SLAC-PUB-3367
July 1984
(T)

QUANTUM COSMOLOGY IN 2 + 1 AND 3 + 1 DIMENSIONS*

TOM BANKS
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

W. FISCHLER
Department of Physics
University of Texas, Austin, Texas 78712

and

LEONARD SUSSKIND
Department of Physics
Stanford University, Stanford, California 94305

Abstract

We study the Wheeler DeWitt equations for the wave function of the universe in $2+1$ and $3+1$ dimensions. Perturbation methods are developed and the first few orders are explicitly calculated for the case of an inflating universe. We find that early quantum fluctuation can produce large scale inhomogeneities in the global structure of the universe at later times.

(submit Nuclear Physics B)

---

*Work supported in part by the Department of Energy, contract DE-AC03-76SF00515, (SLAC), and NSF contract NSF-PHY 83 10654.

Introduction

Quantization of the gravitational field in a flat background has led to a systematic though non-renormalizable perturbation theory for S-matrix elements. For the purpose of laboratory particle physics such a description may eventually prove adequate.

The application of quantum mechanics to the universe as a whole is much more problematic. In particular conceptual problems arise which concern the operational meaning of the coordinate labels of space-time points. In classical gravitation theory we are free to imagine physical coordinate frames composed of rods, clocks, dust particles or other material systems. So long as these objects are sufficiently light their influence on the gravitational field is negligible. In contrast to this, the coordinate frames used to anchor the space time points in flat space quantum theory must be infinitely heavy if they are not to fluctuate. No fundamental principles of special relativity or quantum mechanics forbid such heavy frames arbitrarily weakly coupled to the system under consideration.

The combination of quantum mechanics and gravitation makes it impossible to introduce non-fluctuating material coordinates without disturbing the system or violating the laws of nature. The only alternative is a coordinate independent description. Consider then how coordinate time may be eliminated. The universe is described by a wave function Ψ depending on spatial geometry and matter fields [1,2]. Some dynamical variable such as the total spatial volume may be chosen to replace time. Instead of asking how the probability for some field ϕ varies with time we ask for the conditional probability that ϕ has a specific value given that the universe has volume V. This information is contained in the wave function $\psi(V, \phi, \ldots)$ similarly the wave function should not refer to particular spatial coordinates.

In the case of flat space asymptotic conditions the problem of physically realizable coordinates is less serious if physical questions are restricted to scattering amplitudes. In 3+1 dimensions gravitational influences die off sufficiently rapidly so that asymptotic material coordinates can be introduced without disturbing the interactions between particles. In quantum cosmology we have no such recourse and must be content with a coordinate free description. Such a description can be based on the formulation of Wheeler and Dewitt [1,3]. This paper is primarily concerned with the W-D equations and their perturbative solution.

1. Canonical Formalism

We begin with a review of 2+1 dimensional general relativity and its canonical formalism [4]. Spacetime is described by the 3-bein e^{α}_{μ} where α and μ refer to tangent space and coordinate space respectively. We shall choose coordinates X^0, X^1, X^2 so that the "shifts" vanish [4]

$$g_{01} = g_{02} = 0 \tag{1.1}$$

The time like coordinate will be denoted t and the space-like tangent and coordinate indecies a, m run from 1 to 2. In such coordinates we may choose the 3-bein to satisfy

$$\begin{aligned} e^0_0 &= (g_{00})^{\frac{1}{2}} \\ e^0_m &= e^a_0 = 0 \end{aligned} \tag{1.2}$$

The remaining components form a 2-bein e^a_m. The matrix inverse of e^a_m is called $\tilde{e}^m_a$ and the determinant of e^a_m is called $|e|$. Thus

$$|e| = \frac{1}{2}\, \epsilon^{mn}\, \epsilon_{ab}\, e^a_m\, e^b_n \tag{1.3}$$

$$\tilde{e}^m_a = \frac{\epsilon^{mn}\, \epsilon_{ab}\, e^n_b}{|e|} \tag{1.4}$$

The Einstein Lagrangian with cosmological constant λ is

$$\mathcal{L}_E = \frac{1}{g^2} |e| e^0_0\, R_3 - \frac{\lambda}{g^2}\, |e|\, e^0_0 \tag{1.5}$$

where g is a dimensional coupling with units of $(\text{mass})^{-\frac{1}{2}}$ and R_3 is the 3-dimensional curvature scaler.

The Lagrangian (1.5) contains second time derivatives of e^a_m. These may be eliminated by adding a total time derivative to $\mathcal{L}$. The result is given by

$$\mathcal{L} = \frac{1}{g^2}\left[\frac{-4|\dot{e}|}{e^0_0} + |e|e^0_0 R_2\right] - \frac{\lambda e^0_0|e|}{g^2} \tag{1.6}$$

where $|\dot{e}|$ is the determinant of the time derivative of e^a_m and R_2 is the intrinsic 2-space curvature given by

$$|e|\, R_2 = -\frac{\partial}{\partial X^m}\left[\frac{e^a_n}{|e|}\cdot\frac{\partial e^a_s}{\partial X^r}\right]\epsilon^{mn}\,\epsilon^{rs} \tag{1.7}$$

From (1.6) we may derive a Hamiltonian density given by

$$e^0_0\, H(x) = e^0_0\left\{-\frac{g^2}{4}\,|\pi| - \frac{|e|\,R_2}{g^2} + \frac{\lambda}{g^2}\,|e|\right\} \tag{1.8}$$

Here π^m_a is the canonical conjugate to e^a_m.

The classical field equations are:

1) Euler Lagrange equations of motion for e^a_m.

$$\frac{d}{dt}\,\frac{\dot{e}^b_m}{e^0_0} = \frac{\lambda}{4}\,e^0_0\,e^b_m \tag{1.9}$$

These are the space-space components of Einstein's equations.

If we further restrict the choice of coordinates so that $e^0_0 = 1$ (synchronous gauge) then e satisfies linear equations of motion

$$\ddot{e}^b_m = \frac{\lambda}{4}\,e^b_m \tag{1.10}$$

and the general solution is

$$e^b_m(x,t) = a^b_m(x)e^{\frac{\sqrt{\lambda}t}{2}} + b^b_m(x)e^{-\frac{\sqrt{\lambda}t}{2}} \tag{1.11}$$

2) Vanishing of the generators of local 2-space coordinate transformations. Under an infinitesimal 2-space coordinate transformation

$$X^m \to X^m + f^m(x) \tag{1.12}$$

the 2-bein transforms as

$$\delta e_m^a = \frac{\partial e_m^a}{\partial X^n} f^n + \frac{\partial f^n}{\partial X^m} e_n^a \tag{1.13}$$

The generator is given by

$$\begin{aligned} P_f &= \int (\delta e_m^a) \pi_a^m \, d^2x \\ &= \int \left\{ \frac{\partial e_m^a}{\partial X^n} \pi_a^m - \frac{\partial e_n^a \pi_a^m}{\partial X^m} \right\} f^n \\ &= \int P_n(x) \, f^n(x) \end{aligned} \tag{1.14}$$

The time-space components of the classical field equations are equivalent to

$$0 = P_n(x) = \frac{\partial e_m^a}{\partial X^n} \pi_a^m - \frac{\partial e_n^a \pi_a^m}{\partial X^m} \tag{1.15}$$

3) Vanishing of generators of local tangent space rotations. Under a local 2-space rotation of the tangent space e_m^a transforms as

$$\delta e_m^a = \theta(x) \epsilon^{ab} e_m^b$$

The local generator J is given by

$$0 = J = \frac{(\delta e_m^a) \, \pi_a^m}{\theta} = \epsilon^{ab} e_m^b(x) \pi_a^m(x) \tag{1.16}$$

The vanishing of J is automatic in the usual formulation in terms of the metric g_{mn}.

4) Vanishing of $H(x)$. This is the equation obtained by variation with respect to $e_0^0(x)$. From (1.8) we get

$$H(x) = -g^2 \frac{|\pi|}{4} - \frac{|e|}{g^2} R_2 + \frac{\lambda |e|}{g^2} = 0 \tag{1.17}$$

The vanishing of the Hamiltonian is peculiar to theories which are invariant with respect to time reparametrization [1,2]. Equation (1.17) is the time-time component of Einstein's equations.

The simplest classical solution is open-infinite exponentially expanding space.

$$e^a_m(x,t) = \delta^a_m a(t)$$
$$e^0_0 = 1 \tag{1.18}$$

where $a(t)$ is given by

$$a(t) = c \ \exp \frac{\sqrt{\lambda}\, t}{2} \tag{1.19}$$

The same solution can also be interpreted as a closed universe with the topology of a torus. In this case the system is assumed periodic in x_1 and x_2.

Another class of solutions is given by [5]

$$e^a_m(x,t) = e^{\frac{\sqrt{\lambda}t}{2}} \delta^a_m + M e^{-\frac{\sqrt{\lambda}t}{2}} \tag{1.20}$$

where M is a traceless symmetric matrix satisfying

$$TrM^2 = 2$$

These solutions have the peculiar property that $|e| = 0$ at $t = 0$. This implies a collapsed configuration of zero 2-volume.

Finally we can consider closed spherical space solutions. Let $S^a_m(x)$ be a 2-bein describing a unit 2-sphere. Then solutions of the form

$$e^a_m = A e^{\frac{\sqrt{\lambda}t}{2}} S^a_m + B\, e^{-\frac{\sqrt{\lambda}t}{2}} S^a_m \tag{1.21}$$

exist. A and B must be chosen so that (1.17) is satisfied. Thus we require

$$-4|\,\dot{e}\,| + \lambda|e| - |e|\, R_2 = 0 \tag{1.22}$$

For a sphere of radius $Ae^{\frac{\sqrt{\lambda}t}{2}} + Be^{-\frac{\sqrt{\lambda}t}{2}}$ the curvature R_2 is given by

$$\left(Ae^{\frac{\sqrt{\lambda}t}{2}} + Be^{-\frac{\sqrt{\lambda}t}{2}} \right)^{-2}$$

Thus

$$-4\left(A\frac{\sqrt{\lambda}}{2}e^{\frac{\sqrt{\lambda}t}{2}} - B\frac{\sqrt{\lambda}}{2}e^{-\frac{\sqrt{\lambda}t}{2}}\right)^2|S| + \lambda\left(Ae^{\frac{\sqrt{\lambda}t}{2}}Be^{-\frac{\sqrt{\lambda}t}{2}}\right)^2|S| - |S| = 0 \tag{1.23}$$

This is solved by $A = B = \sqrt{\frac{1}{2\lambda}}$

$$e(x,t)^a_m = S^a_m(x)\sqrt{\frac{2}{\lambda}}\ cosh\frac{\sqrt{\lambda}\,t}{2} \tag{1.24}$$

Equation (1.24) describes a cosmological "bounce" solution in which the universe initially collapses to a radius of order $\lambda^{-\frac{1}{2}}$ and subsequently expands.

In 3 + 1 and higher dimensions solutions of the above type can be modified by adding gravitational waves. In 2 + 1 dimensions the constraints are strong enough to prevent the propagation of real perturbations.

2. The Space Of States

According to Wheeler and DeWitt [1] the quantum wave function of the universe is a functional of intrinsic 2-geometries. A vector in the state space is a function of the infinite collection of geometric invariants needed to specify a 2-geometry. Alternately we may consider vectors to be functionals of the 2-bein $e^a_m(x)$ taking care to make sure that $\psi(e^a_m(x))$ is invariant under coordinate transformations and tangent space rotations of 2-space. Accordingly the physical subspace is defined by

$$\psi(e(x)) = \psi(e'(x)) \tag{2.1}$$

if $e(x)$ and $e'(x)$ describe the same intrinsic 2-geometry. Eq. (2.1) can be written in infinitesemal form

$$\begin{aligned} P_m\ \psi(e) &= 0 \\ J\ \psi(e) &= 0 \end{aligned} \tag{2.2}$$

where P_m and J are given by (1.15) and (1.16) with

$$\pi^m_a = -i\ \frac{\delta}{\delta e^a_m} \tag{2.3}$$

In order to ensure the coordinate independence of the state space we must also require the inner product to be invariant. This is a subtle and difficult problem which is closely connected to the way the theory is regulated. Most of this paper does not depend on these subtleties but for definiteness we shall assume a particular inner product [1]. Thus let us define

$$\langle\psi|\phi\rangle = \int \prod_{x,a,\mu} de^a_m(x)\ \mu(e(x))\psi^*(e)\ \phi(e) \tag{2.4}$$

where $\mu(e)$ is to be determined. Consider a coordinate transformation which leaves the point x fixed. The 2-bein at point x transforms as

$$e^a_m(x)' = \frac{\partial X^n}{\partial X'^m}\ e^a_n(x) \tag{2.5}$$

The Jacobian of this transformation is

$$det\ \frac{\delta e'^a_m}{\delta e^b_n} = det\Big(\delta^a_b\ \frac{\partial X^n}{\partial X'^m}\Big) = det\Big(\frac{\partial X^n}{\partial X'^m}\Big)^2 \tag{2.6}$$

To compensate this Jacobian we choose

$$\mu(e) = \prod_x\ |e|^{-2} \qquad (2.7$$

Thus

$$\langle\psi|\phi\rangle = \int\ \prod\frac{d\,e^a_m(x)}{|e(x)|^2}\ \psi^*\phi \tag{2.8}$$

With respect to this measure some of the usual operators are not hermitian. Any real valued functional of $e(x)$ is hermitian but the conjugate momenta

$$\pi^m_a(x) = -i\frac{\delta}{\delta_{e^a_m}} \tag{2.9}$$

are not. A particularly simple set of operators given by

$$D^a_b(x) = e^a_m(x)\pi^m_b(x) = -ie^a_m(x)\frac{\delta}{\delta e^b_m(x)} \tag{2.10}$$

are hermitian. Alternatively we can define

$$D_n^m = e_n^a \ \pi_a^m \tag{2.11}$$

which is also hermitian. The D_a^b and D_n^m are related by

$$\begin{aligned} D_n^m &= e_n^a \ \tilde{e}_b^m \ D_a^b \\ D_a^b &= e_m^b \ \tilde{e}_a^n \ D_n^m \end{aligned} \tag{2.12}$$

The D_n^m (D_b^a) are generators of the group of general linear transformations on the indecies m (a). The canonical commutation relations between e and π are replaced by

$$\begin{aligned} &[D_b^a,\ D_n^m] = 0 \\ &[D_b^a(x), D_d^c(y)] = i\delta(x-y)\ (\delta_d^a D_b^c(x) - \delta_b^c D_d^a(x)) \\ &[D_n^m(x), D_s^r(y)] = i\delta(x-y)(\delta_n^r D_s^m - \delta_s^m D_n^r) \\ &[e_m^a(x), D_c^b(y)] = i\delta(x-y)\delta_c^a \, e_m^b(x) \\ &[e_m^a(x), D_s^r(y)] = i\delta(x-y)\delta_m^r e_s^a(x) \end{aligned} \tag{2.13}$$

The antisymmetric part of D_b^a is the tangent space rotation generator

$$J = \epsilon^{ab} \ D_b^a \tag{2.14}$$

and annihilates all physical states.

The field variables can be divided into two mutually commuting sets. The first set consists of the trace of D and the determinant of $|e|$.

$$D \ = \ D_a^a \ = \ D_m^m \tag{2.15}$$

The second set contains the traceless part of D called $\bar{D}$ and the part of e which has $det = 1$. Thus define

$$\begin{aligned} \bar{D}_b^a &= D_b^a - \frac{1}{2} \ \delta_b^a \ D \\ \theta_m^a &= \frac{e_m^a}{|e|^{\frac{1}{2}}} \\ \bar{D}_n^m &= D_n^m - \frac{1}{2}\delta_m^n \ D \end{aligned} \tag{2.16}$$

Everything in set 1 commutes with everything in set 2. The D operator generates local Weyl dilatations which rescale $|e|$ while $\bar{D}^a_b$ and $\bar{D}^m_n$ generate the special linear transformations on tangent space and coordinate space.

3. The Wheeler DeWitt Equation

All that is needed now to write down the W-D equation is to expression $H(x)$ in terms of the D's and order it so that it is hermitian. According to (1.17) the classical density $H(x)$ consists of 3 terms

$$H(x) = \quad H_1 + H_2 + H_3 \tag{3.1}$$

$$H_1 = -\frac{g^2}{4} \frac{\epsilon_{mn}\epsilon^{ab}}{2} \pi^m_a \pi^n_b \tag{3.2}$$

$$H_2 = -\frac{|e|}{g^2} R_2 \tag{3.3}$$

$$H_3 = \frac{\lambda}{g^2} |e| \tag{3.4}$$

H_2 and H_3 are manifestly hermitian and may be taken over into the quantum theory without modification. H_1 however is not hermitian as written in (3.2). Using $\pi^m_a = \tilde{e}^m_b D^b_a$ and reordering the operators, H_1 can be made manifestly hermitian

$$H_1 = -D^a_b \frac{g^2}{8|e|} D^c_d \epsilon^{bd}\epsilon_{ac} \tag{3.5}$$

$$= -\frac{g^2}{8} D(x) \frac{1}{|e|} D(x) + \frac{g^2}{4|e|} \bar{D}^a_b \bar{D}^b_a \tag{3.6}$$

The W-D equation is obtained by using

$$D^a_b(x) = -ie^a_m(x) \frac{\delta}{\delta e^b_m(x)} \tag{3.7}$$

and writing

$$\left[-\frac{D^a_b}{8} \frac{g^2}{|e|} D^c_d \epsilon^{bd}\epsilon_{ac} + \frac{\lambda}{g^2}|e| - \frac{R_2}{g^2}\right]\psi(e) = 0 \tag{3.8}$$

Another form for the operator H_1 is given by

$$\begin{aligned} H_1 &= -g^2\Big(e^a_m \pi^m_b \frac{1}{|e|} e^c_n \pi^n_d\Big)\frac{\epsilon^{bd}\epsilon_{ac}}{8} \\ &= -g^2\Big\{\frac{|\pi|}{4} + \frac{i}{8}\frac{\delta(0)\, e^a_m}{|e|}\ \pi^m_a\Big\} \end{aligned} \tag{3.9}$$

In order to see how the formalism operates let us consider a simplified model in which the curvature term H_2 is dropped from the Wheeler DeWitt equation. In this case there is no coupling between spatially distinct points and the W-D equation is statisfied by continuous product wave functions

$$\psi^{\pm} = \exp \pm i2\frac{\sqrt{\lambda}}{g^2}\int\ d^2x\ |e| \tag{3.10}$$

To see that (3.10) solves the W-D equation we write

$$\begin{aligned} H\psi &= -g^2\frac{|\pi|}{4}\ \psi - \frac{ig^2}{8}\frac{\delta(0)e^a_m}{|e|}\ \pi^m_a\psi \\ &= \frac{g^2}{8}\frac{\delta}{\delta e^a_m}\frac{\delta}{\delta e^b_n}\epsilon^{ab}\epsilon^{mn}\psi - \frac{ig^2}{8}\frac{\delta(0)e^a_m}{|e|}\frac{\delta}{\delta e^a_m}\psi + \frac{\lambda}{g^2}\ |e|\psi \end{aligned} \tag{3.11}$$

Now note that

$$\frac{\delta}{\delta e^b_n}\ \psi^{\pm} = \pm 2i\,\epsilon_{bd}\epsilon^{nr}\ e^d_r\frac{\sqrt{\lambda}}{g^2}\ \psi^{\pm}$$

and that

$$\frac{\epsilon_{ab}\epsilon^{mn}\ \delta^2}{2\ \delta e^a_m \delta e^b_n}\ \psi^{\pm}\ =\ -\frac{\lambda}{4g^4}|e|\psi^{\pm} \pm i\frac{\sqrt{\lambda}}{2g^2}\delta(0)\psi^{\pm} \tag{3.12}$$

Substituting in (3.11) shows that $H\psi^{\pm} = 0$. It is interesting to see that the operator ordering (3.9) is required to cancel the singular term proportional to $i\delta(0)$.

Note that the $i\delta(0)$ term cannot be removed by simply subtracting a constant from H because it has opposite sign for ψ^+ and ψ^-. Unfortunately we do not know whether the simple prescription in (3.5) removes all such non hermitian singular effects when the curvature term in H is restored.

4. Linearized Theory

In this section we will linearize and solve the W-D equation to leading order in g. For simplicity we first consider the case of spatial geometries with the topology of a torus, or infinite plane. Our solution corresponds to perturbing about the classical solution given in eqs. (1.18) and (1.19).

We begin by Fourier decomposing e^a_m

$$e^a_m = \sum_{k=0}^{\infty} e^a_m(k) e^{ik\cdot x} \tag{4.1}$$

where the values of k are determined by the appropriate periodicity of the toroidal parameter space x. In the case of infinite space the sum is replaced by an integral.

The fourier amplitudes $e(k)$ will be separated into a background part which is not assumed small and a fluctuating part proportional to g. In particular the fourier amplitudes $e(k)$ for $k \neq 0$ are all assumed small. The amplitudes $e(k=0)$ are separated into a piece proportional to the unit matrix which is not assumed small and a traceless part which is. Thus we write

$$e^a_m = \delta^a_m\, a + \; g\,\bar{h}^a_m \; + \; k \sum_{k\neq 0} e^{ik\cdot x} h^a_m(k) \tag{4.2}$$

where $\bar{h}^a_m$ is traceless. Similarly the canonical momenta are decomposed

$$\pi^m_a = \frac{\delta^m_a Q}{2} + \frac{1}{g}\bar{q}^m_a + \frac{1}{g}\sum_{k\neq 0} e^{ik\cdot x} q^m_a(k) \tag{4.3}$$

The variables (Q, a), $(\bar{h}\,, \bar{q})$ and $(h(k),\, q(k))$ form conjugate pairs. We shall use a simplifying notation

$$\begin{aligned} \tfrac{1}{2}\epsilon_{ab}\epsilon^{mn}\; h^a_m(k) h^b_n(-k) &\equiv |h(k)| \\ \tfrac{1}{2}\epsilon^{ab}\epsilon_{mn}\; q^m_a(k) q^n_b(-k) &\equiv |q(k)| \end{aligned} \tag{4.4}$$

Let us assume that the wave function can be written in the form

$$\psi(h) = exp\Big[\sum_{n=1}^{\infty} g^n\, x_n(h,a)\Big]\psi_0(h,a)$$

where ψ_0 is gaussian in h (but not a) and x_n are polynomials in h. The procedure for finding ψ_0 and x_n is straightforward. We apply the fourier transformed constraint equations

$$\begin{aligned} H(k)\,\psi &= 0 \\ P(k)\,\psi &= 0 \end{aligned} \tag{4.6}$$

and require that they be true to order g^{n-1}. Surprisingly we find that this guarantees that the total hamiltonian

$$H_T = \int d^2x\, H(x) \tag{4.7}$$

annihilates ψ to order g^n. This is not totally unexpected for the following reason. Consider the Einstein equations expanded to a given power in the fluctuations h. The time-time and time-space equations are just the constraints (4.6). The space-space equations are the hamiltonian equations of motion for e_m^a. In order for these to be satisfied to order $n-1$ the total hamiltonian must be correct to order n.

The contribution of the curvature term $-\frac{|e|R_2}{g^2}$ to $H(k)$ is an infinite series in g. We shall need to know the first two terms of this series.

$$\begin{aligned} -\frac{|e|R_2}{g^2} = &-\frac{k^2 Trh(k) \;-\; k_a k_m h_m^a(k)}{ga} \\ &- \epsilon^{mn}\epsilon^{rs}\sum_{\ell p}\frac{k_m \ell_r}{a^2}\left[Tr\; h(p)h_s^n(\ell) - h_n^b(p)h_s^b(\ell)\right] \end{aligned} \tag{4.8}$$

where $\sum_{\ell p}$ means a constrained sum in which $\ell + p - k = 0$.

In analogy with (4.4) we will sometimes write

$$[k^2 Tr\, h(k) - k_a k_m h_m^a(k)]\,[k^2 Tr\, h(-k) - k_b k_n h_n^b(-k)] \;\equiv [k^2 Tr\, h - k_a k_m h_m^a(k)]^2 \tag{4.9}$$

In expanding H in powers of g the conjugate momentum Q should be assumed to be of order $1/g^2$. This can be seen by noting

$$Q = \frac{\partial L}{\partial \dot{a}} = -\frac{8\,\dot{a}}{g^2} \tag{4.10}$$

Thus to order g^0, $H(k)$ and $P(k)$ are given by

$$\begin{aligned}
H(k) =& \delta(k)\Big[-\frac{Q^2 g^2}{16} + \frac{\lambda a^2}{g^2}\Big] \\
& - \Big[g\,Q\,\frac{Tr\,q(k)}{8} - \frac{\lambda}{g}\,a Tr\,h(k) - \frac{k^2 Tr\,h(k) - k_a k_m h^a_m}{ga}\Big] \\
& - \Big\{\frac{|q|_k}{4} - \lambda |h|_k - \sum_{\ell p}\Big[\frac{k_m \ell_r}{a^2}\,(Tr\,h(p)\,h^n_s(\ell) - h^b_n(p) h^b_s(\ell))\Big\}\epsilon^{mn}\epsilon^{rs} \\
& - i\,g^2\,\frac{\delta(0)}{4a}\,Q
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
P_n =& i\Big\{g\,\frac{Q}{2}\Big[k_n h^m_m(k) - k_m h^m_n(k)\Big] - \frac{a}{g} k_m q^m_n(k)\Big\} \\
& + i\sum_{p,\ell}\Big\{\ell_n h^a_m(\ell) q^m_a(p) - k_m h^a_n(\ell) q^m_a(p)\Big\}
\end{aligned} \tag{4.12}$$

We assume that ψ_0 in eq. (4.5) is gaussian in the fluctuations h. Thus we write

$$\psi_0 = exp\Big[\frac{A(a)}{g^2} + \sum B^{mn}_{ab}(a,k) h^a_m(k) h^b_n(-k)\Big] \tag{4.13}$$

Applying H and collecting terms of order g^{-2} gives

$$\frac{1}{16\,g^2}\left(\frac{dA}{da}\right)^2 + \frac{\lambda a^2}{g^2} = 0 \tag{4.14}$$

or

$$A = \pm 2i\,a^2\sqrt{\lambda} \tag{4.15}$$

The ambiguity of sign in (4.15) is related to the possibilities of expanding or contracting universes. The two possibilities are related by time reversal. We shall choose the expanding solution which means the minus sign in (4.15).

Any linear superposition of the expanding and contracting wave functions is also a solution of the W-D equation. The assumption that we can treat each branch separately is equivalent to saying that time reversal (TCP) is spontaneously broken.

The general solution of $P_n\psi = 0$ to order $1/g$ is

$$\begin{aligned}\psi_0 = \exp\Big[-i\frac{A(a)}{g^2} - i\frac{\partial A}{\partial a^2}\sum_k |h(k)|\Big] \\ \times \exp\sum_k c(a,k)\,(k^2 Tr\,h(k) - k_a k_m h^a_m)^2\end{aligned} \tag{4.16}$$

Next, using eq. (4.15) and applying $H(k)\psi_0 = 0$ to order g^{-1} gives $c(a,k)$

$$c(a,k) = \frac{i}{\sqrt{\lambda}\,a^2k^2} \tag{4.17}$$

Thus giving

$$\begin{aligned}\psi_0 = \exp -2i\sqrt{\lambda}\Big(\frac{a^2}{g^2} + \sum_k |h|\Big) \\ \times \exp\frac{i}{\sqrt{\lambda}}\sum\frac{(k^2 Tr\,h - k_a k_m h^a_m)^2}{k^2a^2}\end{aligned} \tag{4.18}$$

The objects in the exponents of (4.18) are the leading contributions to geometric invariants. Consider first $a^2 + g^2\sum_k |h|$.

$$a^2 + g^2\sum_k |h| = \int d^2x\;|e| = V \tag{4.19}$$

where V is the spatial volume.

Next consider $\sum(k^2 Tr\,k - k_a k_m h^a_m)^2/k^2a^2$. From (4.8) we see that this is equivalent in leading order to

$$\frac{1}{g^2}\int |e(x)|\,dx\,|e(y)|\,dy\;R_2(x)R_2(y)G_0(x,y) \tag{4.20}$$

where $G_0(x, y)$ is the 2-dimensional green function given by

$$G_0(x, y) = \int \frac{e^{ik(x-y)}}{k^2} d^2k \ . \tag{4.21}$$

To make (4.20) into a geometric invariant we need only replace $G_0(x, y)$ by the covariant green function $G(xy)$ which satisfies

$$-\partial_m \ |e| \ g^{mn} \, \partial_n G(x, y) = \delta(x - y) \tag{4.22}$$

To zeroeth order in g there is no difference between G_0 and G. We therefore define the invariant

$$I = \int \ |e(x)| \ |e(y)| \, d^2x \ d^2y \ R(x) G(x, y) \, R(y) \tag{4.23}$$

The wave function (4.18) is then given by the leading contribution to

$$\psi_o = exp - i \Big[\frac{2\sqrt{\lambda}\, V}{g^2} - \frac{I}{\sqrt{\lambda}\, g^2} \Big] \tag{4.24}$$

The total Hamiltonian is given by

$$H_T = \Big[-\frac{Q^2 g^2}{16} + \frac{\lambda a^2}{g^2} \Big] - \sum_k \Big[\frac{|q|}{4} - \lambda |h| \Big] - i g^2 \frac{\delta(0)}{4} a Q \tag{4.25}$$

Applying H_T to (4.18) we find that

$$H_T \psi_o = 0 \ + \ \text{order} \ g \tag{4.26}$$

Let us now consider the significance of a wavefunction like (4.24). The first striking feature is that it is a pure phase. Therefore all configurations of 2-space are equally likely. In particular there is no suppression of highly irregular geometries with large values of the local curvature. This however does not mean that space-time is highly irregular. In fact it is a direct consequence of Einsteins equations that the space-time has uniform curvature. The point is that even in an absolutely flat space-time, space-like 2-dimensional surfaces can have arbitrarily complex structure. Our wavefunction has contributions from all space-like submanifolds most of which are very irregular. Indeed the condition that $H(x)=0$ means that the wavefunction is unchanged by an infinitesemal *local* time displacement. Obviously then it can not be concentrated at smooth geometries.

On the other hand invariant measures of the space time irregularities are concentrated around their classical values. For example the 3-space curvature is uniform and proportional to λ as a consequence of equations of motion and constraints. In Section 7 we will explicitly compute the fluctuation in a variable ϕ_H that was introduced [6] to study the gauge invariant fluctuation spectrum of inflating universes. We will find $\phi_H\psi=0$ in the matter free case.

To the next order in g the wavefunction has the form

$$\psi=\Big\{1+g\sum_{p,\ell}K^{nmr}_{bdf}(k,a)\;h^b_m(k)h^d_n(p)h^f_r(\ell)\Big\}\psi_0 \tag{4.27}$$

where $p+\ell+k=0$. The order g correction term has two contributions. The first is implicit in eq. (4.24) and comes from expanding the invariant I to order g. It is given by

$$\sum\frac{-i}{\sqrt{\lambda}}\left[\frac{\delta}{\delta h^a_m(k)}\;\frac{\delta h}{\delta h^b_n(p)}\;\frac{\delta}{\delta h^c_r(\ell)}\;\frac{I}{g^2}\right]_{h=0}h^a_m(k)h^b_n(p)h^c_r(\ell) \tag{4.28}$$

The remaining contribution can be determined by applying $H(k)=P(k)=0$ to zeroth order in g. We find

$$\begin{aligned}&\sum_{\ell,p}\Big\{K^{nmr}_{bdf}+\frac{i}{\sqrt{\lambda}}\left[\frac{\delta}{\delta h^a_m}\;\frac{\delta}{\delta h^b_n}\;\frac{\delta}{\delta h^c_r}\;\frac{I}{g^2}\right]_{h=0}\Big\}h^b_m h^b_n h^f_r\\ &=2\sum\frac{\ell^2p^2-(\ell\cdot p)^2}{3ia^2\lambda^{3/2}}\;\frac{1}{\ell^2p^2k^2}\;\frac{[|e|R]_k}{g}\;\frac{[|e|R]_\ell}{g}\;\frac{[|e|R]_p}{g}\end{aligned} \tag{4.29}$$

where $[|e|R]_k$ means the k^{th} mode of the linearized spatial curvature Equation (4.29) can be written as an integral over 2-space. Define I_0' by

$$\begin{aligned}I_0'=\int\;&d^2x\;d^2y\;d^2z\;[|e|R]_x\;[|e|R]_y[|e|R]_z\\ &\times[\partial_x^2\partial_y^2-(\partial_x\cdot\partial_y)^2]\;G_0(x,y,z)\end{aligned} \tag{4.30}$$

where

$$G_0(x,y,z) = \int du\, G_0(x,u) G_0(y,u) G_0(z,u) \tag{4.31}$$

Then (4.29) becomes

$$\frac{2}{3ia^2\lambda^{3/2}} \frac{I_0'}{g^3} \tag{4.32}$$

we can make I' manifestly invariant by replacing ∂ by covariant derivative ∇, go by the covariant green function and du by $\frac{du|e(u)|}{V}$.

$$I' = \int d^2x\, d^2y\, d^2z [|e|R]_x\, [|e|R]_y [|e|R]_z [\nabla_x^2 \nabla_y^2 - (\nabla_x \cdot \nabla_y)^2] G(x,y,z) \tag{4.33}$$

where

$$G(x,y,z) = \frac{1}{V} \int d^2u |e(u)|\; G(x,u) G(y,u) G(z,u) \tag{4.34}$$

Thus to order g the wave function has the form

$$exp \frac{-i}{g^2} \left[2V\sqrt{\lambda} - \frac{I}{\sqrt{\lambda}} + \frac{2I'}{3V\lambda^{3/2}} \right] \tag{4.35}$$

Notice that when written in terms of dimensionless invariants $I, I' \ldots$ the g^2 dependence factorizes and the dimensionless expansion parameter is $(V\lambda)^{-1}$. The series converges rapidly if the volume is large compared to the Hubble volume. Up to this point no divergences appear in the calculation of ψ. In the next order infinities arise which can be countered by renormalization of g and λ. Beyond that we do not know what happens but it is possible that an increasing number of counter terms is required at each order.

It is also possible that non-hermitian counterterms will be needed to eliminate some infinities. These would be analogous to the $i\delta(0)$ terms we encountered when we used the wrong ordering in the simplified model of section 3. They would indicate quantum corrections to the measure (2.7).

5. Scalar Matter Fields

We now add a minimally coupled scalar matter field. The matter field Lagrangian is

$$\mathcal{L}_m = e_0^0|e|g^{\mu\nu}\partial_\mu\phi\;\partial_\nu\phi = \frac{1}{e_0^0}|e|\,\dot{\phi}^2 - e_0^0|e|g^{mn}\partial_m\phi\partial_n\phi \tag{5.1}$$

which leads to the local Hamiltonian

$$e_0^0 H_m = e_0^0\Big[\frac{P_\phi^2}{4|e|} + |e|g^{mn}\partial_m\phi\;\partial_n\phi\Big] \tag{5.2}$$

where

$$P_\phi = \frac{\partial\mathcal{L}_m}{\partial\dot{\phi}} = 2\frac{\dot{\phi}}{|e|}e_0^0 \tag{5.3}$$

Expanding (5.2) in powers of g gives

$$\begin{aligned} H_m(x) &= \frac{P_\phi^2}{4a^2}(1 - ga^{-1}Trh) + (\nabla\phi)^2 \\ &+ g\frac{Trh}{a}(\nabla\phi)^2 - 2\frac{g}{a}h_n^m\partial_m\phi\partial_n\phi \\ &+ \text{order}\, g^2 \end{aligned} \tag{5.4}$$

Similarly the matter field contributes to the generators P_n.

$$(P_n)_{mat} = \partial_m\phi P_\phi \tag{5.5}$$

Since (5.4) and (5.5) vanish in order g^{-2} and g^{-1} eqs. (4.14) - (4.18) are unaffected. Accordingly we write the leading order wave function as

$$\begin{aligned} \psi_0 &= \exp\Big[-2i\sqrt{\lambda}\Big(a^2 + \sum_k |h(k)\Big)\Big] \\ &\times \exp\Big[\frac{i}{\sqrt{\lambda}\,a^2k^2}\sum_k(k^2Tr\,h - k_ak_mh_m^a)\Big]x(\phi,a) \end{aligned} \tag{5.6}$$

The matter wave function x can be obtained by requiring $H_{tot}\psi_0 = 0$ to zeroth order in g

$$
\begin{aligned}
H_{tot} = & \left\{-\frac{Q^2 g^2}{16}+\frac{\lambda a^2}{g^2}\right\} - ig^2\frac{\delta(0)a}{4}Q + \sum_k\left\{\frac{|q|}{4}+\lambda|h(k)|\right\} \\
& + \sum\frac{P_\phi(k)P_\phi(-k)}{4a^2} + k^2\phi(k)\phi(-k)
\end{aligned}
\tag{5.7}
$$

where P_ϕ is represented by $-i\frac{\delta}{\delta\phi}$. We find that requiring $H_{tot}\psi_0 = 0$ implies a Schroedinger equation for x

$$
i\frac{\sqrt{\lambda}}{2}a\frac{\partial x}{\partial a} = \left[-\frac{1}{4a}\ \frac{d^2}{d\phi(k)^2}+k^2\phi(k)^2\right]X(\phi) \tag{5.8}
$$

The physical meaning of eq. (5.8) can be seen by considering a quantum field ϕ in a classical background geometry given by

$$
\begin{aligned}
e_0^0 &= 1 \\
e_m^o &= e_0^a = 0 \\
e_m^n &= a(t)\delta_m^n = e^{\frac{\sqrt{\lambda}t}{2}}\delta_m^n
\end{aligned}
$$

The fourier modes of ϕ decouple and the schroedinger equation for the mode $\phi(k)$ is

$$
i\frac{\partial x}{\partial t} = \left[-\frac{1}{4a(t)^2}\ \frac{d^2}{d\phi(k)^2}+k^2\phi(k)^2\right]X \tag{5.9}
$$

Substituting

$$
\frac{\partial}{\partial t} = \dot{a}\frac{\partial}{\partial a} = \frac{\sqrt{\lambda}}{2}\ a\ \frac{\partial}{\partial a}
$$

we see that (5.9) and (5.8) are the same. Thus we see how the W-D wave function contains information about the evolution of ϕ in the form of correlations between the dynamical variables a and ϕ. We also see that in the linearized theory these correlations are obtained by solving the field dynamics in a classical background geometry.

This is the first place that the classical notion of time has entered our formalism. It is a convenient description of the correlations in the W-D wave function, within the semiclassical approximation. Once the geometry has large quantum fluctuations, time loses all meaning. Many author's [7] have attempted to define time variables which make sense in the quantum region, but with no success. The W-D equation is second order in all of its variables and it is impossible to convert it exactly into a Schrodinger equation. We accept the argument of DeWitt [1] that time is only an approximate concept in quantum gravity.

Gaussian solutions to (5.9) can be found which span the entire space of solutions. We therefore look for solutions of the form

$$x(\phi) = exp - \frac{1}{2}[F(t,k)\phi(k)\phi(-k)] \tag{5.11}$$

Applying (5.9) to (5.11) we get an equation for $F(t,k)$

$$\frac{i}{2}\frac{dF}{dt} + \frac{F^2}{4a(t)^2} - k^2 = 0 \tag{5.12}$$

The general solution to (5.12) is of the form

$$F = a\,k\left[\frac{W\,e^{2ik/a\sqrt{\lambda}} - e^{-2ik/a\sqrt{\lambda}}}{W\,e^{2ik/a\sqrt{\lambda}} + e^{-2ik/a\sqrt{\lambda}}}\right] \tag{5.13}$$

in the limit of small a. W is a complex integration constant.

In general (5.13) is violently oscillating as $a \to 0$. There exists a particular solution which is well behaved, namely $W = 0$. This state can be thought of as the state of minimal excitation and we will concentrate on it for the rest of this paper. The choice W in (5.13) is equivalent to requiring that the energy in the field ϕ not blow up as $a \to 0$. To see this we note that if the field ϕ is replaced by the dimensionless field X

$$X = \phi a^{\frac{1}{2}}$$

then the matter field hamiltonian is

$$H_m = \sum_k \frac{P_X^2(k) + k^2 X(k)^2 - a\sqrt{\lambda}\,X(k)^2}{a}$$

When $a \to 0$, H_m becomes

$$H_m = \frac{1}{a}\sum_k P_X^2(k) + k^2 X(k)^2$$

which has the eigenvalues

$$E_m = \frac{1}{a}\sum_k |k| n(k)$$

Thus to keep E finite as $a \to 0$ requires all the occupation numbers n to vanish. This is equivalent to requiring $W = 0$.

The solution of (5.12) with this initial condition is given by

$$F = -\frac{2ia|k|\, H_0^{(1)}\left(\frac{2k}{a\sqrt{\lambda}}\right)}{H_1^{(1)}\left(\frac{2k}{a\sqrt{\lambda}}\right)}$$

where H is the Hankel function. For large a, F behaves like

$$-\frac{4ik^2}{\sqrt{\lambda}}\ell n\frac{2a\sqrt{\lambda}}{k} + \pi\frac{k^2}{\sqrt{\lambda}} \tag{5.14}$$

It is interesting that for large a the probability distribution for $\phi(k)$ becomes a-independent.

We have carried out the calculation of ψ to order g when the matter field is included. In this order the quantum fluctuations in ϕ begin to become correlated with the metrical perturbation h. The wave function to order g has the form

$$\begin{aligned}\psi = \exp&\left[-2i\frac{V\sqrt{\lambda}}{g^2} + \frac{iI}{\sqrt{\lambda}\,g^2} - 2i\frac{I'}{3\lambda V\sqrt{\lambda}\,g^2}\right]\\ \times \exp&\sum\left[-\frac{F(ka)}{2}\phi(k)\phi(-k)\right]\\ \times&\left[1 + g\sum K_a^m(\ell,p)\phi(\ell)\phi(p)h_m^a(-\ell-p)\right]\end{aligned} \tag{5.15}$$

The procedure for finding K^a_m is by now familiar. Apply $H(k)$ and $P(k)$ to ψ and require the result to vanish to order g^0. The result is

$$
\begin{aligned}
K_a^m(p,\ell) = & \frac{\epsilon^{ba}\epsilon^{nm}k_b k_n}{4i\sqrt{\lambda}\,a^3k^2}\left[F_\ell F_p + 4p\cdot\ell a^2\right] \\
& + \frac{a}{2k^2}\Big\{\delta_{am}(k\cdot p\,F_\ell + k\cdot\ell\,F_p) - (k_a\ell_m + k_m\ell_a)F_p \\
& + (k_a\,p_m + k_m\,p_a)F_\ell\Big\}
\end{aligned}
\tag{5.16}
$$

where F is defined by (5.11). The first term in (5.16) can be written in the form

$$
\frac{1}{4i\sqrt{\lambda}}\begin{Bmatrix} \int dx\,dy\,dz\ \frac{du}{|e(u)|}\ F(u-x)F(u-y)G_0(u-z)\phi(x)\phi(y)|e(z)|R(z) \\ -4\int dx dz\frac{|e(x)|}{a^2}(\nabla\phi(x))^2 G_0(x-z)|e(z)|R(z)| \end{Bmatrix}
\tag{5.17}
$$

where $F(x)$ is the Fourier transform of $F(p)$. This is the lowest order contribution to a new reparametrization invariant term in the wave function. There are many ways to make it invariant and we would have to go to higher order to figure out the correct one.

The term linear in F is a correction to the invariant whose lowest order term is the logarithm of (5.11).

6. Spherical Geometries

Thus far we have considered 2-spaces with the topology of a torus or plane. We now turn our attention to topologically spherical geometries. We again decompose $e^a_m(x)$ into a dominant term describing the homogeneous isotropic component plus fluctuations. Let

$$
e^a_m(x) = E^a_m\ a(t)\ +\ gh^a_m
$$

where E^a_m describes a sphere of unit surface area. Such a sphere has uniform Gaussian curvature given by

$$
R = 4\pi \tag{6.1}
$$

For a simple description it is convenient to map the sphere onto the plane by stereographic projection. Then E^a_m is given by

$$E^a_m = \delta^a_m \; f(r) \equiv \frac{\delta^a_m}{(1+\pi r^2)} \tag{6.2}$$

The fluctuation $gh^a_m(x)$ is assumed orthogonal to E^a_m.

$$\int d^2x \; E^a_m(x) \; h^b_m(x) = 0$$

or

$$\int d^2x \; f(r) \; Tr \; h(x) = 0 \tag{6.3}$$

To find the wave function we follow the same procedure as in section 4. To leading order the total hamiltonian is

$$H = -g^2 \frac{Q^2}{16} + \frac{\lambda a^2}{g^2} - \frac{4\pi}{g^2} \tag{6.4}$$

where Q is conjugate to a and the constant 4π is obtained by integrating $|e|R_2$. We apply (6.4) to $\psi(a)$ and require the result to vanish to order $\frac{1}{g^2}$. We write

$$\psi(a) = e^{\;A(a)/g^2}$$

and find

$$\left[\frac{d\,A(a)}{da}\right]^2 = 4\left[16\pi - 4\,\lambda\, a^2\right] \tag{6.5}$$

There are two solutions

$$A = \pm 2 \int \sqrt{16\pi - 4\lambda a^2} \; da \tag{6.6}$$

We next apply the local hamiltonian and $P(x)$ and require them to vanish to order g^{-1}. The momentum constraint requires ψ to have the form

$$\begin{aligned} \psi = \exp \; & \frac{A(a^2)}{g^2} + \frac{A'}{2a}\int \; |h(x)| \; d^2x \\ \times \exp & \left[\int \int \; d^2x\, d^2y\, K^{m\,n}_{a\,b}(x,y)\, h^a_m(x) h^b_n(y)\right] \end{aligned} \tag{6.7}$$

where $A' = \frac{dA}{da} = \pm\sqrt{16\pi - 4\lambda a^2}$, and the kernel K must satisfy

$$\frac{\partial f}{\partial x^n} K_{m\,b}^{m\,r} - \frac{\partial}{\partial x^m}\left[f K_{n\,b}^{m\,r}\right] = 0 \tag{6.8}$$

Equation (6.8) is the condition that the second factor be invariant with respect to spatial reparametrization. It can be solved by invariants constructed from the curvature R. Define the substracted curvature

$$\bar{R} = R - \frac{4\pi}{a^2} \tag{6.9}$$

The density

$$\bar{R}\,|e| = R|e| - \frac{4\pi}{a^2}|e| \tag{6.10}$$

when expanded in powers of h_m^a has no zeroeth order term.

The general solution to the constraint $P = 0$ is given by

$$\int\int K(x,y)h(x)h(y) = \int\int [\bar{R}\,|e|]_x\,\mathcal{G}(x,y)\,[\bar{R}\,|e|]_y \frac{a}{g^2 A'} \tag{6.11}$$

where $\mathcal{G}$ is any function of the invariant distance between x and y in the background geometry. The lowest order contribution to $\bar{R}\,|e|$ is given by

$$\begin{aligned} -[\bar{R}\,|e|]_x &= g\frac{\partial_m}{a}\left[\frac{1}{f(x)}\partial_r h_s^n + \frac{h_n^s \partial_r f}{f^2}\right]\epsilon^{mn}\epsilon^{rs} \\ &\quad - g\frac{\partial_m}{a}\left\{\left[\frac{1}{f^2}\,\partial_m f\right] Trh\right\} - g\frac{4\pi}{a}\;Tr\,h\;f(x) \end{aligned} \tag{6.12}$$

The next step is to solve for $\mathcal{G}$ by using $H(x) = 0$ to order g^{-1}. We find

$$\begin{aligned} H(x) \approx & -\frac{gQ}{8} Trq(x)\; f(x) + \frac{\lambda a}{g}\; Trh(x) f(x) \\ & + \frac{\partial m}{ag}\left[\frac{1}{f(x)}\partial_r h_s^n + \frac{h_n^s\,\partial_r f}{f^2}\right]\epsilon^{mn}\epsilon^{rs} - \frac{\partial m}{ag}\left\{\left[\frac{1}{f^2}\,\partial_m f\right]\;Tr\;h\right\} \end{aligned} \tag{6.13}$$

Applying (6.13) to (6.7) gives

$$4\pi f(x) Tr\ h(x) + \frac{aA'}{4} f(x) \int K^{mn}_{mb}(x,y)\, h^b_n(y)$$
$$+ \partial_m \Big[\frac{1}{f}\partial_r h^n_s + \frac{h^s_n \partial_r f}{f^2}\Big]\epsilon^{mn}\epsilon^{rs} - \partial_m\Big\{\Big[\frac{1}{f^2}\partial_m f\Big] Tr\ h\Big\} = 0 \tag{6.14}$$

The easiest way to solve (6.14) is to evaluate it for h of the form $h(x)\delta^a_m$. In fact there is no real loss of generality because any spatial metric in 2-dimensions can be brought to this form. In this case eq. (6.14) takes the form

$$0 = (\nabla^2 + 8\pi f^2)\Big(\frac{h}{f}\Big) + \frac{aA'}{4} f(x) \int K^{mn}_{mn}(x,y) f(y) \left(\frac{h(y)}{f(y)}\right) \tag{6.15}$$

Furthermore

$$-\bar{R}\,|e| = \frac{g}{a}[\nabla^2 + 8\pi f^2]\Big(\frac{h}{f}\Big) \tag{6.16}$$

Thus $\mathcal{G}(x,y)$ can be identified as

$$\frac{4}{aA'}\mathcal{G}(x,y) = \langle x\,|(\nabla^2 + 8\pi f^2)^{-1}[f(x)K^{mn}_{mn}(x,y)f(y)](\nabla^2 + 8\pi f^2)^{-1}|\,y\rangle \tag{6.17}$$

Using (6.15) this implies that

$$\mathcal{G}(x,y) = (\nabla^2 + 8\pi)^{-1}\delta(x-y) \tag{6.18}$$

Thus we see that there are two differences between the toroidal and spherical cases.

First, the term $\frac{2\sqrt{\lambda}V}{g^2}$ is replaced by $\frac{A(V)}{g^2}$ with A given by (6.6). The second difference is the replacement of $I = -\int \bar{R}|e| \frac{1}{\nabla^2} \bar{R}|e|$ by

$$I = -\int \bar{R}\,|e|\, \frac{1}{(\nabla^2 + 8\pi)}\, \bar{R}\,|e| \tag{6.19}$$

or more generally

$$I = -\int \bar{R}\,|e|\ G\ \bar{R}\,|e| \tag{6.20}$$

where G satisfies the "massive" equation

$$-\partial_m |e| g^{mn} \partial_n G(x,y) + 8\pi |e| G(x,y) = \delta(x-y) \tag{6.21}$$

In this case the wave function of the linearized theory is no longer a pure phase. For $a \gg \frac{4\pi}{\lambda}$

$$\begin{aligned} A' &= \pm 4i\sqrt{\lambda}\, a \\ A &= \pm 2i\sqrt{\lambda}\, a^2 \end{aligned} \tag{6.22}$$

The large a wave function then has the form

$$\psi = exp - i\Big[\frac{2\sqrt{\lambda}\, V}{g^2} + \frac{I}{\sqrt{\lambda}\, g^2}\Big]$$

However for $a^2 \ll \frac{4\pi}{\lambda}$ we find

$$\begin{aligned} A' &= 8\sqrt{\pi} \\ A &= 8\sqrt{\pi}\, a \end{aligned} \tag{6.23}$$

and

$$\psi = \exp\Big[\pm\frac{8\sqrt{\pi V}}{g^2} \pm \frac{\sqrt{V}}{8g^2\sqrt{\pi}} I\Big] \tag{6.24}$$

The reason that ψ becomes real for $a^2 \ll \frac{4\pi}{\lambda}$ is that this is the classically forbidden region. To determine which linear combination of wave functions in (6.24) to use with (6.22) we must extrapolate through the region $a^2 \sim \frac{4\pi}{\lambda}$. This however is the region in which our approximations break down.

In the limit $\lambda \to 0$ the wave function has the form of (6.24). In the case

$$\psi = exp - \frac{8\sqrt{\pi V}}{g^2} - \frac{\sqrt{V}}{8g^2\sqrt{\pi}} I$$

the wave function is well behaved since $I > 0$. However if

$$\psi = exp + \Big(\frac{8\sqrt{\pi V}}{g^2} + \frac{1}{8g^2}\sqrt{\frac{V}{\pi}} I\Big)$$

it blows up for large volume and for wildly fluctuating ($I \gg 1$) geometries of any volume (of course our approximation breaks down for $I \gg 1$.

Therefore requiring the wave function to be well behaved for wildly fluctuating geometries requires us to choose the wave function for which large volumes are exponentially suppressed.

Adding matter fields can be done as in the toroidal case. In the lowest order the wave function factorizes

$$\psi = exp\left[\frac{A(V)}{g^2} + \frac{aI}{g^2 A'(V)}\right] \psi_{\text{matter}} \tag{6.25}$$

Applying $\int H(x) d^2x$ and requiring the result to vanish to order g^0 we obtain a Schroedinger equation for ψ_{matter}.

$$i \frac{\partial \psi_{\text{matter}}}{\partial \tau} = H_{\text{matter}} \psi_{\text{matter}} \tag{6.26}$$

where τ is defined by

$$i \, \partial_\tau = A' \frac{\partial}{\partial a} \tag{6.27}$$

For large a

$$\frac{\partial}{\partial \tau} = 4a \sqrt{\lambda} \frac{\partial}{\partial a} \tag{6.28}$$

as in the flat or toroidal case. For small a

$$i \partial \tau = \sqrt{8\pi} \frac{\partial}{\partial a} \tag{6.29}$$

Thus we see that in the classically forbidden region ψ_m satisfies an elliptic or diffusion type equation with an imaginary time

$$\tau = \frac{ia}{\sqrt{8\pi}} \tag{6.30}$$

7. Quantum Fluctuations In The Large Scale Structure of Space-Time

In this section we shall study the anisotropy and inhomogeneity in the large scale structure of space time. Let us begin by considering a particular fourier component of $|e(x)|\ R(x)$ in the case of a toroidal universe.

$$[|e|\,R]_k = \int |e(x)|R(x)\ e^{ik\cdot x}\ d^2k \tag{7.1}$$

The magnitude of $[|e|\,R]_k$ is a measure of the degree of fluctuation in the spatial geometry. Since the wave functions that we have computed are pure phases, at least to order g, the fluctuations in $[|e|\,R]_k$ are infinite. However this violent fluctuation is not a signal of a meaningful fluctuation of space time. It is due to the ambiquity of the 2-dimensional spatial surface imbedded in $2+1$ dimensional space-time. Accordingly we shall modify $[|e|\,R]_k$ so that it is independent of the choice of spatial surface, at least to leading order in g.

It is convenient but not essential to work in "synchronous" coordinates defined by

$$\begin{aligned} e_0^0 &= 1 \\ e_0^a &= e_m^o = 0 \end{aligned} \tag{7.2}$$

Given any initial space-like surface Γ synchronous coordinates can be constructed so that Γ is the origin of time. Let us consider a coordinate transformation which displaces the initial surface by amount $gf^0(x,t)$ and which preserves the conditions (7.2).

$$\begin{aligned} t &\rightarrow t + gf^0(t,x) \\ X^m &\rightarrow X^m + g\,f^m(t,x) \end{aligned} \tag{7.3}$$

Imposing (7.2) on the transformed 3-bein we find

$$\partial_t\ f^0 = 0 \tag{7.4}$$

$$\partial_t\ f^m = -g^{sm}\ \frac{\partial f^0}{\partial X^s} \tag{7.5}$$

To leading order in the coupling constant g eq. (7.5) becomes

$$f^m(x,t) = -\frac{\partial f^0}{\partial X^m}\int_0^t \frac{1}{a^2(t)} \tag{7.6}$$

Let us now consider the variation of e^a_m.

$$\begin{aligned}\delta e^a_m &= g\Big[\frac{\partial e^a_m}{\partial X^\nu} f^\nu + \frac{\partial f^\nu}{\partial X^m} e^a_\nu\Big] \\ &= \dot{e}^a_m\, f^0 + (\partial_n e^a_m) f^n + \frac{\partial f^n}{\partial X^m} e^a_n\end{aligned} \tag{7.7}$$

which to leading order becomes

$$\begin{aligned}\delta e^a_m = g\,\delta h^a_m &= g\Big[\frac{\partial e^a_m}{\partial X^\nu} f^\nu + \frac{\partial f^\nu}{\partial X^m} e^a_\nu\Big] \\ &= g\Big[\delta^a_m\, \dot{a}\, f^0 + a\frac{\partial f^a}{\partial X^m}\Big] \\ &= g\Big[\delta^a_m\, \dot{a}\, f^0 - a\frac{\partial^2 f^0}{\partial X^m \partial X^a}\int_0^t \frac{1}{a^2}\Big]\end{aligned} \tag{7.8}$$

It is straight forward to show that the quantity[1]

$$Z_g = -|e|R(x) + ga\,\dot{a}\frac{d}{dt}\Big[\frac{2}{\nabla^2}\frac{\partial_a\partial_m h^a_m}{a} - \frac{Trh}{a}\Big] \tag{7.9}$$

is unchanged by (7.8). To leading order we can replace $\dot{a}$ by $\sqrt{\lambda}\ a$. Thus define

$$\begin{aligned}Z_g =& g\frac{\sqrt{\lambda}\,a}{2}\Big[\frac{2}{\nabla^2}\partial_a\partial_m\, \dot{h}^a_m - Tr\,\dot{h}\Big] \\ &- g\frac{a\lambda}{4}\Big[\frac{2}{\nabla^2}\partial_a\partial_m h^a_m - Trh\Big] - |e|\,R(x)\end{aligned} \tag{7.10}$$

Using $\dot{h}^a_m = -4\epsilon^{ab}\epsilon_{mn}q^n_b$ it is easy to see that Z is given by

$$Z_g = g^2 H_g(x) + 2\,g^2\sqrt{\lambda}\frac{\vec{\nabla}\vec{P}_g}{\nabla^2} \tag{7.11}$$

Equation (7.11) defines a measure of the geometric fluctuation at point x which does not depend on the choice of space-like surface through x. For an

---

[1] Z_g ia equivalent to $\nabla^2\Phi_H$ defined in (Ref. [6]) where $H_g(x)$ and $P_g(x)$ are the purely gravitational hamiltonian and space-translation generators.

expanding torus with metric $e^a_m = a(t)\delta^a_m$, Z obviously vanishes. Furthermore for any geometry which uniformly expands in the sense that

$$e^a_m(x,t) = a(t)\, e^a_m(x,0) \tag{7.12}$$

we find

$$Z_g = -|e|\, R(x) \tag{7.13}$$

Classically the general solution including matter fields asymptotically satisfies (7.12) from which it follows that Z_g asymptotically become time independent. Such a geometry asymptotically tends to a fixed shape which dilates with time.

The question we shall consider is how quantum fluctuations of Z_g behave as the volume tends to infinity. Before beginning the calculation let us discuss the meaning of the possible results. The simplest possibility is that $Z_g \to 0$ with volume. This would indicate that the geometry becomes homogeneous. Quantum fluctuations in the global structure of space die in time.

A more interesting possibility is that Z_g is not zero but has a limit as volume goes to ∞. In this case the universe approaches a fixed shape which grows in time.

Finally one may find that Z_g grows with volume. In this case quantum fluctuations cause bigger and bigger inhomogeneities and anisotropies as the universe grows.

We shall compute the quantity

$$W_v(x,y) = \langle\psi_v|Z(x)Z(y)|\psi_v\rangle_{\text{connected}} \tag{7.14}$$

where ψ_v stands for the projection of (5.6) on the subspace with given volume V. This quantity measures the correlation between fluctuations at points x and y as a function of the the total volume. In particular we will be interested in the limit of large volume. Notice that if $|x-y|$ is kept fixed as $V \to \infty$ the proper distance between the points grows like the scale factor a. Accordingly if (7.14) does not vanish as $V \to \infty$ then quantum fluctuations distort the global large scale geometry of the universe even when it grows to arbitrarily large size.

To begin the calculation we note that eq. (5.6) defines a product state with the gravitational factor being unaffected by the matter field. Thus $H_g(x)$ and $P_g(x)$ annihilate it. This means that (7.14) vanishes identically to order g^2. The first non-vanishing contribution occurs at order g^4. To compute it the wave function must be calculated to order g^2. However we can avoid this by observing that the constraints $H = P = 0$ require

$$\begin{aligned} g^2 H_g(x) &= -g^2 H_m(x) \\ g^2 P_g(x) &= -g^2 P_m(x) \end{aligned} \tag{7.15}$$

where H_m and P_m refer to the matter field. Therefore we can calculate the quantity

$$\langle \psi_v | Z_m(x) Z_m(y) | \psi_v \rangle_{\text{connected}} \tag{7.16}$$

by using

$$\frac{1}{g^2} Z_m(x) = \left[H_m(x) + 2\sqrt{\lambda} \frac{1}{\nabla^2} \nabla \cdot P_m \right] \tag{7.17}$$

Note that $\frac{1}{g^2} Z_m$ is the matter energy density in a "co-moving" frame where $\vec{P}_m(x) = 0$. The operators H_m and $\vec{P}_m$ are explicitly order zero in g. Therefore to calculate (7.16) to order g^4 requires only the lowest order wave function given in (5.6). In fact since to lowest order in g, H_m and P_m are independent of $h^a_m(x)$ we can ignore the gravitational part of ψ and write

$$\psi_m = exp\left[-\frac{1}{2} \sum_k F(k, a)\ \phi(k)\phi(-k) \right] \tag{7.18}$$

where a is identified with $V^{\frac{1}{3}}$. Since ψ_m satisfies the Schroedinger equation in a classical background geometry the computation is identical to calculating the corresponding quantity for a quantum field in a classical exponentially inflating universe.

Let us consider a typical contribution to (7.17). The matter hamiltonian has a contribution $(\nabla\phi)^2$. Thus consider the expectation value

$$g^4\langle(\nabla\phi(x))^2\ (\nabla\phi(y))^2\rangle \tag{7.19}$$

Because (5.6) is gaussian in ϕ Eq. (7.19) can be written

$$\begin{aligned}\sim g^4\langle\partial_m\phi(x)\ \partial_n\phi(y)\rangle^2 &= g^4\Big[\frac{\partial}{\partial X^m}\ \frac{\partial}{\partial X^n}\langle\phi(x)\ \phi(y)\rangle\Big]^2\\ &= g^4\Big[\int\ dk\ k_m\,k_n\ \phi(k)\,\phi(-k)\ e^{ik(x-y)}\Big]^2\end{aligned} \tag{7.20}$$

For a gaussian wave function

$$\langle\phi(\ell)\phi(-\ell)\rangle\ \sim \frac{1}{F(\ell,a)+F^*(\ell,a)}$$

so that

$$W \sim g^4\Big[\sum_\ell \frac{\ell m\ \ell n}{Re\,F(\ell,a)}e^{i\ell(x-y)}\Big]^2 \tag{7.21}$$

Equation (7.21) is ultra violet divergent. The divergence can be handled by standard renormalization techniques. The subtractions are proportional to $\delta(x-y)$ and its derivatives. They do not affect the part of the k integral ($k < a\sqrt{\lambda}$) which corresponds to physical wavelengths of order the horizon size or larger. Thus we can estimate the fluctuations by cutting off the ℓ integral at $\ell \sim a\sqrt{\lambda}$. From eq. (5.14) we see that $Re\,F$ is given by $\frac{\pi\ell^2}{\sqrt{\lambda}}$ for $\ell \gg a\sqrt{\lambda}$. Accordingly W behaves like

$$\begin{aligned}&g^4\int\ d^2\ell\,d^2q\ \frac{(\ell\cdot q)^2}{\ell^2q^2}\ \delta(\ell+q+k)\ e^{ik(x-y)}\ d^2k\\ &= g^4\int^{a\sqrt{\lambda}}\ d^2\ell\ \frac{[\ell\cdot(\ell+k)]^2}{\ell^2(\ell+k)^2}\ e^{ik(x-y)}\,d^2k\\ &= g^4\,\lambda a^2\,\delta_H(x-y)+\int^{a\sqrt{\lambda}}\ d^2\ell\,\Big\{\frac{[\ell\cdot(\ell+k)]}{\ell^2(\ell+k)^2}-1\Big\}e^{ik(x-y)}\,d^2k\end{aligned} \tag{7.22}$$

where δ_H is a smeared δ - function.

The first term of (7.22) comes mostly from modes with $\ell \sim a\sqrt{\lambda}$. In terms of proper momentum this means momentum of order $\sqrt{\lambda}$. The second term is insensitive to the cutoff and arises from modes with $\ell \sim k$. The remaining terms in W give similar contributions. Thus we see that W consists of a volume independent contribution plus a growing term given by

$$\lambda\, g^4\, a^2\, \delta_H(x-y) \tag{7.23}$$

This term has a simple intuitive significance. The total co-moving energy in a coordinate volume v is

$$\epsilon_v = \int_v \frac{Z_m(x)}{g^2} \tag{7.24}$$

The fluctuation in ϵ_v is

$$(\Delta\epsilon_v)^2 = \langle \epsilon_v^2 \rangle - \langle \epsilon_v \rangle^2 = \int_v d^2x\, d^2y\, W(x,y)\left(\frac{1}{g^4}\right) \tag{7.25}$$

From (7.22) we see that

$$(\Delta\epsilon_v)^2 \sim (v\, a^2)\lambda \tag{7.26}$$

The quanity va^2 is the total proper volume of the region v.

If we consider v to be composed of disjoint volumes of order the horizon size then eq. (7.26) means that the energy fluctuations in these volumes are statistically independent and of order $\sqrt{\lambda}$. Evidently the rapid inflation produces significant large scale distortions of the background geometry which do not disappear as the volume expands. Indeed as the low k modes are inflated past the horizon new fluctuations keep replacing them.

8. 3 + 1 Dimensions

The Lagrangian for $3+1$ dimensional gravity is (in synchronous gauge)

$$L = -\frac{\epsilon^{abc}\epsilon_{mnr}}{g^2}\; e^a_m\, \dot{e}^b_n\, \dot{e}^c_r - \frac{\lambda}{g^2}|e| + \frac{R|e|}{g^2} \tag{8.1}$$

and the Hamiltonian

$$H = -g^2\epsilon^{abc}\epsilon_{mnr}\ \tilde{e}^m_a\ \pi^n_b\pi^r_c + \frac{\lambda|e|}{g^2} - \frac{|e|R}{g^2} \tag{8.2}$$

which to zeroth order in g is

$$\begin{aligned} H = &\left\{-\frac{2}{3a}\ Q^2g^2 + \frac{\lambda a^3}{g^2}\right\} \\ &+ \left\{\frac{2g^3}{9a^2}Q^2 Tr\ h - \frac{4gQ}{3a} Tr\ q + \frac{\lambda a^2 Tr\ h}{g} + \frac{2\nabla^2 Tr\ \bar{h}}{g}\right\} \\ &+ \left\{-\frac{2g^4}{9a^3}Q^2 h^a_m h^m_a + \frac{2g^2}{a^2}\left(Tr\ h\ Tr\ q - h^a_m q^m_a\right)\frac{Q}{3}\right. \\ &\left. -\frac{1}{a}\left[(Tr\ q)^2 - q^m_a a^a_m\right] + \frac{a\lambda[(Tr\ h)^2 - h^a_m h^m_a]}{2}\right\} - \frac{2k^2}{a}\,|\,\bar{h}\,| \end{aligned} \tag{8.3}$$

where $Tr\ \bar{h} \equiv Tr\ h - \frac{k_a k_m}{k^2} h^m_a$ and $|\,\bar{h}\,| = \frac{1}{2}\epsilon^{mnr}\epsilon_{abc}\, h^a_m h^b_n\ \frac{k_r k_c}{k^2}$.

It is helpful to consider a single fourier coefficient with wave vector along an axis which we arbitrarily label x_1. The 2-space orthogonal to k we call $X^M = (X^2, X^3)$. The independent components of h^a_m are

$$\begin{aligned} &a)\ \ h^1_1 \\ &b)\ \ h^A_1\,,\ h^1_M \\ &c)\ \ h^2_2 + h^3_3 = h^M_M = Tr\,\bar{h} \\ &d)\ \ h^2_2 - h^3_3 \equiv \phi_1 a \\ &e)\ \ h^2_3 + h^3_2 \equiv \phi_2 a \\ &f)\ \ h^2_3 - h^3_2 \end{aligned} \tag{8.4}$$

The components of (8.4) labelled (a, b, c, f) are nonpropagating degrees of and ϕ_1, ϕ_2 are the graviton degrees of freedom. We shall see that $\phi_{1,2}$ behave in the same way as minimally coupled scalar matter fields.

The linearized wave function is obtained in the same manner as for $2+1$ dimensions. In this case we find

$$\psi = \exp -i\frac{\sqrt{\lambda/3}}{g^2}\Big[a^3 + \frac{a}{2}(Tr\ h)^2 - \frac{a}{2}h^a_m h^m_a\Big]$$
$$\times\ \exp\Big[\frac{2ik^2}{8a\sqrt{\lambda/6}}(h^M_M)^2\Big]\ x(\phi_1,\phi_2) \tag{8.5}$$

where $x(\phi_1, \phi_2)$ satisfies the minimally coupled schroedinger equation

$$i\frac{\partial\psi}{\partial t} = -\Big[\frac{1}{2a^3}\frac{d^2}{d\phi_i^2} - a\ \frac{k^2}{2}\phi_i^2\Big]\psi \tag{8.6}$$

where t is defined by

$$a = \exp\ 2\sqrt{\frac{\lambda}{3}}\ t \tag{8.7}$$

In other words the graviton fields ϕ behave like a pair of minimally coupled scalars.

As in the $2+1$ dimensional case a particular gaussian solution to (8.6) exists which is nonsingular as $a \to 0$. It is given by

$$x = e^{-F(a,k)/2}\ \phi_k\phi_{-k}$$

where

$$F = \frac{ik^2}{\sqrt{\frac{4\lambda}{3}}}\ a\Big[-1 + \frac{i|k|}{\sqrt{\frac{4}{3}}\lambda\, a}\Big/\ 1 + \Big(\frac{|k|}{\sqrt{\frac{4}{3}}\lambda\, a}\Big)^2\Big] \tag{8.8}$$

As in $2+1$ dimensions a measure of fluctuations is provided by Z_g where Z_g

$$Z_g = -\frac{g^2 H_g}{a} - 3g^2\frac{\lambda}{a}\ \nabla^{-2}\nabla\cdot P_g \tag{8.9}$$

where H_g and P_g are the graviton energy and momentum densities,

$$H_g(x) = \frac{a^3\ \dot{\phi}^2}{2} + \frac{a\,(\nabla\phi)^2}{2}$$
$$P_g(x) = a^3\,\dot{\phi}\ \nabla\phi$$

As in the $2+1$ dimensional case with matter fields we find that

$$\langle Z_g(x)\ Z_g(y)\rangle$$

consists of two contributions. The first contribution arises from graviton modes with coordinate momenta $\sim \frac{1}{|x-y|}$. This term is non growing. The second contribution arises from modes with physical momenta of order $\sqrt{\lambda}$ and has the form

$$ag^4\lambda^{5/2}\ \delta^3(x-y) \tag{8.10}$$

This leads to the same conclusions as in the $2+1$ dimensional case. In particular inflation leads to large scale inhomogeneity and anisotropy. It is an open question whether this leads to observable effects.

Acknowledgements

We are deeply indebted to Emil Matrinec, Michael Peskin, Nick Tsamis, Barat Ratra and Mark Mueller and Joseph Attik.

References

1. J. A. Wheeler, Superspace and the Nature of Quantum Geometrodynamics, Battelle Rencontres, (W. A. Benjamin) 1968, B. S. DeWitt, PR 160, 1113 (1967).
2. S. W. Hawking and J. B. Hartle, PRD 28, 2960 (1983).
3. M. Hermeaux, M. Pilati and C. Teitelboüm PL 110B, 123 (1982).
4. R. Arnairtt, S. Deser and C. W. Misner, the dynamics of general relativity in "Gravitation", an introduction to current research ed. L. Witten, Wiley, New York (1962).
5. E. J. Martinec, Soluble Systems in Quantum Gravity, PRD 30, Sept. 15^{th} issue.
6. J. Bardeen, PRD 22, 1882 (1980).
7. R. Brandenberger, Generation and Evolution of Fluctuations in energy density in inflationary universe models. NSF ITP 84-76 (to be published in 8^{th} John Hopkins), Workshop on current problems (June 1984).
8. B. S. De Witt, see Ref. 1.