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HYPERMULTIPLET COUPLINGS IN $N = 2$ SUPERGRAVITY*

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ABSTRACT

An arbitrary number of massless spin $(0, \frac{1}{2})$ multiplets are coupled to $N = 2$ supergravity, and the scalar fields are found to lie on a negatively curved quaternionic manifold.

In order to build realistic theories, it is useful to know what locally supersymmetric Lagrangians are possible. There are many results on this question for $N = 1$. For $N = 2$, much less is known. The general $N = 2$ matter couplings are only now being derived.[1–3] In this talk I will report on some recent work done with Edward Witten. I will show how to couple an arbitrary number of massless spin $(0, \frac{1}{2})$ hypermultiplets to $N = 2$ supergravity.[2] I will give what we believe is the most general form of the kinetic Lagrangian. I will not, however, attempt to include mass terms or potentials. Nor will I discuss the spin $(0, \frac{1}{2}, 1)$ gauge field multiplet.

Matter couplings in supergravity theories give rise to complicated nonlinear interactions. These interactions have a natural interpretation in the language of nonlinear sigma models, where the scalar fields ϕ^i are viewed as the coordinates of a Riemannian manifold M. Different matter couplings correspond to different manifolds, and constraints on the matter couplings arise as restrictions on the possible manifolds.

As shown in Table 1, $N = 1$ globally supersymmetric models exist for all Kahler manifolds M.[4] Only a subclass of these models may be coupled to supergravity, the so-called Kahler manifolds of restricted type, or Hodge manifolds. This fact is related to the quantization of Newton's constant in certain $N = 1$ theories.[5]

Table 1: Sigma Model Manifolds In Four Spacetime Dimensions.

	global supersymmetry	local supersymmetry
$N = 1$	Kahler	Hodge
$N = 2$	hyperkahler	quaternionic

In $N = 2$ global supersymmetry, sigma models exist for all hyperkahler manifolds M.[6] The purpose of this talk is to show that $N = 2$ supergravity restricts M to be quaternionic.[2] Hyperkahler and quaternionic manifolds are related, but they are, in fact, different objects. This tells us immediately that the matter couplings allowed in $N = 2$ supersymmetry are forbidden in $N = 2$ supergravity, and vice versa. It also tells us that the local $N = 2$ matter/supergravity Lagrangian cannot be trivially reduced to $N = 1$.

To understand the $N = 2$ matter couplings, we shall first consider rigid supersymmetry and hyperkahler manifolds. A hyperkahler manifold is a $4n$-dimensional real Riemannian manifold whose holonomy group G is contained in $\mathrm{Sp}(n)$. The holonomy group of a manifold is the group of transformations generated by parallel transporting all vectors around all possible closed curves in the manifold. In general, the holonomy group of a $4n$-dimensional Riemannian manifold is contained in $\mathrm{O}(4n)$. A hyperkahler manifold is defined to have $G \subseteq \mathrm{Sp}(n) \subseteq \mathrm{O}(4n)$.

Since hyperkahler manifolds are $4n$-dimensional, we shall focus our attention on theories with $4n$ real scalar fields ϕ^i, $2n$ Majorana spinors χ^Z, and 2 supersymmetry parameters ϵ^A. The supersymmetry transformation of ϕ^i,

$$\delta \phi^i = \gamma^i_{AZ} \left(\bar{\epsilon}_R{}^A \chi_L{}^Z + \bar{\epsilon}_L{}^A \chi_R{}^Z \right) , \qquad (1)$$

shows that there must exist nonsingular objects $\gamma^i_{AZ}(\phi)$ that split the tangent space in two, $T = H \otimes P$. From (1) we see that H and P are 2 and $2n$-dimensional bundles, respectively. The splitting $T = H \otimes P$ gives a strong restriction on the allowed manifolds M. Supersymmetry implies an even stronger condition: The γ^i_{AZ} must also be covariantly constant.

The objects γ^i_{AZ} are not as mysterious as they might seem. Let us specialize for the moment to the case $n = 1$. In four dimensions, the γ^i_{AZ} are just the Dirac γ-matrices. The tangent space is the product of two spinor bundles, and the γ-matrices are indeed covariantly constant.

Having split T into $H \otimes P$, we must now invoke the constraint that T is real. This implies that H and P must both be real or pseudoreal. As far as we know, H and P real does not give a sigma model. Therefore, we take H and P to be pseudoreal. This means that A is an $\mathrm{Sp}(1)$ index, and Z is an $\mathrm{Sp}(n)$ index.

In global supersymmetry, the ϵ^A are constants, so the bundle H is trivial (flat). This implies that the holonomy group $G \subseteq \mathrm{Sp}(n)$, so M is hyperkahler. In local supersymmetry, we shall see that H cannot be trivial, so $G = \mathrm{Sp}(1) \times K$, where $K \subseteq \mathrm{Sp}(n)$. Such a manifold is by definition quaternionic.

To actually verify these assertions, we must do some work. The fact that $[\nabla_i, \nabla_j]\gamma^k_{AZ} = 0$ implies that the Riemann curvature is given in terms of the $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$ curvatures,

$$R_{ijkl}\, \gamma^l_{AY}\, \gamma^k_{BZ} = \epsilon_{AB}\, R_{ijYZ} + \epsilon_{YZ}\, R_{ijAB} , \qquad (2)$$

Here ϵ_{YZ} and ϵ_{AB} are the covariantly constant $\mathrm{Sp}(n)$and $\mathrm{Sp}(1)$ antisymmetric tensors. I shall first consider the case of M

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hyperkahler, so $R_{ijAB} = 0$. Using relations similar to the Dirac algebra,

$$\gamma^i_{AY}\,\gamma^j_{BZ}\,g_{ij} = \epsilon_{AB}\epsilon_{YZ}$$

$$\gamma^i_{AZ}\,\gamma^{jBZ} + \gamma^j_{AZ}\,\gamma^{iBZ} = g^{ij}\,\delta_A{}^B \qquad (3)$$

$$\gamma^i_{AY}\,\gamma^{jAZ} + \gamma^j_{AY}\,\gamma^{iAZ} = n^{-1}\,g^{ij}\,\delta_Y{}^Z \ ,$$

we find

$$R_{ijk\ell} = \gamma_k{}^{AZ}\,\gamma_{\ell A}{}^Y\,R_{ijYZ} \ . \qquad (4)$$

This simply expresses the fact that the holonomy group a hyperkahler manifold is contained in Sp(n). The cyclic identity on the curvature implies

$$R_{ijYZ} = \gamma_i^{AW}\,\gamma_{jA}{}^X\,\Omega_{YZXW} \ , \qquad (5)$$

where Ω_{YZXW} is a totally symmetric Sp(n) tensor, called the hyperkahler curvature.

The relations (2)-(5) are all that are needed to show that

$$\mathcal{L} = -\,g_{ij}\,\partial_\mu\phi^i\,\partial^\mu\phi^j - \frac{1}{2}\bar{\chi}_Z\,\gamma^\mu\,D_\mu\,\chi^Z + \frac{1}{16}\,\Omega_{XYZW}\,(\bar{\chi}_L{}^X\,\gamma_\mu\,\chi_L{}^Y)(\bar{\chi}_L{}^Z\gamma^\mu\,\chi_L{}^W) \qquad (6)$$

is invariant under the following $N = 2$ supersymmetry transformations:

$$\delta\phi^i = \gamma^i_{AZ}\,(\bar{\varepsilon}_R{}^A\,\chi_L{}^Z + \bar{\varepsilon}_L{}^A\,\chi_R{}^Z)$$
$$\delta\chi_L{}^Z = 2\,\partial_\mu\phi^i\,\gamma_i^{AZ}\,\gamma^\mu\epsilon_{RA} - \Gamma_i{}^Z{}_Y\,\delta\phi^i\chi_L{}^Y \ . \qquad (7)$$

To render the Lagrangian (6) invariant under $N = 2$ local supersymmetry, we first let $\epsilon^A \to \epsilon^A(x)$. We then include the spin $(1, \frac{3}{2}, 2)$ supergravity multiplet. We ensure invariance by the Noether procedure: we change the Lagrangian and transformation laws order by order in $\kappa^2 = 8\pi\, G_N$, so that all variations vanish. To order κ^0, we find $\nabla_i\,\gamma^j_{AZ} = 0$, as before. We also find

$$R_{ijAB} = \kappa^2(\gamma_{iAZ}\,\gamma_{jB}{}^Z - \gamma_{jAZ}\,\gamma_{iB}{}^Z) \neq 0 \ . \qquad (8)$$

The cyclic identity implies

$$R_{ijXY} = \kappa^2(\gamma_{iAX}\,\gamma_j{}^A{}_Y - \gamma_{jAX}\,\gamma_i{}^A{}_Y) + \gamma_i{}^{AW}\,\gamma_{jA}{}^Z\,\Omega_{XYZW} \ . \qquad (9)$$

Since both $R_{ijAB} \neq 0$ and $R_{ijXY} \neq 0$, the holonomy group $G \subseteq$ Sp(1)×Sp(n), and M is quaternionic. Note that the curvatures depend on Newton's constant. As $\kappa \to 0$, the Sp(1) curvature vanishes, and M changes from quaternionic to hyperkahler.

Contracting the Riemann curvature $R_{ijk\ell}$, we find

$$R = -8\kappa^2(n^2 + 2n) \ . \qquad (10)$$

For a given dimension of M, precisely *one* value of R is allowed, and this value is negative. Equation (10) is the $N = 2$ analogue of the $N = 1$ quantization condition.

All this information is contained in the order κ^0 terms of the Noether coupling. The higher order terms do not lead to any new restrictions. The full Lagrangian and transformation laws are given in Ref. 2.

The Noether procedure tells us that in $N = 2$ supergravity, the scalar fields in $(0, \frac{1}{2})$ multiplets must lie on negatively curved quaternionic manifolds. Relatively few such manifolds are known, but particularly intriguing cases include

$$\frac{\mathrm{Sp}(n,1)}{\mathrm{Sp}(n)\times\mathrm{Sp}(1)} \ , \qquad \frac{\mathrm{SU}(n,2)}{\mathrm{SU}(n)\times\mathrm{SU}(2)\times\mathrm{U}(1)} \qquad (11)$$

and

$$\frac{\mathrm{SO}(n,4)}{\mathrm{SO}(n)\times\mathrm{SO}(4)} \ ,$$

for $n > 1$.

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