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EVOLUTION EQUATION AND RELATIVISTIC BOUND STATE WAVE FUNCTIONS FOR SCALAR FIELD MODELS IN FOUR AND SIX DIMENSIONS*

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ABSTRACT

We investigate the evolution equation for distribution amplitudes in the framework of a scalar theory quantized on the light cone. We find general solutions for the cases of 4 and 6 dimensions and use them to reconstruct two-body relativistic bound state wave functions at small distances. The relation between the light-cone bound state equation and the Bethe-Salpeter equation is discussed.

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1. Introduction

A long-range goal in the study of quantum chromodynamics is to actually calculate the spectrum and wavefunctions of hadrons from first principles. An important theoretical tool to this end is lattice gauge theory which may eventually provide accurate numerical values for hadronic properties.[1] The Bethe-Salpeter approach is primarily suited to weak-binding bound state problems where higher irreducible kernel corrections and non-perturbative vacuum effects can be neglected.

An alternative approach[2] to relativistic bound state problems is the light-cone quantization method, which provides a Hamiltonian formalism and Fock-state representation of QCD at equal light-cone time $\tau = t + z/c$. The momentum-space bound-state solutions to this system of relativistic equations $\psi(x, k_\perp, \lambda)$ are functions of the light-cone variables $x_i = (k_i^0 + k_i^z)/(p^0 + p^z)$ and $\vec{k}_{\perp i}$, and the particle helicities λ_i. They are immediately suitable for calculations of covariant observables, such as structure functions, distribution amplitudes, form factors, anomalous moments, correlations and other hadronic properties.

The first step in solving the full set of coupled Fock state equations on the light-cone is to find a simple, analytically tractable equation for the valence, lowest-particle-number sector, and to develop a systematic perturbation theory for obtaining higher particle number states and higher accuracy. These requirements are satisfied by the simplest approximation, corresponding to the lowest order irreducible kernel; i.e. the light cone ladder approximation. Furthermore, one can prove that higher Fock-state contributions in light-cone ladder approximation in a renormalizable theory are negligible at large relative transverse momentum momenta. In gauge theories this statement is true for physical gauges for the vector fields, such as light-cone gauge. Thus the covariant ladder approximation is equivalent to light-cone ladder approximation at large $\vec{k}_\perp$. This equivalence eliminates any possibility of cusp-like (non-analytic) behavior of the distribution amplitude $\phi(x_i, Q)$ or light-cone wavefunction of the type described by Karmanov.[3] Since $\phi(x_i, Q)$ satisfies the evolution equation which is derived by taking large $\vec{k}_\perp$ limit of the light-cone projected Bethe-Salpeter equation, one can prove that $\phi(x_i, Q)$ is analytic (i.e. cusp free) in the whole x_i region and can be expanded by Gegenbauer polynomials in two-body bound state problems.[4] The cusp behavior is induced by the artificial limit of taking the binding energy to zero.

The initial problem to be examined in this paper is the behavior of the two-body bound-state wave function at large values of relative momentum. To this end we study first the properties of the corresponding valence-quark "distribution amplitudes" which control high-momentum transfer exclusive reactions.[4] The distribution amplitude $\phi(x_i, Q)$ is the amplitude for finding the $|q\bar{q}\rangle$ Fock-state in the bound state collinear up to scale Q. Its variation with Q will be described by an evolution equation. We find the solution of the evolution equation and use it to reconstruct a detailed form of the wave function at short distances. Although we will deal here with a simple spinless model, the methods are valid independent of spin. The scalar models are also of interest to the extent that they give a first look at the nature of wave functions for relativistic, strongly bound system. We perform the analysis within the Wick-Cutkosky model and discuss two different cases with $N = 4$ and $N = 6$ dimensions. Working in 4 dimensions, we obtain the bound-state wave function which asymptotically matches the simplest approximation to the light-cone wave function. The case of

6 dimensions is more interesting. Even though this case is a nonphysical one, it has mathematical and graphical similarity with the more physical non-Abelian quark-gluon theories. In particular, it has a fundamental trilinear coupling, it is renormalizable, and furthermore it happens to be asymptotically free. Because of asymptotic freedom, the higher order kernels can be neglected at short distances. The asymptotic behavior of the bound state Bethe-Salpeter wave function in such theory has also been investigated by Appelquist and Poggio.[5] In this paper the analysis is based on the light-cone approach and the resulting light-cone wave function exhibits calculable anomalous dimension corrections to a naive asymptotic behavior.

The paper is organized as follows: In Section 2 we present the evolution equation for the distribution amplitudes within the Wick-Cutkosky model. In section 3 we solve this equation for the case of $N = 4$ dimensions and use this solution to reconstruct the behavior of the two-body bound-state wave function at short distances. The QCD-like case of $N = 6$ dimensions is discussed in Section 5. The light-cone equation and its relation to the Bethe-Salpeter equation and the distinction between their respective ladder approximations are discussed in Appendices A and B.

2. The Evolution Equation

We shall consider the light-cone description of the relativistic composite system of two scalar particles interaction via the exchange of a massless scalar particle (Wick-Cutkosky model).[6,7] The interaction Lagrangian is $\mathcal{L} = g\psi^2\chi$, where ψ is a "quark" field with mass m, and χ is a massless "gluon" field. The bound state wave function can be described by means of the Fock-space components of the state vector $|\pi\rangle$ (see Appendix A).

The light-cone equation for the two-body wave function in N dimensions reads (See A(6))

$$\psi(x_i,\vec{k}_\perp) = \frac{1}{x_1x_2}\,\frac{1}{M^2 - \frac{m^2+k_\perp^2}{x_1x_2}}\,g^2(k_\perp) \cdot \int_0^1 [dy] \int \left[d^{N-2}\ell_\perp\right]\; K(x_i,\vec{k}_\perp;y_i,\vec{\ell}_\perp)\psi(y_i,\vec{\ell}_\perp)\ , \tag{2.1}$$

where x_i, y_i are the fractions of the total P^+ momentum of bound state carried by the i-th valence quark ($x_1+x_2=1$, $[dy] = dy_1\cdot dy_2\cdot\delta(1-y_1-y_2)$), $\vec{k}_\perp$, $\vec{\ell}_\perp$ are the N-2 dimensional perpendicular momenta, $[d^{N-2}\ell_\perp] = \frac{1}{2(2\pi)^{N-1}}\, d^{N-2}\ell_\perp$ and M is the mass of the bound state.

The asymptotic behavior of the coupling constant $g^2(Q)$ has the form

$$g^2(Q) = \begin{cases} g^2 & \text{for } N=4\ , \\ \frac{g_\Lambda^2}{\log(Q^2/\Lambda^2)} & \text{for } N=6\ , \end{cases} \tag{2.2}$$

i.e. there is a "running coupling constant" for the theory in 6 dimensions.[5] Restricting ourselves to the one-gluon-exchange only we obtain the light cone ladder approximation (LCLA) to the kernel of Eq. (2.1),

$$K(x_i,\vec{k}_\perp;y_i,\vec{\ell}_\perp) = \frac{\theta(y_1-x_1)}{y_1-x_1}\,\frac{1}{M^2-\frac{m^2+k_\perp^2}{x_1}-\frac{(\vec{k}_\perp-\vec{\ell}_\perp)^2}{y_1-x_1}-\frac{m^2+\ell_\perp^2}{y_2}} + (1\leftrightarrow 2)\ . \tag{2.3}$$

The high-momentum transfer exclusive reactions are controlled by "distribution amplitudes" $\phi(x_i,Q)$, which give the probability for finding the valence quarks in bound state with momentum fraction x_i at relative perpendicular distance not smaller than $b_\perp \sim O(1/Q)$, i.e. collinear up to scale Q

$$\phi(x_i,Q) = \int^{|k_\perp|<Q} \left[d^{N-2}k_\perp\right]\,\psi^{(Q)}(x_i,\vec{k}_\perp)\ . \tag{2.4}$$

The variation of ϕ with Q comes from the upper limit of the integration as well as from renormalization scale dependence of the wave function

$$\psi^{(Q)}(x_i,\vec{k}_\perp) = \frac{Z_2(Q)}{Z_2(Q_0)}\,\psi^{(Q_0)}(x_i,\vec{k}_\perp)\ , \tag{2.5}$$

due to vertex and self-energy insertions. For the $N=4$ case $d^2k_\perp = \pi dk_\perp^2$ and for the $N=6$ case $d^4k_\perp = \pi^2k_\perp^2\,dk_\perp^2$, so that the differentiation of Eq. (2.4) yields the

$$Q^2\,\frac{\partial}{\partial Q^2}\,\phi(x_i,Q) = Q^2\,\frac{\partial \log Z_2(Q)}{\partial Q^2}\,\phi(x_i,Q) + (Q^2\pi)^{(N-2)/2}\frac{1}{2(2\pi)^{N-1}}\,\psi^{(Q)}(x_i,Q)\ ,\quad N=4 \text{ or } 6\ . \tag{2.6}$$

We compute now the $\psi^{(Q)}$ from one-gluon exchange LCLA kernel (2.3). The dominant behavior of the wave function for $k_\perp\to\infty$ is obtained from (2.1)-(2.3) by neglecting $m, \ell_\perp$ relative to $k_\perp$ in the kernel and integrating over $\ell_\perp \leq Q$. One obtains then

$$\psi^{(Q)}(x_i, Q) = \frac{1}{Q^4}\, g^2(Q) \cdot \int_0^1 [dy]\, V(x_i, y_i) \int^Q \left[d^{N-2}\ell_\perp\right]\, \psi^{(Q)}(y, \ell_\perp)\ , \tag{2.7}$$

where the evolution kernel is given by

$$V(x_i, y_i) = \theta(y_1 - x_1)\, \frac{x_1}{y_1} + (1 \leftrightarrow 2)\ . \tag{2.8}$$

Substituting (2.7) into (2.6) and using (2.4) one obtains an evolution equation for $\phi(x_i, Q)$

$$Q^2 \frac{\partial}{\partial Q^2}\, \phi(x_i, Q) = Q^2\, \frac{\partial \log Z_2(Q)}{\partial Q^2}\, \phi(x_i, Q)$$
$$+ (\pi Q^2)^{(N-2)/2} \cdot \frac{g^2(Q)}{Q^4} \cdot \frac{1}{2(2\pi)^{N-1}} \int_0^1 [dy]\, V(x_i, y_i) \phi(y_i, Q)\ . \tag{2.9}$$

The vertex and self-energy insertions combine to the following form

$$Z_2(Q) = 1 + \int_0^1 [dy] \int_Q^\infty \left[d^{N-2}\ell_\perp\right] \cdot g^2(\ell_\perp) \cdot \frac{1}{y_1 y_2} \cdot \frac{1}{\left(m^2 - \frac{m^2+\ell_\perp^2}{y_2} - \frac{\ell_\perp^2}{y_1}\right)^2}\ . \tag{2.10}$$

The dominant behavior for $Q \to \infty$ is then given by

$$Z_2(Q) = 1 + \int_0^1 [dy] \cdot y_1 y_2 \cdot \xi(Q) = 1 + \frac{1}{6}\, \xi(Q) \simeq e^{(1/6)\xi(Q)}\ , \tag{2.11}$$

where

$$\xi(Q) = \int_Q^\infty \left[d^{N-2}\ell_\perp\right]\, \frac{g^2(\ell_\perp)}{\ell_\perp^4}\ . \tag{2.12}$$

In the following we find the most general solution of Eq. (2.9), which is next used to reconstruct the short-range behavior of the bound state wave function ψ by means of Eq. (2.6). This is done separately for $N = 4$ and $N = 6$ case.

3. The Case of $N = 4$ Dimensions

For the case of $N = 4$ dimensions the evolution equation (2.9) upon substitution of (2.2) and (2.12) reads

$$Q^2 \frac{\partial}{\partial Q^2} \phi(x_i, Q) = -\frac{1}{Q^2} \frac{\pi}{2(2\pi)^3} g^2 \left\{ \frac{1}{6} \phi(x_i, Q) - \int [dy] V(x_i, y_i) \phi(y_i, Q) \right\} . \tag{3.1}$$

The most general solution of this equation can be expressed as a superposition of separable functions. The dependence on x_i is then given in terms of Gegenbauer polynomials $c_n^{3/2}$ in analogy to the true QCD solution;[4] the functional dependence at large Q^2, however, is quite different

$$\phi(x_i, Q) = x_1 x_2 \sum_{n=0}^{\infty} a_n c_n^{3/2}(x_1 - x_2)\, e^{+\gamma_n \cdot g^2 \cdot \frac{\pi}{2(2\pi)^3} \cdot \frac{1}{Q^2}} , \tag{3.2}$$

where

$$\gamma_n = +\frac{1}{6} - \frac{1}{(n+1)(n+2)} . \tag{3.3}$$

To derive this result we used the orthogonality and recurrence relations of the Gegenbauer polynomials.

For very large Q^2 the distribution amplitude (3.2) behaves like

$$\phi(x_i, Q) \to x_1 x_2 \sum_{n=0}^{\infty} a_n c_n^{3/2}(x_1 - x_2) \cdot \left\{ 1 + \frac{1}{Q^2} g^2 \frac{\pi}{2(2\pi)^3} \gamma_n + O(1/Q^4) \right\} . \tag{3.4}$$

Given the distribution amplitude $\phi(x_i, Q_0)$ at some momentum Q_0, the coefficients a_n can be determinant by using the orthogonality relations of Gegenbauer polynomials

$$a_n e^{+\gamma_n g^2 \frac{\pi}{2(2\pi)^3} \frac{1}{Q_0^2}} = \frac{4(2n+3)}{(n+2)(n+1)} \int_0^1 [dx]\, c_n^{3/2}(x_1 - x_2)\, \phi(x_i, Q_0) \ , \tag{3.5}$$

thus determining the behavior of $\phi(x, Q)$ for any $Q^2 > Q_0^2$. Alternatively, we show below how the coefficients a_n can be directly related to the asymptotic behavior of the two-body bound-state wave function $\psi(x_i, k_\perp)$ with $k_\perp^2 = Q^2$. Using (3.2) as an input and solving Eq. (2.6) for $\psi(x_i, Q)$ we obtain with help of Eq. (2.5)

$$\begin{aligned} \psi^{(Q_0)}(x_i, Q) &= g^2 \cdot \frac{Z_2(Q_0)}{Z_2(Q)} \frac{x_1 x_2}{Q^4} \sum_{n=0}^{\infty} a_n \cdot c_n^{3/2}(x_1 - x_2) \\ &\quad \cdot \frac{1}{(n+1)(n+2)}\, e^{\gamma_n \cdot g^2 \cdot \frac{\pi}{2(2\pi)^3} \frac{1}{Q^2}} \\ &= g^2 e^{\frac{1}{6} \cdot g^2 \cdot \frac{\pi}{2(2\pi)^3} \frac{1}{Q_0^2}} \cdot \frac{x_1 x_2}{Q^4} \sum_{n=0}^{\infty} a_n c_n^{3/2}(x_1 - x_2) \\ &\quad \cdot \frac{1}{(n+1)(n+2)}\, e^{-\frac{1}{(n+1)(n+2)} \cdot g^2 \cdot \frac{\pi}{2(2\pi)^3} \frac{1}{Q^2}} \end{aligned} \tag{3.6}$$

As the model of the two-body light-cone wave function (without self energy and vertex corrections) we take the expression

$$\psi(x, k_\perp) \sim \frac{1}{x_1 x_2} \frac{1}{\left(M^2 - \frac{m^2 + k_\perp^2}{x_1 x_2}\right)^2} \ . \tag{3.7}$$

This expression is the light-cone projection of the Bethe-Salpeter wave function found as the exact solution[8] to the ladder approximation Bethe-Salpeter equation for the case of $M^2 = 0$, (see Appendix B, especially Eq. (B.18)). Since in the limit $Q^2 \to \infty$ the light-cone projection of the $M^2 = 0$ ladder Bethe-Salpeter equation and the LCLA coincide, we expect the form (3.7) to serve as an excellent approximation of the two-body LCLA wave function in the limit $Q^2 \to \infty$:

$$\psi(x,Q) \sim \frac{x_1 x_2}{Q^4\left(1+\frac{m^2}{Q^2}\right)^2} \; . \tag{3.8}$$

Upon comparison of the $Q \to \infty$ limit of (3.6) with (3.8) one may obtain at once

$$a_n = 0 \quad \text{for } n \geq 1 \; , \qquad g^2 = \frac{8(2\pi)^3}{\pi} m^2 \; , \tag{3.9}$$

so that Eq. (3.6) reduces to

$$\psi^{(Q_0)}(x,Q) \propto \frac{x_1 x_2}{Q^4} e^{-2m^2/Q^2} \; . \tag{3.10}$$

The expression (3.9) constitutes the condition on the strength of the coupling constant required to produce the bound state mass $M^2 = 0$. We note that in the nonrelativistic limit the LCLA kernel reduces to the Coulomb potential $V(r) = -\alpha/r$ in momentum space where $\alpha \equiv g^2/16\pi\, m^2$. Thus the result (3.9) corresponds to $\alpha = 4\pi$. We note here that within the ladder Bethe-Salpeter equation the mass M^2 of the ground state vanishes for $\alpha = 2\pi$.[6] The reason for the factor of 2 discrepancy is due to the fact that we take the condition of g^2 at the next leading Q^2 term while we take for the wavefunction only the leading Gegenbauer polynomial corresponding to the large Q^2 behavior of the kernel.[9] Actually, one can prove that the Cutkosky condition $\alpha = 2\pi$ is obtained from the light-cone ladder approximation at large $\vec{k}_\perp$ limit.

4. The Case of $N = 6$ Dimensions

As we pointed out previously, the case of $N = 6$ dimensions exhibits mathematical similarity with the QCD problem.

The evolution equation (2.9) takes now the form

$$Q^2 \frac{\partial}{\partial Q^2} \phi(x_i, Q) = -\frac{\pi^2}{2(2\pi)^5} \frac{g_\Lambda^2}{\log(Q^2/\Lambda^2)} \left\{ \frac{1}{6} \phi(x_i, Q) - \int [dy] V(x_i, y_i) \phi(y_i, Q) \right\} \tag{4.1}$$

and the general solution of (4.1) can be written as

$$\phi(x_i, Q) = x_1 x_2 \cdot \sum_{n=0}^{\infty} a_n c_n^{3/2}(x_1 - x_2) \cdot \left(\log \frac{Q^2}{\Lambda^2} \right)^{-\gamma_n \cdot g_\Lambda^2 \cdot \frac{\pi^2}{2(2\pi)^5}} , \tag{4.2}$$

where the anomalous dimensions γ_n are again given by (3.3). Note that $\gamma_0 = -1/3$, $\gamma_1 = 0$, $\gamma_2 = 1/12$ and $\gamma_n < \gamma_{n+1} < 1/6$ so that the leading ($n = 0$) term in (4.2) grows as $Q^2 \to \infty$, whereas in the true QCD model of pion[4] all anomalous dimensions $\gamma_n \geq 0$. However, this leading term does not contribute if we calculate the decay of our bound state into two spinless particles. Indeed, calculating the matrix element of the electromagnetic current controlling such a decay, $\langle 0 | J^\mu | \pi \rangle$, we have to take into account the particular form of the coupling of the spin-zero boson to electromagnetic field, given by the vertex $ie(p + p')^\mu$. Therefore,

$$\langle 0 | J^+ | \pi \rangle \sim e \int [dx] (x_1 - x_2) \phi(x_i, Q) , \tag{4.3}$$

where the index 1(2) stands for positively (negatively) charged constituents. Therefore, only odd terms in (4.2) contribute and in the limit $Q^2 \to \infty$ the leading contribution to (4.3) comes from $n = 1$ term. Since $\gamma_1 = 0$ this yields just the decay constant, in analogy to the QCD model of pion.

Repeating the procedure used in the last chapter we find the two-body wave function

$$\psi^{(Q_0)}(x_i, Q) = g_\Lambda^2 \, \frac{Z_2(Q)}{Z_2(Q_0)} \, \frac{x_1 x_2}{Q^4 \log(Q^2/\Lambda^2)}$$

$$\cdot \sum_{n=0}^{\infty} a_n \, \frac{1}{(n+1)(n+2)} \, c_n^{3/2}(x_1 - x_2) \left(\log \frac{Q^2}{\Lambda^2} \right)^{-\gamma_n g_\Lambda^2 \frac{\pi^2}{2(2\pi)^5}}$$

$$= g_\Lambda^2 \left(\log \frac{Q_0^2}{\Lambda^2} \right)^{-\frac{1}{6} g_\Lambda^2 \frac{\pi^2}{2(2\pi)^5}} \cdot \frac{x_1 x_2}{Q^4 \log(Q^2/\Lambda^2)}$$

$$\cdot \sum_{n=0}^{\infty} a_n \frac{1}{(n+1)(n+2)} \, c_n^{3/2}(x_1 - x_2) \left(\log \frac{Q^2}{\Lambda^2} \right)^{\frac{1}{(n+1)(n+2)} g_\Lambda^2 \frac{\pi^2}{2(2\pi)^5}} . \tag{4.4}$$

Again, the coefficients a_n can be determined from the knowledge of the distribution amplitude at some momentum scale Q_0, or upon comparison with the asymptotic form of the bound state wave function. More interestingly, one can easily see that the asymptotic form of the wavefunction (4.4),

$$\psi(x_i, Q) \propto \frac{x_1 x_2}{Q^4} \left(\log \frac{Q^2}{\Lambda^2} \right)^{-1 + \frac{g_\Lambda^2 \cdot \pi^2}{4(2\pi)^5}} \left\{ 1 + \frac{a_1}{3 a_0} (x_1 - x_2) \left(\log \frac{Q^2}{\Lambda^2} \right)^{-\frac{g_\Lambda^2 \cdot \pi^2}{6(2\pi)^5}} \right\} , \tag{4.5}$$

which is derived from the distribution amplitude $\phi(x_i, Q)$ is consistent with the asymptotic Bethe-Salpeter wavefunction up to logarithmic corrections.[5]

5. Summary

Working within the light-cone quantization scheme, we are able to obtain a deeper insight into a behavior of the two-body bound state wave function at short distances than was so far provided by alternative methods[3,5] exploiting asymptotic properties of the integral kernel itself. In particular we have exploited the relationship between exact solution to the evolution equation for the distribution amplitude and wavefunction equations to extract the large momentum behavior of the light cone and Bethe-Salpeter wavefunctions. Using these constraints we are led to convenient, analytic forms of the two-body wave function which can serve as a basis for a perturbation theory for higher particle-number states. Finally, as discussed in Appendix B, we identify the differences in content between the ladder approximations for the Bethe-Salpeter and light-cone wavefunctions in terms of interaction retardation and higher Fock state components.

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APPENDIX A

LIGHT-CONE BOUND-STATE EQUATION

We shall consider the light-cone description of the relativistic composite system of two scalar particles interacting via exchange of a massless scalar particle (Wick-Cutkosky model).[5,6] The interaction Lagrangian is $\mathcal{L} = g\ \psi^2\chi$, where ψ is a "quark" field with mass m, and χ is a massless "gluon" field. The bound state can be described by means of the Fock-space components of the state vector $|\pi\rangle$.

At any given light cone time $\tau = t + z$ we can define a set of basis states

$$
\begin{aligned}
&|0\rangle \\
&|q\bar{q} : \underline{k}_i\rangle = q^+(\underline{k}_1)\bar{q}^+(\underline{k}_2)|0\rangle \qquad \text{(A1)} \\
&|q\bar{q}g : \underline{k}_i\rangle = q^+(\underline{k}_1)\bar{q}^+(\underline{k}_2)b^+(\underline{k}_3)|0\rangle
\end{aligned}
$$

where $q^+, \bar{q}^+$ and b^+ are the Fourier transforms of the unrenormalized operators at time τ of massive field ψ and massless field χ, respectively, and where $\underline{k}_i = (k^+ = k^0 + k^3, \vec{k}_\perp)_i$ is the momentum of the i-th parton. Of course the elements, other than the vacuum (we ignore here the possibility of zero modes as would be characteristic of spontaneous symmetry breaking.), of this Fock-space basis are not eigenstates of the full Hamiltonian $H_{LC} = P^- = P^0 - P^3$. However, they form a useful basis for studying the physical states of the theory. The bound system under consideration (a "pion") is described by a state

$$
|\pi\rangle = \sum_{q\bar{q}} |q\bar{q}\rangle\psi_{q\bar{q}} + \sum_{q\bar{q}g} |q\bar{q}g\rangle\psi_{q\bar{q}g} + \ldots \qquad \text{(A2)}
$$

Any bound state, such as $|\pi\rangle$ must be an eigenstate of the full Hamiltonian. Working in a frame where $\underline{P} = (P^+, \overrightarrow{0}_\perp)$ and $P^- = M^2/P^+$, the state $|\pi\rangle$

satisfies the equation

$$(M^2 - H_{LC})|\pi\rangle = 0 \, . \tag{A3}$$

Projecting this onto various Fock-states $\langle q\bar{q}|$, $\langle q\bar{q}g|, \ldots$ results in infinite number of coupled integral equations[10,11]

$$\begin{aligned} &\left[M^2 - \sum_i \left(\frac{k_\perp^2 + m^2}{x}\right)_i\right] \cdot \begin{bmatrix} \psi_{q\bar{q}} \\ \psi_{q\bar{q}g} \\ \cdot \\ \cdot \end{bmatrix} \\ &= \begin{bmatrix} 0 & \langle q\bar{q}|V|q\bar{q}g\rangle & \ldots \\ \langle q\bar{q}g|V|q\bar{q}\rangle & 0 & \\ \cdot & & \\ \cdot & & \end{bmatrix} \begin{bmatrix} \psi_{q\bar{q}} \\ \psi_{q\bar{q}g} \\ \cdot \\ \cdot \end{bmatrix} \end{aligned} \tag{A4}$$

where V is the interaction part of H_{LC}.

Parameterizing the individual momenta of both partons by

$$\underline{k}_1 = (x_1 P^+, \vec{k}_\perp) \ , \ \underline{k}_2 = (x_2 P^+, -\vec{k}_\perp), k_i^- = \frac{m_i^2 + k_\perp^2}{x_i P^+} \tag{A5}$$

and taking into account only the 2- and 3-body sectors of the equation (A4) we arrive at the effective equation for the two-body wave function ($[dy] \equiv dy_1\, dy_2\, \delta(1 - y_1 - y_2)$)

$$\begin{aligned} \psi(x_i, \vec{k}_\perp) = &\frac{1}{x_1 x_2} \frac{1}{M^2 - \frac{m^2 + k_\perp^2}{x_1 x_2}} \cdot \frac{g^2}{2 \cdot (2\pi)^{N-1}} \int_0^1 [dy] \int d^{N-2}\ell_\perp \\ &\left\{ \frac{\theta(y_1 - x_1)}{y_1 - x_1} \frac{1}{M^2 - \frac{m^2 + k_\perp^2}{x_1} - \frac{m^2 + \ell_\perp^2}{y_2} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1}} + (1 \leftrightarrow 2) \right\} \psi(y_1, \vec{\ell}_\perp). \end{aligned} \tag{A6}$$

which we call the light cone ladder approximation (LCLA). This equation is symbolically depicted in Fig. 1. In the case of $N = 4$ dimensions it is useful to introduce the dimensionless coupling constant α, defined by $\alpha = g^2/16\pi m^2$. At $|\vec{k}_\perp|$, $|\vec{\ell}_\perp| \ll m$, $|x_1 - x_2| \ll 1$, $|y_1 - y_2| \ll 1$ the kernel of Eq. (A.6) reduces to the Coulomb potential $V(r) = -\alpha/r$ in momentum space. At $\alpha \ll 1$ we can parametrize $M^2 = 4m^2 - 4m\epsilon$, $|\epsilon| = |M - 2m| \ll m$ and we have $\epsilon = \frac{1}{4}m\alpha^2$, whereas the solution of (A2) reduces to nonrelativistic wave function of the ground state in the Coulomb potential.

APPENDIX B

RELATION TO THE BETHE-SALPETER EQUATION

Let us consider the ladder approximation Bethe-Salpeter equation for the wave function of bound state in the Wick-Cutkosky model.[6,7] in the case of $N = 4$ dimensions.

$$(k_1^2 - m^2)(k_2^2 - m^2)\psi^{BS}(k_i) = -\frac{ig^2}{(2\pi)^4} \int d^4\ell_1\, d^4\ell_2\, \delta(k_1 + k_2 - \ell_1 - \ell_2) \times \frac{1}{(k_1 - \ell_1)^2 + i\epsilon}\psi^{BS}(\ell_i) \tag{B1}$$

We denote $P = k_1 + k_2$, $P^2 = M^2$, $k = \frac{1}{2}(k_1 - k_2)$, $\ell = \frac{1}{2}(\ell_1 - \ell_2)$. The solution of (B.1) can be formally written as

$$\psi^{BS}(k_i) = -i\int_{-1}^{+1} dz\, \frac{h(z, M^2)}{(k^2 + \frac{1}{4}M^2 - m^2 + kPz + i\epsilon)^3} \tag{B2}$$

where the function $h(z)$ satisfies the second-order differential equation[8,11] with boundary condition $h(\pm 1) = 0$. Parametrizing the individual momenta by

$(P^+ = 1)$

$$\underline{k}_1 = (x_1, \vec{k}_\perp) \qquad \underline{k}_2 = (x_2, -\vec{k}_\perp) \qquad x_1 + x_2 = 1$$
$$\ell_1 = (y_1, \vec{\ell}_\perp) \qquad \underline{\ell}_2 = (y_2, -\vec{\ell}_\perp) \qquad y_1 + y_2 = 1 \tag{B3}$$

we rewrite (B.1) in the form

$$\psi^{BS}(x_1, \vec{k}_\perp, p_i^-) = \frac{1}{x_1 x_2 \left(k_1^- - \frac{m^2+k_\perp^2}{x_1} + \frac{i\epsilon}{x_2}\right)\left(k_2^- - \frac{m^2+k_\perp^2}{x_2} + \frac{i\epsilon}{x_2}\right)}$$
$$\cdot \frac{ig^2}{2\cdot(2\pi)^4} \int d\ell_2^- \, d\ell_2^- \, [dy] d^2\ell_\perp \cdot \delta(\ell_1^- + \ell_2^- - M^2) \tag{B4}$$
$$\cdot \frac{1}{(x_1 - y_1)(k_1^- - \ell_1^-) - (\vec{k}_\perp - \vec{\ell}_\perp)^2 + i\epsilon} \psi^{BS} \; (y_i, \vec{\ell}_\perp, \ell_i^-)$$

We now define the light-cone projection of Bethe-Salpeter equation by

$$\psi(x_i, \vec{k}_\perp) \equiv \int dk_1^- \, dk_2^- \, \delta(k_1^- + k_2^- - M^2) \, \psi^{BS} \; (x_i, \vec{k}_\perp, k_i^-) \tag{B5}$$

Thus the light-cone projection of (B.4) gives

$$\psi(x_i, \vec{k}_\perp) = \frac{-ig^2}{2\cdot(2\pi)^4} \cdot \int dk_1^- \, dk_2^- \, \delta(k_1^- + k_2^- - M^2)$$
$$\frac{1}{x_1 x_2 \left(k_1^- - \frac{m^2+k_\perp^2}{x_1}\right)\left(k_2^- - \frac{m^2+k_\perp^2}{x_2} + \frac{i\epsilon}{x_2}\right)}$$
$$\cdot \int d\ell_1^- \, d\ell_2^- [dy] \; d^2\ell_\perp \delta(\ell_1^- + \ell_2^- - M^2) \tag{B6}$$
$$\frac{1}{(x_1 - y_1)\left[k_1^- - \ell_1^- - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{x_1 - y_1} + \frac{i\epsilon}{x_1 - y_1}\right]} \psi^{BS} \; (y_i, \vec{\ell}_\perp)$$

Again, we perform the dk_i^- integration closing the contour of integration in the upper half-plane for $y_1 < x_1$ and in the lower half plane for $y_1 > x_1$ (see Ref.

12).The integral vanishes unless $0 < x_1,\ x_2 < 1$, and the result is

$$\psi(x_i, \vec{k}_\perp) = \frac{g^2}{2\cdot(2\pi)^3}\,\frac{1}{x_1 x_2}\,\frac{1}{M^2 - \frac{m^2+k_\perp^2}{x_1 x_2}}$$

$$\int d\ell_1^-\, d\ell_2^-\ \delta(\ell_1^- + \ell_2^- - M^2)\ \int [dy] \int d^2\ell_\perp$$

$$\left\{ \frac{\theta(y_1 - x_1)}{y_1 - x_1}\ \frac{1}{M^2 - \ell_2^- - \frac{m^2+k_\perp^2}{x_1} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1} + \frac{i\epsilon}{y_1 - x_2}} \right. \tag{B7}$$

$$\left. + \frac{\theta(y_2 - x_2)}{y_2 - x_2}\ \frac{1}{M^2 - \ell_1^- - \frac{m^2+k_\perp^2}{x_2} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_2 - x_2} + \frac{i\epsilon}{y_2 - x_2}} \right\}$$

$$\times\, \psi^{BS}\ (y_i, \vec{\ell}_\perp, \ell_i^-)$$

This equation involves the light-cone wave function on the left-hand side, but the Bethe-Salpeter wave function on the right hand side. We now use the identity

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B - A)\frac{1}{A} \tag{B8}$$

which implies

$$\frac{1}{M^2 - \ell_2^- - \frac{m^2+k_\perp^2}{x_1} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1}} = \frac{1}{M^2 - \ell_{20}^- - \frac{m^2+k_\perp^2}{x_1} - \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1}}$$

$$+ \frac{1}{M^2 - \ell_{20}^- - \frac{m^2+k_\perp^2}{x_1} + \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1}}\ (\ell_2^- - \ell_{20}^-) \tag{B9}$$

$$\times \frac{1}{M^2 - \ell_2^- - \frac{m^2+k_\perp^2}{x_1} + \frac{(\vec{k}_\perp - \vec{\ell}_\perp)^2}{y_1 - x_1}},$$

where ℓ_{20}^- is an arbitrary constant. Now we can carry out the $d\ell_i^-$ integration over the first term in (B.9). An identical trick is applied to the second term in kernel of (B.7).

Choosing the fixed points to be

$$\ell_{20}^- \equiv \frac{m^2+\ell_\perp^2}{y_2} \qquad ; \qquad \ell_{10}^- \equiv \frac{m^2+\ell_\perp^2}{y_1} \tag{B10}$$

we obtain from (B.7)

$$\psi(x_i,\vec{k}_\perp) = \frac{g^2}{2\cdot(2\pi)^3}\,\frac{1}{x_1x_2}\,\frac{1}{M^2-\frac{m^2+k_\perp^2}{x_1x_2}}\int[dy]\int d^2\ell_\perp$$

$$\left[\frac{\theta(y_1-x_1)}{y_1-x_1}\,\frac{1}{M^2-\frac{m^2+\ell_\perp^2}{y_2}-\frac{m^2+k_\perp^2}{x_1}-\frac{(\vec{k}_\perp-\vec{\ell}_\perp)^2}{y_1-x_1}}\right.$$

$$\cdot\left\{\psi(y_i,\vec{\ell}_\perp)+\int d\ell_1^-\,d\ell_2^-\,\delta(\ell_1^-+\ell_2^--M^2)\cdot\left(\ell_2^--\frac{m^2+\ell_\perp^2}{y_2}\right)\right.$$

$$\left.\left.\frac{1}{M^2-\ell_2^--\frac{m^2+k_\perp^2}{x_1}-\frac{(\vec{k}_\perp-\vec{\ell}_\perp)^2}{y_1-x_1}}\psi^{BS}\,(y_i,\vec{\ell}_\perp,\ell_i^-)\right\}+1\leftrightarrow 2\right]. \tag{B11}$$

Neglecting the terms involving ψ^{BS} on the right hand side we immediately arrive at the LCLA, Eq. (A.6).

To obtain an insight into the structure of the discarded higher sectors terms we could alternatively start from Eq. (B.4), iterate it one time and then project onto light-cone. Again, on the right hand side we would have terms involving ψ^{BS}, but some of them, with help of Eq. (B.6), could be reexpressed in terms of light-cone wave function. Then we would arrive at the equation

$$\psi(x_i,\vec{k}_\perp) = \frac{g^2}{2\cdot(2\pi)^3}\,\frac{1}{x_1x_2}\,\frac{1}{M^2-\frac{m^2+k_\perp^2}{x_1x_2}}\int[dy]\cdot d^2\ell_\perp\cdot\frac{g^2}{2(2\pi)^3}\,\frac{1}{y_1y_2}\int[dz]d^2p_\perp$$

$$\Bigg\{\Bigg\{\frac{\theta(y_1-x_1)}{y_1-x_1}\cdot\frac{1}{M^2-\frac{m^2+k_\perp^2}{x_1}-\frac{(\vec{k}^\perp-\vec{\ell}_\perp)^2}{y_1-x_1}-\frac{m^2+\ell_\perp^2}{y_2}}\Bigg[\frac{\theta(z_1-y_1)}{z_1-y_1}$$

$$\cdot\left(\frac{1}{M^2-\frac{m^2+\ell_\perp^2}{y_1y_2}}+\frac{1}{M^2-\frac{m^2+k_\perp^2}{x_1}-\frac{(\vec{k}_\perp-\vec{\ell}_\perp)^2}{y_1-x_1}-\frac{(\vec{p}_\perp-\vec{\ell}_\perp)^2}{z_1-y_1}-\frac{m^2+p_\perp^2}{z_2}}\right)$$

$$\cdot\frac{1}{M^2-\frac{m^2+\ell_\perp^2}{y_1}-\frac{(\vec{p}_\perp-\vec{\ell}_\perp)^2}{z_1-y_1}-\frac{m^2+p_\perp^2}{z_2}}$$

$$+\frac{\theta(z_2-y_2)}{z_2-y_2}\cdot\frac{1}{M^2-\frac{m^2+\ell_\perp^2}{y_1y_2}}\,\frac{1}{M^2-\frac{m^2+\ell_\perp^2}{y_2}-\frac{(\vec{p}_\perp-\vec{\ell}_\perp)^2}{z_2-y_2}-\frac{m^2+p_\perp^2}{z_1}}\Bigg]\Bigg\}$$

$$+\,(1\leftrightarrow 2)\Bigg\}\;\psi(z_i,\vec{p}_\perp)$$

$$+\,(\text{terms involving }\psi^{BS})\quad . \tag{B12}$$

This equation is depicted in Fig. 2. It is easy to recognize that in addition to the single gluon exchange we have now the diagrams corresponding to the exchange of two gluons at a given light-cone time. Likewise, the terms involving ψ^{BS} in (B.12) correspond to three- and many-gluon exchange diagrams, as it can be easily demonstrated upon one more iteration of Eq. (B.4). Neglecting these terms we still have irreducible two-gluon exchange diagrams. Dropping also these terms (i.e. the second term in the curly bracket in Eq. (B.12)) we obtain the once-iterated LCLA.

However, even taking into account the full right hand side of Eq. (B.12) (or, equivalently, Eq. (B.11)) we drastically differ from what we would have obtained taking into account the higher Fock-space sector of Eq. (A.4). Although (B.11) contains any number of exchanged bosons in the intermediate states, there are no diagrams where any two of exchanged boson lines cross each other. On the other hand, all such diagrams are automatically generated from Eq. (A.4), when we eliminate its higher sectors.

It is interesting to note, however, that in the weak-binding limit the approximate solutions of (B.11) and LCLA coincide.[3] This can be easily demonstrated, if we project the Bethe-Salpeter function (B.2) onto light-cone. For weak-binding the spectral function can be approximated by[8] $h(z, M^2) = 1 - |z|$ and the light-cone projection of the Bethe-Salpeter wave function (B.2) is given by

$$\psi^{\ell.c.}_{BS,}(x_i, \vec{k}_\perp) = -i \int dk^- \int_{-1}^{+1} dz \, \frac{1 - |z|}{(k^2 - \frac{1}{4}M^2 - m^2 + kPz + i\epsilon)^3} \; . \qquad \text{(B13)}$$

We rewrite the denominator as

$$\begin{aligned} k^2 + \frac{1}{4}M^2 - m^2 + kPz &= (k + \frac{1}{2}Pz)^2 - \left(m^2 + \frac{1}{4}M^2(z^2 - 1)\right) \\ &= (k')^2 - C \end{aligned} \qquad \text{(B14)}$$

where

$$k' = k + \frac{1}{2}Pz \; , \qquad C = m^2 + \frac{1}{4}M^2(z^2 - 1) \; .$$

Since $dk^- \equiv dk'^-$ the integration is straightforward

$$\int dk^- \frac{1}{(k'^+ k'^- - k_\perp^2 - C)^3} \equiv -\pi \, i \, \frac{1}{(k_\perp^2 + C)^2} \, \delta(k'^+) \; . \qquad \text{(B15)}$$

but

$$k'^{+} = k^{+} + \frac{1}{2}P^{+}z \equiv \frac{1}{2}(k_1^{+} - k_2^{+}) + \frac{1}{2}z = \frac{1}{2}(x_1 - x_2 + z) \tag{B16}$$

and we arrive at

$$\begin{aligned} \psi_{BS}^{\ell.c.}(x_i, \vec{k}_\perp) &= -2\pi \cdot \int_{-1}^{+1} dz \frac{1-|z|}{(k_\perp^2 + m^2 + \frac{1}{4}M^2(z^2-1))^2}\delta(x_1 - x_2 + z) \\ &= +8\pi \frac{1}{x_1 x_2 \left[M^2 - \frac{k_\perp^2 + m^2}{x_1 x_2}\right]^2 (1 + |x_1 - x_2|)} \end{aligned} \tag{B17}$$

which coincides with the rough approximation to the solution of (A.6) found by Karmanov.

We mention here, that in another extreme case, when the mass of the bound system vanishes, the spectral function is also known and is given[8] by $h(z, M^2) = 1 - z^2$. The light-cone projection of the Bethe-Salpeter wave function is then easily evaluated and takes the form

$$\psi_{BS}^{\ell.c.}(x_i, \vec{k}_\perp) = 8\pi \frac{1}{x_1 x_2 \left(M^2 - \frac{m^2 + k_\perp^2}{x_1 x_2}\right)^2} . \tag{B18}$$

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FIGURE CAPTIONS

1. The Light Cone Ladder Approximation (LCLA) for the bound state wavefunction.

2. The structure of the light-cone bound state equation (B.12).

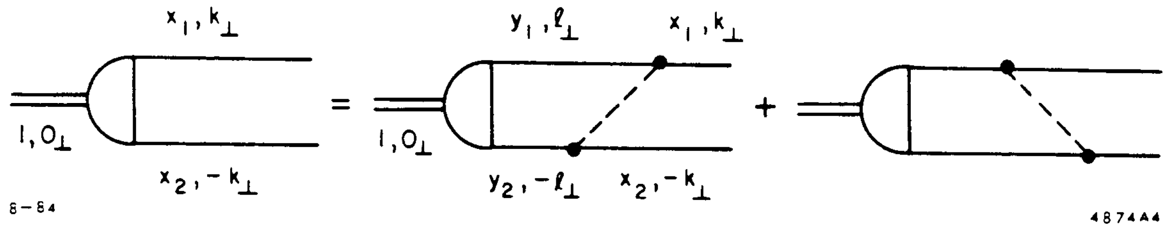

Fig. 1

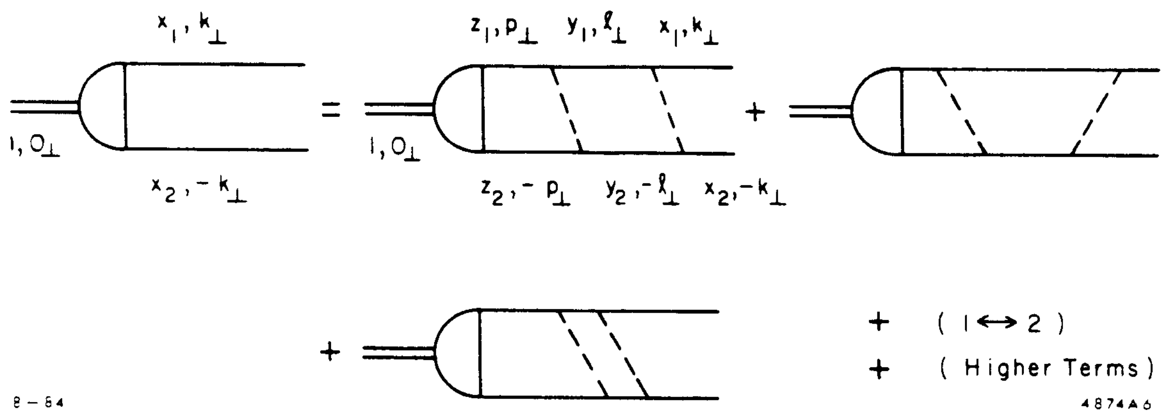
x₁, k⊥
1, 0⊥
x₂, − k⊥
=
z₁, p⊥ y₁, ℓ⊥ x₁, k⊥
1, 0⊥
z₂, − p⊥ y₂, − ℓ⊥ x₂, − k⊥
+
+
+ (1 ↔ 2)
+ (Higher Terms)
8 − 84
4874A6

Fig. 2