SLAC – PUB – 3391
July 1984
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ANALYTIC STUDY OF θ VACUA ON THE LATTICE†

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ABSTRACT

A new definition of the topological charge density for four dimensional lattice gauge theory is given. Using a systematic expansion we find a cusp in the vacuum energy at $\theta = \pi$ signaling the spontaneous breaking of CP there. Unlike its two dimensional analogue (QED_2), QCD confines at $\theta = \pi$. As a by-product an expression for the topological mass term for $2 + 1$ dimensional lattice gauge theory is obtained.

Submitted to *Physics Letters*

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† This work was supported by the Department of Energy under Contract Number DE-AC03-76SF00515 and under grant No. DOE-AC02-76ER02220.

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We present here a variant of Luscher's[1] definition of the topological charge density for four dimensional lattice gauge theories.[2,3] The definition, although quite complicated, enables analytic study of θ vacua[4] in a systematic expansion. Following Ref. [5]we impose the following requirements:

1. q_n has the correct continuum limit.

2. $\sum_n q_n =$ integer in a finite volume with periodic boundary conditions.

The second requirement guarantees the absence of a "perturbative tail" in various topological quantities.

We start by considering the higher dimensional generalization of QED_2, namely the theory of a gauge field of the fourth kind in four dimensions. The seemingly unrelated problem is based on the abelian gauge potential $K_{\mu\nu\rho}$ which transforms as $K_{\mu\nu\rho} \rightarrow K_{\mu\nu\rho} + \partial_{[\mu}\Lambda_{\nu\rho]}$ ($\Lambda_{\nu\rho}$ - the gauge function). The gauge invariant electric field $F = \mathcal{E}_{\mu\nu\rho\sigma}\partial_\mu K_{\nu\rho\sigma}$ is a Lorentz singlet. Like QED_2 this is a trivial field theory and it has a topological charge

$$Q = \int d^4x\, q(x) = \frac{1}{2\pi}\int d^4x\, F(x)\ . \tag{1}$$

In finite volume with periodic boundary conditions Q is quantized if the $U(1)$ gauge symmetry is compact. A θ-term is possible and it has the interpretation of a background electric field.[6]

It is easy to construct a lattice version of this theory.[7] The gauge fields are elements of $U(1)$ $\phi_{n,\mu} = e^{ik_{n,\mu}}$ defined on the cubes (labelled by a site index n and the direction μ which is not in the cube). The gauge functions are defined on the plaquettes and the simplest gauge invariant object is the oriented product of $\phi_{n,\mu}$ around a hypercube

$$\phi_n = e^{i\sum_\mu (-1)^\mu (k_{n,\mu} - k_{n+\rho,\mu})}\ . \tag{2}$$

In analogy to QED_2[8,9] the topological charge density is

$$q_n = \frac{1}{2\pi i}\,\ell n\,\phi_n \qquad -\pi \leq -i\,\ell n\,\phi_n < \pi\ . \tag{3}$$

It has the correct continuum limit and $\sum_n q_n$ = integer when periodic boundary conditions are imposed.

The Euclidean lattice theory defined by

$$e^{-VF(\theta)} = \int \prod_{n,\mu} \frac{dk_{n,\mu}}{2\pi} \, e^{i\theta \sum_n q_n} \tag{4}$$

(V – the number of sites in the lattice) is easily analyzed. Following Ref. [9] we find with periodic boundary conditions

$$F(\theta) = \begin{cases} -\ell n\left(\frac{2}{\theta} \sin \frac{\theta}{2}\right) & |\theta| \leq \pi \\ F(\theta + 2\pi\, k) & \text{otherwise} . \end{cases} \tag{5}$$

The cusp at $\theta = \pi$ is a result of the periodicity in θ. The vacuum is two-fold degenerate at this point and CP is spontaneously broken. The analogue of the Wilson loop for this theory is the "Wilson bag"[10] — the oriented product of $\phi_{n,\mu}$ around a four volume. Two "volume tensions" are obtained (depending on the orientation of the bag)

$$\sigma_{k1} = \ell n \, \frac{2\pi - \theta}{\theta} , \qquad \sigma_{k2} = \ell n \, \frac{2\pi + \theta}{\theta} . \tag{6}$$

Note that $\sigma_{k1}(\pi) = 0$ and the theory does not confine at that point. The physical interpretation of all these phenomena is identical to that in QED_2.[6,9]

The relation between this trivial model and QCD with θ[4] is made clear by writing the topological charge density as

$$\begin{aligned} q(x) &= \frac{1}{32\pi^2} \, \mathcal{E}_{\mu\nu\rho\sigma} \operatorname{Tr} F_{\mu\nu} F_{\rho\sigma} = \frac{1}{2\pi} F(A) = \frac{1}{2\pi} \, \mathcal{E}_{\mu\nu\rho\sigma} \partial_\mu K_{\nu\rho\sigma}(A) \\ K_{\nu\rho\sigma}(A) &= \frac{1}{4\pi} \operatorname{Tr} A_\nu \Big(\partial_\rho A_\sigma + \frac{2}{3} A_\rho A_\sigma \Big) \end{aligned} \tag{7}$$

where A_μ and $F_{\mu\nu}$ are matrices. When a non-abelian gauge transformation $\Lambda(x)$ is performed on A_μ, $K_{\nu\rho\sigma}$ transforms as a gauge field of the fourth kind.[10,11,12]

Our definition of the topological charge density is based on this feature. We assign a $U(1)$ element $\phi_{n,\mu}(U) = e^{ik_{n,\mu}(U)}$ which depends on the gauge fields U to every cube and, as in the previous example, we define

$$q_n(U) = \frac{1}{2\pi i}\, \ell n\, \phi_n(U) \qquad -\pi \leq -i\, \ell n\, \phi_n(U) < \pi\,. \tag{8}$$

$\phi_n(U)$ is the oriented product of $\phi_{n,\mu}(U)$ around the hypercube labelled by n. Regardless of the particular form of $k_{n,\mu}(U)$, $\sum_n q_n$ = integer. It is not easy, however, to find a function $k_{n,\mu}(U)$ which has the correct continuum limit and transforms under a gauge transformation such that $q_n(U)$ of Eq. (8) is gauge invariant. Such a construction will be presented later.

We first discuss some of the physical consequences which are independent of the detailed construction of $k_{n,\mu}(U)$. We introduce an auxiliary field $\widetilde{\phi}_{n,\mu} = e^{i\widetilde{k}_{n,\mu}}$ in terms of which we write the topological charge density $\widetilde{q}_n = \frac{1}{2\pi i}\, \ell n\, \widetilde{\phi}_n$ $(-\pi \leq -i\, \ell n\, \widetilde{\phi}_n < \pi)$. $\widetilde{\phi}_n$ is the oriented product of $\widetilde{\phi}_{n,\mu}$ around the hypercube. Consider the lattice model

$$\begin{aligned} e^{-VF(\beta,\alpha,\theta)} = \int \prod_{n,\mu} \frac{d\widetilde{k}_{n,\mu}}{2\pi}\, dU_{n,\mu}\ \exp\ \Big\{\beta \sum_p \mathrm{Re\ Tr}\, U_p \\ + \alpha \sum_{n,\mu} \cos\Big[\widetilde{k}_{n,\mu} - k_{n,\mu}(U)\Big] + i\theta \sum_n \widetilde{q}_n\Big\} \end{aligned} \tag{9}$$

(U_p – the oriented product of the link variables around the plaquette p). For $\alpha \to \infty$ the auxiliary field $\widetilde{\phi}_{n,\mu}$ "freezes" on $\phi_{n,\mu}(U)$. In this limit the theory becomes an ordinary non-abelian gauge theory with a θ-term based on the topological charge of Eq. (8). On the other hand, for small α the θ dependence can be easily analyzed. There are two almost decoupled sectors: a no-abelian gauge theory (sector I) and an abelian gauge theory of the fourth kind (sector II). With periodic boundary conditions

$$F(\beta,\alpha,\theta) = \begin{cases} F(\beta,0,0) - \ell n\left(\frac{2}{\theta}\sin\frac{\theta}{2}\right) + O(\alpha^2) & |\theta| \leq \pi \\ F(\beta,\alpha,\theta + 2\pi k) & \text{otherwise.} \end{cases} \tag{10}$$

For zero α the two sectors are completely decoupled. The observables of sector

I are independent of θ and sector II is identical to the previously considered example.

When α is non-zero but small, it can be treated perturbatively. The expansion in α is a strong-coupling like expansion and hence it has a finite radius of convergence. The θ dependence of every diagram in the expansion can be found exactly (for a similar computation in two dimensions, see ref. 9). The β dependence of every diagram is much more difficult to find. It is given by expectation values of operators which depend on $k_{n,\mu}(U)$ in the non-abelian theory at $\theta = 0$. This calculation depends, of course, on the precise form of $k_{n,\mu}(U)$ and is practically impossible unless $k_{n,\mu}(U)$ is a simple function, and even then Monte-Carlo techniques are needed when β is large.

The cusp in $F(\beta, \alpha, \theta)$ at $\theta = \pi$ which represents a first order phase transition cannot be destroyed by small $O(\alpha^n)$ corrections. Hence, at least for small α CP is spontaneously broken at $\theta = \pi$. For zero α, the string tension of sector I is independent of θ. When $\alpha \neq 0$ and small, it has a weak θ dependence. By examining the possible diagrams taking into account the gauge invariance of sector II, it becomes clear that the Wilson loop cannot be screened and the string tension remains non-zero for all values of θ including $\theta = \pi$. The Wilson bag, on the other hand,is screened by the gluons; contributions of order α^L in the expansion (L—the number of cubes around the bag) lead to a "perimeter" law with a coefficient $f(\beta)\ell n \frac{1}{\alpha} + O(\alpha^2)$.

It is instructive to make an analogy between our results and the two dimensional CP^{N-1} model at large N[13] or in strong coupling.[9] The dummy gauge field of the CP^{N-1} model is analogous to the $\widetilde{k}_{n,\mu}$ field.[10] The Wilson loop in two dimensions is similar to our Wilson bag.[10] The inverse correlation length in the CP^{N-1} case is the mass of the z^*z bound state and in the four dimensional case it is related to the string tension. This quantity does not vanish in the CP^{N-1} model at $\theta = \pi$ neither for large N[13] nor for strong coupling,[9] and it does not vanish in our case either.

We cannot prove that the structure observed for small α persists all the way to the interesting point $\alpha = \infty$. We can show, however, that at least for $\theta \simeq 0$, $F(\beta, \alpha, \theta)$ is smooth

$$F(\beta, \alpha, \theta) = F(\beta, 0, 0) - 4\ell n\, I_0(\alpha) + O(\theta^2)\ . \tag{11}$$

So far we have not needed the exact form of $k_{n,\mu}(U)$, we only had to know that such a construction exists. We now present one possible definition of $k_{n,\mu}(U)$. It leads to an expression for $q_n(U)$ which is very similar but not identical to that given by Luscher.[1] For simplicity we consider an SU(2) gauge theory. The generalization to other groups is straightforward. Following Ref. [1] we first introduce a standard labelling of the corners of the cube (Fig. 1) and of the corners of each of the six plaquettes ($i = 1, \ldots, 6$) around it (Fig. 2). The orientation is picked such that $\alpha < \beta < \gamma$ and $\delta < \epsilon$. The coordinates x_λ ($\lambda = \alpha, \beta, \gamma, \delta, \epsilon$) vary between zero and one and label the interior of the cube.

For every plaquette ($i = 1, \ldots, 6$) we assign a group element $P^{(i)}(n)$ at the corners ($n = a, b, c, d$) which is the gauge transformation to a complete axial gauge in the plaquette, i.e. the three link elements on the thick lines in Fig. 2 are transformed to $U = 1$ by $P^{(i)}(n)$ and $P^i(a) = 1$. We also assign a group element $S(n)$ $n = 0, \ldots, 7$ to every corner of the cube. $S(n)$ transforms the cube to a complete axial gauge; i.e. the 7 thick links in Fig. 1 are transformed to $U = 1$ by $S(n)$ and $S(0) = 1$.

If a gauge transformation $\Omega(n)$ is performed on U, $S(n) \rightarrow \Omega(0)\, S(n)\, \Omega^+(n)$ and $P^{(i)}(n) \rightarrow \Omega(a)\, P^{(i)}(n)\, \Omega^+(n)$.

We now continuously interpolate $S(n)$ and $P^{(i)}(n)$ on the plaquettes such that for every $\Omega(n)$ (except possibly a set of zero measure) at the corners, there exists a continuous interpolation $\Omega(x)$ on the plaquettes such that

1. $S(x) \rightarrow S'(x) = \Omega(0)\ S(x)\ \Omega^+(x)$
2. $P^{(i)}(x) \rightarrow P^{(i)\prime}(x) = \Omega(a)\ P^{(i)}(x)\ \Omega^+(x)$

3. $\Omega(x)$ on a plaquette depends only on Ω at its corners and the gauge fields around it.

Such an interpolation exists for all gauge configurations except for a set of zero measures:

$$P^{(i)}(x_\delta, x_\epsilon) = U_{ac}{}^{x_\epsilon}\left[U_{ca}{}^{x_\epsilon}(U_{ac}U_{cd}U_{db}U_{ba})^{x_\epsilon}U_{ab}U_{bd}{}^{x_\epsilon}\right]^{x_\delta} \tag{12a}$$

$$S(x) = \begin{cases} P^{(i)}(x) & \text{for } i = 1,2,3 \\ R^{(i)}(x)P^{(i)}(x) & \text{for } i = 4,5,6 \end{cases} \tag{12b}$$

$$\begin{aligned} R^{(4)}(x_\beta, x_\gamma) = &\Big[(U_{03}U_{37}U_{72}U_{20})^{x_\gamma}U_{02}(U_{27}U_{74}U_{46}U_{62})^{x_\gamma} \\ &\cdot U_{26}U_{61}(U_{16}U_{64}U_{45}U_{51})^{x_\gamma}U_{10}(U_{01}U_{15}U_{53}U_{30})^{x_\gamma}\Big]^{x_\beta} \\ &(U_{03}U_{35}U_{51}U_{10})^{x_\gamma}U_{01} \\ R^{(5)}(x_\alpha, x_\gamma) = &(U_{03}U_{37}U_{72}U_{20})^{x_\gamma}U_{02} \\ R^{(6)}(x_\alpha, x_\beta) = &U_{03} \end{aligned}$$

where $U^x = \left(e^{i\vec{t}\cdot\vec{\sigma}}\right)^x \equiv e^{ix\vec{t}\cdot\vec{\sigma}}$ $|\vec{t}| < \pi$ and U^x is ambiguous for $U = -1$. Due to this ambiguity, we cannot define $k_{n,\mu}$ for a set of "exceptional configurations". Luckily, this set is of zero measure. Note that every configuration can be gauge transformed to an exceptional configuration. Such gauge transformations are on lines in the group space and hence, they are of zero measure.

It is straightforward to check that $S(x)$ of Eq. (12b) is continuous even on the links. The interpolation of $\Omega(x) = \Omega(0)\,S(x)\,S'^{+}(x)$ is, therefore, continuous too. The requirement that $P^{(i)}(x)$ and $S(x)$ are transformed by the same $\Omega(x)$ is trivially obeyed for $i = 1,2,3$. For the other plaquettes it follows from $R^{(4)}(x) \to \Omega(0)\,R^{(4)}(x)\,\Omega^+(1)$, $R^{(5)}(x) \to \Omega(0)\,R^{(5)}(x)\,\Omega^+(2)$ and $R^{(6)}(x) \to \Omega(0)\,R^{(6)}(x)\,\Omega^+(3)$. The requirement that $\Omega(x)$ on a plaquette depends only on Ω at its corners and the gauge fields around it is satisfied since $\Omega(x) = \Omega(0)\,P^{(i)}(x)\,P^{(i)\,\prime +}(x)$. In particular, $\Omega(x)$ on the link between $n = r$ and $n = s$ depends only on $\Omega(r)$, $\Omega(s)$ and U_{rs}: $\Omega(x) = [\Omega(s)\,U_{sr}\Omega^+(r)]^x\Omega(r)\,U_{rs}^x$.

We now interpolate $S(x)$ in the cube and define

$$k_{n,\mu}(U) = (-1)^{\mu} \frac{\mathcal{E}_{\mu\nu\rho\sigma}}{12\pi} \left[\int_{c} d^3x \,\mathrm{Tr}(S\partial_\nu S^{-1})\,(S\partial_\rho S^{-1})\,(S\partial_\sigma S^{-1}) \right.$$
$$\left. + 3 \int_{\partial c} d^2x \,\mathrm{Tr}(P^{-1}\partial_\rho P)\,(S^{-1}\partial_\sigma S) \right] . \tag{13}$$

The first term is the volume in the group enclosed by $S(x)$ on ∂c. It is independent of the precise interpolation of $S(x)$ in the cube up to adding a multiple of 2π which results form the total volume of the group. This ambiguity does not affect $e^{ik_{n,\mu}}$. The second term contains a sign depending on the orientation of the plaquette. ρ and σ are the two directions in the plaquette. Note that actually only two out of the six plaquettes ($i = 4, 5$) contribute to this term.

If a gauge transformation $\Omega(n)$ is performed,

$$\delta k_{n,\mu}(U) = (-1)^{\mu} \frac{\mathcal{E}_{\mu\nu\rho\sigma}}{12\pi} \left[\int_{c} d^3x \,\mathrm{Tr}(\Omega^{+}\partial_\nu\Omega)\;(\Omega^{+}\partial_\rho\Omega)\;(\Omega^{+}\partial_\sigma\Omega) \right.$$
$$\left. + 3 \int_{\partial c} d^2x \,\mathrm{Tr}(\Omega^{+}\partial_\rho\Omega)\;(P^{-1}\partial_\sigma P) \right] . \tag{14}$$

Hence $k_{n,\mu}(U)$ has the correct transformation properties. Note that the oriented sum $f_n(U)$ of $k_{n,\mu}(U)$ around a hypercube is not gauge invariant; it may change by 2π·integer. However, $e^{if_n(U)}$ and therefore $q_n(U) = \frac{1}{2\pi i}\, \ell n\, e^{if_n}$ $(-\frac{1}{2} \leq q_n < \frac{1}{2})$ are gauge invariant.

It is tedious but straightforward to verify that $q_n(U)$ has the correct continuum limit.[1]

The construction presented here is very similar to that of Luscher's.[1] His result, however, is not identical to ours. He fixes a different gauge for every hypercube and all his objects are defined in this gauge. Therefore, he does not assign a unique $k_{n,\mu}(U)$ to every cube. Moreover, his value for $q_n(U)$ may differ from ours by an integer. This difference becomes irrelevant in the continuum limit. It should be pointed out that both definitions are not manifestly cubic invariant.

We would like to mention that the object $k_{n,\mu}(U)$ defined earlier can be used as a lattice version of the topological mass term in $2+1$ dimensional QCD.[14] Just as in the continuum $k_{n,0}(U)$ and even $\sum_n k_{n,0}(U)$ (the sum is over all cubes in the three dimensional lattice orthogonal to $\hat{0}$) are not gauge invariant. $\sum_n k_{n,0}(U)$ may change by 2π·integer (for periodic boundary conditions) and hence its coefficient must be quantized in order to have a gauge invariant theory.[14]

ACKNOWLEDGEMENTS

It is a pleasure to thank T. Banks for collaboration in the early stages of this work. I have also benefited from discussions with S. Elitzur, M. Peskin, E. Rabinovici and M. Weinstein. I thank the SLAC theory group for their hospitality where this work was completed.

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FIGURE CAPTIONS

1. Standard labelling of the corners of the cube and the six plaquettes.

2. Standard labelling of the corners of a plaquette.

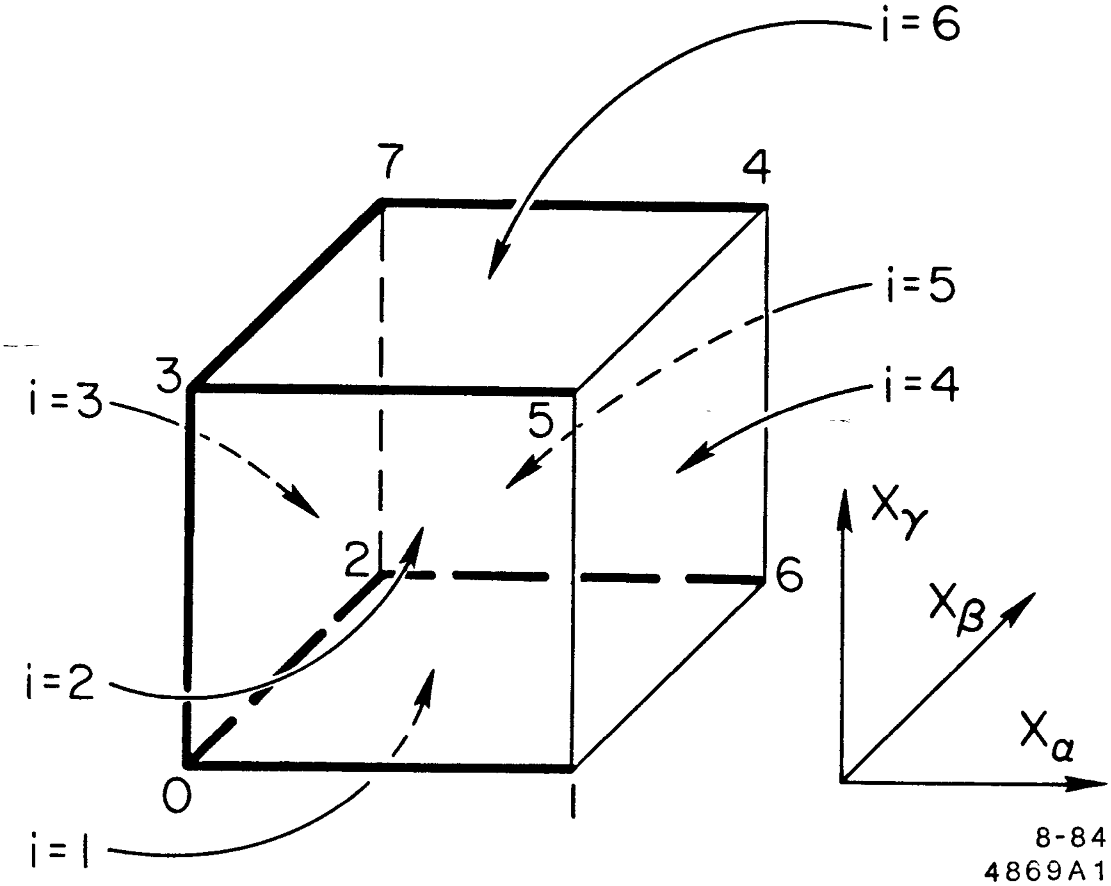

Fig. 1

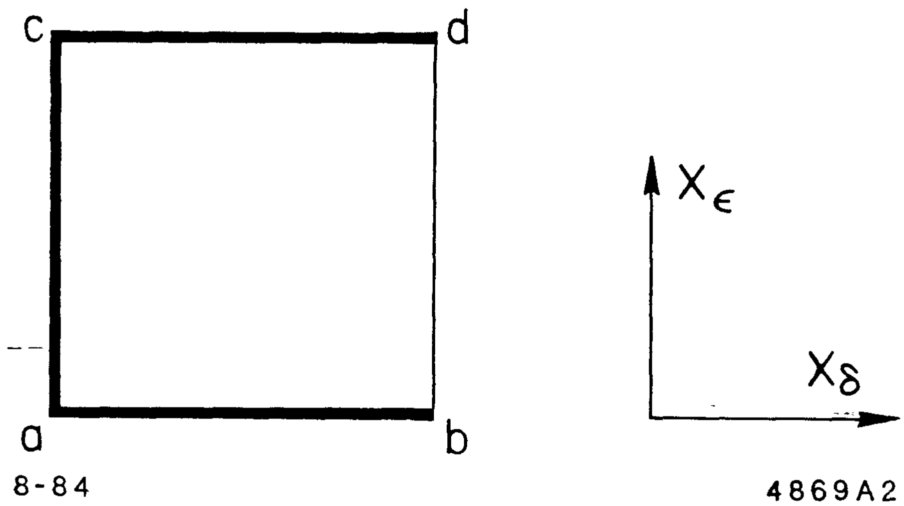

Fig. 2