce a rate of proton decays too abundant to be consistent with experimental result. Secondly, there are four–fermion interactions different from those induced by either $SU(5)$ or $SO(10)$ grand unifying gauge interactions. They may be a source of some phenomena not predictable by the standard renormalizable theory.

One can get a better understanding of the above mentioned property of the four–fermion interactions by invoking the concept of holonomy group. The curvature 2–form R^I_J provides a representation of the generators of holonomy group. Furthermore the components $R_{I^*JMN^*}$ obey the symmetry relation $R_{I^*JMN^*} = R_{N^*MJI^*}$. When the four–fermion interactions are pictured as current–current interactions, the currents have the property of the generators of the holonomy group. In the case of Eq. (33), we say the holonomy group is identical to the isotropy group. But in general, according to Eq. (34), they are not identical.

6. Explicit Breaking of G Symmetry by Supergravity

The global symmetry G of a supersymmetric nonlinear sigma model has to be explicitly broken in order for the model to approximate reality to a higher order. One source of such breaking will come from the gauging of a subgroup $S \subset H$. It breaks G to $S \times R \subset G$ at the tree level, where R commutes with S. Further effects resulting from gauging of supersymmetric nonlinear sigma model have been studied before.[17,18] Here we point out two plausible mechanisms which are available when the model is coupled to supergravity.

It is an intrinsic property of a Kahler manifold that there exists at least one linear combination, denoted Q, of the generators of the center of the isotropy group H, such that the sum of the Q–charges of σ–fermions is nonvanishing i.e.

$$\sum_I Q_I \neq 0 \tag{35}$$

where I runs through all σ–fermions. The $U(1)$ generated by Q is a chiral $U(1)$ because the fermions are two–component Weyl fermions. Therefore it happens necessarily that the fermion triangle with one Q–current and two energy momentum tensors has an anomaly. The Q–current, J_μ^Q, of massless σ–fermions is not conserved, but obeys

$$D_\mu J_\mu^Q = \frac{-1}{384\pi^2} \sum_I Q_I \, R\tilde{R} \tag{36}$$

where $R\widetilde{R} = 1/2\epsilon^{\mu\nu\alpha\beta} R_{\mu\nu\sigma\tau} R^{\sigma\tau}_{\alpha\beta}$, $R_{\mu\nu\sigma\tau}$ being the usual Riemann curvature tensor of curved space–time.[19]

Q is an element of the algebra of G. The violation of global conservation of Q implies the simultaneous violation of the symmetries generated by those generators, denoted by X generically, which carry nonvanishing Q–charge, because Q is required for the closure of the commutators $[X, \bar{X}]$, where $\bar{X}$ denotes the antihermitian conjugate of X. For the E_8 model with $H = SO(10) \times SU(3) \times U(1)$, Q is generator of the $U(1)$ center of H, i.e. $Q = T$ of Eq. (10), and $\sum_I Q_I = 56$. The E_8 is broken explicity to $SO(10) \times SU(3) \subset H$ by this mechanism. In general one may choose $Q_I \propto C_I$.

The second plausible mechanism is the following. When a supersymmetric nonlinear sigma model is coupled to supergravity, general Kahler invariance, Eq. (2), of the action can be preserved at the classical level if and only if a Kahler transformation, Eq. (2), is accompanied by a chiral transformation of the Fermi fields:

$$\begin{aligned} \chi^i &\rightarrow EXP[+\frac{1}{4}(F - F^*)\gamma_5]\ \chi^i \\ \psi_\mu &\rightarrow EXP[-\frac{1}{4}(F - F^*)\gamma_5]\psi_\mu \end{aligned} \tag{37}$$

where ψ_μ is the gravitino (Rarita–Schwinger field).[20] The chiral transformation, Eq. (37), can be traced back to a R–symmetry. Each χ^i carries $+1/4$ units of R–charge while ψ_μ carries $-1/4$. Modulo the uncertainty caused by the field–dependent $(F - F^*)$, we expect a violation of the R-symmetry by the anomaly of the fermion (χ^i, and ψ_μ) triangle with one R–current and two energy-momentum tensors. This anomaly thus breaks the Kahler invariance. The understanding of this mechanism is still in a preliminary stage, further progress is needed.

7. Discussions

Contrasting the structure of supersymmetric nonlinear sigma model studied here, particularly Sections 2 and 3, with that of ordinary nonlinear sigma model we cannot help but be impressed by the power of supersymmetry. Besides pairing up the Fermi fields with Bose fields, supersymmetry demands that the manifold, on which the Bose fields take value, should carry an invariant complex structure, which in turn leads to nontrivial handedness assignment of the Fermi fields. Furthermore, the number of independent squashing parameters is greatly reduced in the supersymmetric models. Consequently, we anticipate a crucial role to be played by mechanism responsible for breakdown of supersymmetry, especially in the matter of confronting the E_8 models with reality. Of course the E_8 symmetry, and parts of the isotropy groups $H = SO(10) \times SU(3) \times U(1)$, $SO(10) \times SU(2) \times U(1)^2$, and $SO(10) \times U(1)^3$ need to be broken too. We pointed out two plausible symmetry breaking mechanisms, which involve gravity, in Section 6. They deserve further study.

In Section 5, through an analysis of the four–fermion interactions inherent in the E_8 models we concluded that the characteristic mass scale Λ_σ should be around or beyond the grand unification mass scale $\Lambda_{GUT} \approx 10^{15}$ GeV. Indeed there is an additional argument which suggests that $\Lambda_\sigma > \Lambda_{GUT}$. This argument originates from the fact that, in the E_8 models, the $SU(3)_C \times SU(2)_L \times U(1)$ gauge group as well as the grand unifying gauge group $SU(5)$ (or $SO(10)$) is embedded in the isotropy group of each E_8 model. Spontaneous breakdown of the grand unifying gauge group to $SU(3)_C \times SU(2)_L \times U(1)$ at Λ_{GUT} therefore necessarily means simultaneously spontaneous breakdown of the $SU(5)$ (or $SO(10)$) subgroup of isotropy group H. Thus, the structure E_8/H loses its meaning if

$\Lambda_\sigma \lesssim \Lambda_{GUT}$. This phenomenological lower bound on Λ_σ suggests naturally that Λ_σ may in fact be related to the Planck mass M_P.

Now let us assume that one of the three E_8 models is the correct one. What conclusion can one draw from it about the dynamical degrees of freedom proper to the matter in the preonic phase? First of all, the model implies a supersymmetric preonic phase. Secondly, since E_8 is not a classical group, the new degrees of freedom cannot be the commonly speculated point–like preons. (The global symmetry of ordinary preon model is necessarily a classical group, or a product of classical groups.) Thirdly, taking the suggestion that Λ_σ be related to M_p, we expect the new degrees of freedom to live in a new phase of space–time. In the present state of the art, the superstrings in higher dimensional space–time seem to be the only plausible candidate for the new dynamical degrees of freedom.

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APPENDIX I

The E_8 algebra has 248 generators. In terms of the basis $\{\ L_{AB},\ T^I_J,\ T,\ \bar{X}_I,\ \bar{Y}^I_A,\ \bar{W}_{I\dot{\alpha}},\ \bar{Z}_\alpha,\ X^I,\ Y_{IA},\ W^I_\alpha,\ Z_{\dot{\alpha}}\ \}$ defined in Section 2, we obtain the following non-vanishing commutators for the algebra.

$$[L_{AB},\ L_{CD}] = \delta_{BC}L_{AD} + \delta_{AD}L_{BC} - \delta_{AC}L_{BD} - \delta_{BD}L_{AC}$$

$$[T^I_J,\ T^K_L] = i\delta^K_J T^I_J - i\delta^I_L T^K_J$$

$$[L_{AB},\ Y_{IC}] = -\delta^{AB}_{CD} Y_{ID}$$

$$[L_{AB},\ W^I_\alpha] = -(\Sigma_{AB})_{\alpha\dot{\beta}}\ W^I_\beta$$

$$[L_{AB},\ Z_{\dot{\alpha}}] = -(\Sigma^*_{AB})_{\dot{\alpha}\beta}\ Z_{\dot{\beta}}$$

$$[T^I_J,\ X^K] = i\delta^K_J X^I - \frac{1}{3}i\delta^I_J X^K$$

$$[T^I_J,\ Y_{KA}] = -i\delta^I_K Y_{JA} + \frac{1}{3}i\delta^I_J Y_{KA}$$

$$[T^I_J,\ W^K_\alpha] = i\delta^K_J W^I_\alpha - \frac{1}{3}i\delta^I_J W^K_\alpha$$

$$[T,\ X^I] = \frac{4}{3}i\ X^I$$

$$[T,\ Y_{IA}] = \frac{2}{3}i\ Y_{IA}$$

$$[T,\ W^I_\alpha] = \frac{1}{3}i\ W^I_\alpha$$

$$[T,\ Z_\alpha] = i\ Z_\alpha$$

$$[X^I,\ \bar{X}_J] = iT^I_J + i\delta^I_J T$$

$$[\bar{X}_I,\ Y_{JA}] = \epsilon_{IJK}\bar{Y}^K_A$$

$$[\bar{X}_I,\ W^J_\alpha] = \delta^J_I \bar{Z}_\alpha$$

$$[\bar{X}_I,\ Z_{\dot\alpha}] = -\bar{W}_{I\dot\alpha}$$

$$[\bar{Y}^I_A,\ \bar{Y}^J_B] = \delta_{AB}\epsilon^{IJK}\bar{X}_K$$

$$[\bar{Y}^I_A,\ \bar{W}_{J\dot\alpha}] = -\frac{1}{\sqrt{2}}\ \delta^I_J(\Gamma^*_A)_{\dot\alpha\dot\beta}\bar{Z}_\beta$$

$$[\bar{Y}^I_A,\ X^J] = \epsilon^{IJK}Y_{KA}$$

$$[\bar{Y}^I_A,\ W^J_\alpha] = -\frac{1}{\sqrt{2}}\ \epsilon^{IJK}(\Gamma_A)_{\alpha\beta}\bar{W}_{K\dot\beta}$$

$$[\bar{Y}^I_A,\ Z_{\dot\alpha}] = \frac{1}{\sqrt{2}}\ (\Gamma^*_A)_{\dot\alpha\dot\beta}W^I_\beta$$

$$[\bar{Y}^I_A,\ Y_{JB}] = -\delta^I_J L_{AB} + i\delta_{AB}T^I_J - \frac{1}{2}i\delta_{AB}\delta^I_J T$$

$$[\bar{W}_{I\dot\alpha},\ \bar{W}_{J\dot\beta}] = -\frac{1}{\sqrt{2}}\epsilon_{IJK}(\Gamma^*_A)_{\dot\alpha\dot\beta}\bar{Y}^K_A$$

$$[\bar{W}_{I\dot\alpha},\ \bar{Z}_\beta] = \delta_{\dot\alpha\beta}\bar{X}_I$$

$$[\bar{W}_{I\dot\alpha},\ X^J] = -\delta^I_J\delta_{\dot\alpha\beta}Z_{\dot\beta}$$

$$[\bar{W}_{I\dot\alpha},\ Y_{JA}] = -\frac{1}{\sqrt{2}}\epsilon_{IJK}(\Gamma^*_A)_{\dot\alpha\dot\beta}W^K_\beta$$

$$[\bar{W}_{I\dot\alpha},\ W^J_\beta] = -i\delta_{\dot\alpha\beta}T^J_I - \frac{1}{4}i\delta^J_I\delta_{\dot\alpha\beta}T - \frac{1}{2}\delta^J_I(\Sigma^*_{AB})_{\dot\alpha\beta}L_{AB}$$

$$[\bar{W}_{I\dot\alpha},\ Z_{\dot\beta}] = -\frac{1}{\sqrt{2}}(\Gamma^*_A)_{\dot\alpha\dot\beta}Y_{IA}$$

$$[\bar{Z}_\alpha,\ X^I] = \delta_{\alpha\dot{\beta}} W^I_\beta$$

$$[\bar{Z}_\alpha,\ Y_{IA}] = -\frac{1}{\sqrt{2}}(\Gamma_A)_{\alpha\beta}\bar{W}_{I\dot{\beta}}$$

$$[\bar{Z}_\alpha,\ W^I_\beta] = \frac{1}{\sqrt{2}}(\Gamma_A)_{\alpha\beta}\bar{Y}^I_A$$

$$[\bar{Z}_\alpha,\ Z_{\dot{\beta}}] = -\frac{3}{4} i\delta_{\alpha\dot{\beta}} T - \frac{1}{2}(\Sigma_{AB})_{\alpha\dot{\beta}} L_{AB}$$

$$[Y_{IA},\ Y_{JB}] = \delta_{AB}\epsilon_{IJK} X^K$$

$$[Y_{IA},\ W^J_\alpha] = -\frac{1}{\sqrt{2}}\delta^I_J(\Gamma_A)_{\alpha\beta} Z_{\dot{\beta}}$$

$$[W^I_\alpha,\ W^J_\beta] = -\frac{1}{\sqrt{2}}\epsilon^{IJL}(\Gamma_A)_{\alpha\beta} Y_{KA}$$

$$[W^I_\alpha,\ Z_{\dot{\beta}}] = \delta_{\alpha\dot{\beta}} X^I$$

and their hermitian conjugates. The objects Γ_A, $A = 1,\ 2, \ldots,\ 10$, are 16×16–dimensional gamma matrices for $SO(10)$. The explicit form for Γ_A is

$$\Gamma_A = \begin{cases} \begin{pmatrix} 0 & -\delta^A_\ell & 0 \\ -\delta^A_i & 0 & -\delta^{Ai}_{mn} \\ 0 & -\delta^{A\ell}_{jk} & \epsilon_{Ajkmn} \end{pmatrix}, & \text{for } A = 1, 2, \ldots, 5; \\ \\ i\begin{pmatrix} 0 & -\delta^{A-5}_\ell & 0 \\ -\delta^{A-5}_i & 0 & -\delta^{(A-5)i}_{mn} \\ 0 & -\delta^{(A-5)\ell}_{jk} & \epsilon_{(A-5)jkmn} \end{pmatrix}, & \text{for } A = 6, 7, \ldots, 10. \end{cases}$$

where $(0, i, [jk])$ is the row index, and $(0, \ell, [mn])$ is the column index, in accordance with the splitting $\underline{16} = \underline{1} + \underline{5} + \underline{10}$ with respect to $SO(10) \supset SU(5)$

branching. One can introduce 32×32 – dimensional gamma matrices

$$\gamma_A = \begin{pmatrix} 0 & \Gamma_A \\ \Gamma_A^\dagger & 0 \end{pmatrix} \quad , A = 1, 2, \ldots, \; 10 \quad ,$$

which obey the Clifford Algebra $\{ \gamma_A, \; \gamma_B \} \; = 2\delta_{AB}$. The objects Σ_{AB} are defined as

$$\Sigma_{AB} = \frac{1}{4}(\Gamma_A \Gamma_B^\dagger - \Gamma_B \Gamma_A^\dagger)$$