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THE EVALUATION OF MASSIVE MULTILOOP FEYNMAN DIAGRAMS*

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ABSTRACT

Within the framework of dimensional regularization, a new technique for evaluating multiloop propagator-type Feynman diagrams involving massive propagators is outlined and discussed.

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1. INTRODUCTION

Remarkable progress has recently been made in the analytic computation of multiloop Feynman diagrams. In particular, multiloop Feynman integrals have been evaluated via Gegenbauer polynomial x-space and p-space techniques (GPXT and GPPT respectively) [1] and via integration by parts [2]. The GPPT has been used with success in computing

- the $O(\alpha^3)$ photon renormalization constant [3];
- the hyperfine splitting in positronium [4].

Alternatively, GPXT has found even more usefulness in computations. It's conquests include

- the 3 loop contribution to $\sigma_{tot}(e^+e^- \rightarrow \text{hadrons})$ [5];
- the 4 loop β function in ϕ^4 theory [6].

As a consequence of the translational invariance (in x or p space) of dimensionally regulated integrals, a straightforward recursion formula has been derived for multiloop massless propagator-type graphs [2]. This method is defined from a simple integration by parts. It has been used to reduce the complexity of many of the cumbersome GPXT calculations. In particular, the 5 loop contribution to the anomalous dimension in ϕ^4 theory has been evaluated analytically via the integration by parts technique [7].

Currently, the computation of a given multiloop massless propagator-type diagram involves using the integration by parts method to reduce it to a sum of primitive plus nonprimitive[1] integrals. The remaining nonprimitive terms are much easier to evaluate via GPXT than was the original expression.

The general applicability of the recursive technique has been restricted because of the requirement that all masses in a given integral are to be ignored. Levine and Roskies [1] have used GPPT alone to evaluate massive sixth-order

[1]A particular integral is called nonprimitive if it cannot be evaluated via a repeated application of a 1 - loop formula.

vertex graphs in QED. However, in their calculation they set the photon mass to zero and as such avoid the complicated square root integrands normally associated with massive GPPT. Recently, the author has discovered a way of extending the integration by parts method to include mass. The method has been used with considerable success in computing two loop massive gluonic corrections to the electromagnetic polarization tensor [8].

Because this new technique complements the recursive method it reduces the role of GPPT or GPXT to integrals which are easy to evaluate. It should be said at the outset that one possible disadvantage of the method, as currently formulated, is that it only allows one to isolate particular $[\ell n(q^2/m^2)]^a/(q^2/m^2)^b$ behavior (a and b positive or negative constants).

2. THE MELLIN TRANSFORM TECHNIQUE (MTT)

Consider a given finite scalar[2] multiloop Feynman integral with N massive and M massless propagators. The propagators all have arbitrary multiplicity. Following Itzykson and Zuber [9], we write in n dimensional Euclidean space

$$I(q) = \int \prod_{\ell n=1}^{N} \frac{d^n k_\ell}{(2\pi)^n} \left(\frac{1}{k_\ell^2 + m^2}\right)^{\rho_\ell} \prod_{\ell'=1+N}^{I} \frac{d^n k_{\ell'}}{(2\pi)^n} \left(\frac{1}{k_{\ell'}^2}\right)^{\rho_{\ell'}} \times \prod_{v=1}^{V} (2\pi)^n \, \delta^n \left(t_v - \sum_{\ell=1}^{I} \epsilon_{v\ell} \, k_\ell\right) \tag{1}$$

where we assume one external momentum q^2 and one mass scale m. V and I

[2]The requirement that the lines be scalar is only a trivial restriction used to simplify the presentation. The fact that the integral be finite is a necessary condition for the method to work. Therefore, the MTT is to be applied only after appropriate renormalizations have been performed. This point will be elaborated on later in the present work. It is assumed $I(q)$ does not possess an expansion about $m = 0$.

denote the number of vertices and internal lines respectively.

$$\epsilon_{v\ell} = \begin{cases} (\text{-})1 & \text{if the } vth \text{ vertex is the starting (ending)} \\ & \text{point of the line } \ell \\ 0 & \text{if } \ell \text{ is not incident on } v \end{cases}$$

t_v denotes the sum of incoming momenta at the vertex v and an overall momentum conservation is assumed. Defining the dimensionless variables

$$k_\ell^2 = \frac{k_\ell^2}{q^2} \qquad s = \frac{q^2}{m^2} \qquad T_v = \frac{t_v}{(q^2)^{1/2}}$$

eq. (1) becomes

$$\begin{aligned} I(s) = (m^2)^{\frac{Ln}{2}}\, s^{\frac{Ln}{2}} \int \prod_{\ell=1}^{N} \frac{d^n K_\ell}{(2\pi)^n}\, (m^2)^{\rho_\ell} \left(\frac{1}{1+sK_\ell^2}\right)^{\rho_\ell} \\ \times \prod_{\ell'=1+N}^{I} \frac{d^n K_{\ell'}}{(2\pi)^n}\, s^{-\rho_{\ell'}}\, (m^2)^{-\rho_{\ell'}} \left(\frac{1}{K_{\ell'}^2}\right)^{\rho_{\ell'}} \\ \times \prod_{v=1}^{V} (2\pi)^n\, \delta^n(T_v - \sum_{\ell=1}^{I} \epsilon_{v\ell}\, K_\ell) \end{aligned} \tag{2}$$

where L denotes the number of loops.

The product of massive propagators may now be expanded in terms of partial fractions.

$$\begin{aligned} I(s) = (m^2)^{\frac{Ln}{2}}\, s^{\frac{Ln}{2}} \int \frac{d^n K_1}{(2\pi)^n} \cdots \frac{d^n K_N}{(2\pi)^n}\, (m^2)^{-\rho_1-\ldots-\rho_N} \\ \left\{ \frac{A_{\rho_1}(K_1^2,\ldots K_N^2)}{(1+sK_1^2)^{\rho_1}} + \ldots + \frac{A_1(K_1^2,\ldots K_N^2)}{(1+sK_1^2)} + \ldots + \frac{J_{\rho_j}(K_1^2,\ldots K_N^2)}{(1+sK_j^2)^{\rho_j}} \right. \\ \left. + \ldots + \frac{J_1(K_1^2,\ldots K_N^2)}{(1+sK_1^2)} + \ldots \right\} \prod_{\ell'=1+N}^{I} \frac{d^n K_{\ell'}}{(2\pi)^n}\, s^{-\rho_{\ell'}}\, (m^2)^{-\rho_{\ell'}} \left(\frac{1}{K_{\ell'}^2}\right)^{\rho_{\ell'}} \\ \times \prod_{v=1}^{V} (2\pi)^n\, \delta^n\, (T_v - \sum_{\ell=1}^{I} \epsilon_{v\ell}\, K_\ell) \end{aligned} \tag{3}$$

where the coefficients $A_{\rho_1}(K_1^2, ...K_N^2)$, ... are rational functions of $K_1^2, ...K_N^2$ [10].

The Mellin transform [11] and its inverse are defined from

$$\tilde{I}(\alpha) = \int_0^\infty ds\ s^{-\alpha-1}\ I(s) \tag{4a}$$

$$I(s) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} d\alpha\ s^\alpha\ \tilde{I}(\alpha) \tag{4b}$$

(σ is chosen so that $\tilde{I}(\alpha)$ is analytic along the contour). Performing the s integration over eq. (3),

$$\begin{aligned}
\tilde{I}(\alpha) = \int \frac{d^nK_1}{(2\pi)^n} \cdots \frac{d^nK_N}{(2\pi)^n} \cdots\cdot \frac{d^nK_I}{(2\pi)^n} (m^2)^{\frac{Ln}{2}-\rho_1-...-\rho_I} \left(\frac{1}{K_{N+1}^2}\right)^{\rho_{N+1}} \cdots\cdot \left(\frac{1}{K_I^2}\right)^{\rho_I} \\
\times \Bigg\{ \frac{A_{\rho_1}(K_1^2...K_N^2)\Gamma(\frac{Ln}{2} - \rho_{N+1} - ... - \rho_I - \alpha)\Gamma(\rho_1 + \alpha + \rho_{N+1} + ... + \rho_I - \frac{Ln}{2})}{(K_1^2)^{\frac{Ln}{2}-\rho_{N+1}-...-\rho_I-\alpha}\ \Gamma(\rho_1)} \\
+ \cdots\cdot \\
+ \frac{J_1(K_1^2...K_N^2)\Gamma(\frac{Ln}{2} - \rho_{N+1} - ... - \rho_I - \alpha)\Gamma(1 + \alpha + \rho_{N+1} + ... + \rho_I - \frac{Ln}{2})}{(K_j^2)^{\frac{Ln}{2}-\rho_{N+1}-...-\rho_I-\alpha}} + \cdots\cdot \Bigg\} \\
\times \prod_{v=1}^{V} (2\pi)^n\ \delta^n\ (T_v - \sum_{\ell=1}^{I}\ \epsilon_{v\ell}\ K_\ell)
\end{aligned} \tag{5}$$

The massive Feynman integral has been transformed into a sum of massless α-space Feynman integrals. The mass parameter (or equivalently s), has been converted to α; a power of the massless propagator. The one complication which remains has to do with the coefficients $A_{\rho_1}(K_1^2...K_N^2)$, ... Aside from trivial numerator factors,

$$J_{\rho_j}(K_1^2...K_N^2) \sim \frac{1}{(K_1^2 - K_j^2)^{\text{power}_1} \cdots\cdot (K_N^2 - K_j^2)^{\text{power}_N}}$$

Therefore, for $N \geq 2$ the given integral $I(s)$ does not strictly have a massless equivalent in the α-plane. However, unlike $I(s)$, a recursive formula defined via an integration by parts can be defined for $\tilde{I}(\alpha)$. This statement we offer without proof. It is certainly true for $N = 1$ and a two loop example with $N = 2$ will be outlined in the next section.

Using the integration by parts method in conjunction with GPXT allows one to evaluate all K_ℓ integrations. This part of the calculation is by no means trivial but the technique is well covered in the existing literature and therefore it will not be discussed here.

Once all momentum space integrations have been performed the inverse transform can be applied to yield $I(s)$. In general $\tilde{I}(\alpha)$ will possess poles in the α plane coming from the Γ functions that arise from K_ℓ and s integrations. By virtue of the identity

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{d\alpha s^\alpha}{(\alpha + a)^{b+1}} = \frac{1}{\Gamma(b+1)} \frac{(\ell n\ s)^b}{s^a} \tag{6}$$

an n^{th} order singularity at $\alpha = -a$ will become in the s plane

$$\frac{1}{\Gamma(n)} \frac{(\ell n\ s)^{n-1}}{s^a} \tag{7}$$

The fact that $I(s)$ is finite means the singularities in $\tilde{I}(\alpha)$ are due solely to $(\ell n\ s)^b/s^a$ terms. If $I(s)$ had $1/n-4$ divergences, it wouldn't be clear whether the poles in the α plane are reflective of this singularity or $(\ell n\ s)^b/s^a$. In addition, if $I(s)$ is finite, the transforms eqs. (4a) and (4b) are mathematically well defined functions in the $n \to 4$ limit. It may be that this finiteness restriction is not necessary but it makes calculations simpler to interpret if the $1/n-4$ divergences are removed prior to computing $\tilde{I}(\alpha)$.

Because of the use of dimensional regularization in handling all K_ℓ integrations there will be poles in $\tilde{I}(\alpha)$ at $\alpha = -[m + K(n-4)]$ (m and integer and K a constant) and not only at $\alpha = -m$. These poles in the α-plane become in the

s-plane

$$\frac{1}{\Gamma(b+1)} \frac{(\ell n\ s)^b}{s^m} \left[1 - K(n-4)\ell n\ s + ...\right]$$

Therefore, in the neighborhood of $\alpha = -m$ (the neighborhood of $\alpha = -m$ is defined by $\alpha = -[m + K(n-4)]$) there exist poles all contributing to the same $1/s^m$ behavior. However, the residues at each singularity must be calculated separately and the $n \rightarrow 4$ limit taken at the end. The pole location and order determines the $\ell n\ s/s$ behavior, the residue at the pole determines the numerical coefficient of $\ell n\ s/s$. In general, in the vicinity of $\alpha = -m$

$$\tilde{I}(\alpha) = \frac{R_0}{(\alpha+m)^{b+1}} + \frac{R_1}{(\alpha + m + K_1(n-4))^{b+1}} + ... \\ + \frac{R_s}{(\alpha + m + K_s(n-4))^{b+1}} \tag{8}$$

with residues going like

$$R_\ell = \sum_{j=0}^{L} \frac{a_j^\ell}{(n-4)^j}$$

Performing the inverse transform on eq. (8) yields

$$I(s) = \frac{1}{\Gamma(b+1)} \frac{(\ell n\ s)^b}{s^m} \sum_{\ell=0}^{s} \sum_{j=0}^{L} (1 - K_\ell(n-4)\ell n\ s + ...) \frac{a_j^\ell}{(n-4)^j} \tag{9}$$

Because $\frac{1}{n-4}$ factors coming from the residue tend to cancel $(n-4)$ factors coming from the $s^{-K(n-4)}$ expansion it should be obvious why the $n \rightarrow 4$ limit is to be taken at the finish of the calculation. In addition, because $I(s)$ is finite as $n \rightarrow 4$, the $\frac{1}{n-4}$ divergences that appear in the residues will completely cancel in eq (9). For an explicit example of this residue computation and cancellation see [8].

As mentioned in the introduction, this Mellin transform technique has been used in the $O(\alpha_s)$ evaluation of the massive gluonic correction to the electromagnetic polarization tensor. In particular, the $1/s^2$ term has been computed. The result is

$$\Pi(s) = \frac{1}{s^2} \cdot \frac{5\alpha_s}{1152\pi^3} \left\{ 48\varsigma(3) + 81 - \frac{8\pi^2}{3} + 4\ell n\ s - 8(\ell n\ s)^2 \right\}$$

($\varsigma(x)$ is the Riemann Zeta function).

3. $N = 2$ TWO - LOOP EXAMPLE

In this section we will briefly outline the Mellin transform technique applied to a two loop nonprimitive integral. The evaluation of primitive expressions and the computation of residues will not be given. It is felt that discussions of the GPXT applied to primitive integrals along with the associated residue computation all exist in the present literature [1],[8].

Define $I(q)$ by

$$I(q) = \int \frac{d^n\ p\ d^n\ k}{(p^2 + m^2)(k^2 + m^2)(p - q)^2(k - q)^2(p - k)^2}$$

Defining the dimensionless variables $P,\ K,\ s$

$$I(s) = (m^2)^{n-5} \int \frac{d^n\ P\ d^n\ K}{(1 + sP^2)(1 + sK^2)(P - 1)^2(K - 1)^2(P - K)^2} \qquad (10)$$

Expanding $\frac{1}{(1+sP^2)(1+sK^2)}$ in terms of partial fractions, and performing the Mellin transform over s yields for eq. (10),

$$\tilde{I}(\alpha) = (m^2)^{n-5} \int \frac{d^n\ P\ d^n\ K}{(P - 1)^2(K - 1)^2(P - K)^2}\ \Gamma(n - \alpha - 3)\Gamma(4 + \alpha - n)\ \frac{1}{(P^2 - K^2)}$$
$$\times \left[\frac{1}{(P^2)^{n-\alpha-4}} - \frac{1}{(K^2)^{n-\alpha-4}}\right]$$

The momentum space integrations may now be performed. Define the recursion formula from

$$O = \int d^n\ P\ d^n\ K\ \frac{\partial}{\partial P_\mu} \left\{ \frac{(P - K)_\mu}{(P - 1)^2(K - 1)^2(P - K)^2(P^2 - K^2)(K^2)^\gamma} \right\} \qquad (11)$$

Let

$$F(\gamma) = \int \frac{d^n\, P\, d^n\, K}{(P-1)^2(K-1)^2(P-K)^2(P^2-K^2)(K^2)^\gamma}$$

where

$$\tilde{I}(\alpha) = -2F(n-\alpha-4)$$

Evaluating eq. (11) the recursion formula is

$$(n-4)F(\gamma) = \int d^n\, P\, d^n\, K \left\{ \frac{1}{(P-1)^4(K-1)^2(P^2-K^2)(K^2)^\gamma} + \frac{1}{(P-1)^2(K-1)^2(P^2-K^2)(K^2)^\gamma} - \frac{1}{(P-1)^4(P-K)^2(P^2-K^2)(K^2)^\gamma} \right\} \tag{12}$$

the remaining integrals in eq. (12) are primitive and can be evaluated via GPPT or GPXT. Once this has been performed, $\tilde{I}(\alpha)$ can be analyzed for its poles in the α plane. In the neighborhood of the singularity of interest, residues can be computed and the result inverse transformed via eq. (9). The $n \to 4$ limit is taken at the end.

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