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SOLUBLE SYSTEMS IN QUANTUM GRAVITY*

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ABSTRACT

$1+1$ and $2+1$ dimensional gravity are quantized in a gauge where the dynamics reduces to a finite number of physical degrees of freedom. The inclusion of scalar matter fields in the $1+1$ dimensional case is also considered.

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1. Introduction

The quantization of gravity remains one of the outstanding problems of theoretical physics despite more than 30 years of research. There are indeed many aspects we need to understand: questions of covariant regularization and renormalization;[1] measurement;[2] quantum coherence;[3] gravitational collapse and singularities;[4] and the potential for change of topology;[5-6] to name but a few. These problems are sufficiently formidable in 2 and 3 spacetime dimensions, let alone 10 or 11, to warrant a probe of simple soluble systems in order to provide a sound basis for the investigation of more realistic theories. The work reported here is an attempt to quantize gravitation in 1+1 and 2+1 dimensions using canonical path integral methods; we will find that the only gauge invariant degrees of freedom are a finite number of global variables. In 1+1 dimensions, only the volume of space cannot be gauged away; in 2+1 dimensions, only the volume and a set global metric parameters remain.[7,5] In section 2, we review Hamiltonian methods for gravity and path integrals of constrained systems. We proceed in section 3 to apply this formalism to the quantization of 1+1 gravity in a particular gauge.

In section 4 we add scalar matter to 1+1 gravity and discuss some implications for the theory of quantized strings, and conclude our discussion in section 5 with quantization of 2+1 gravity. The appendix contains a mathematical exercise needed to construct the 2+1 wavefunction.

2. Path Integrals, Constraints, and Gravity in the Hamiltonian Formalism

A direct route to the path integral expression for the transition amplitude in quantum mechanics is to partition the evolution time T into N steps and approximate $e^{i\hat{H}_{cl}(\hat{\pi},\hat{\phi})\,T/N}$; one finds[8]

$$\langle \phi', T \left| e^{i\hat{H}T} \right| \phi, 0\rangle = \int\limits_{\substack{\phi(0)=\phi \\ \phi(T)=\phi'}} \mathcal{D}\pi\, \mathcal{D}\phi\; e^{i\int_0^T (\pi\dot{\phi}-H_{cl})dt} \tag{2.1}$$

If the action is invariant under some continuous symmetry, then the paths in the functional integral will be highly degenerate. The symmetry is characteristically generated by some constraint variable χ, for example in a gauge theory

$$\chi = \nabla \cdot E \qquad \longrightarrow \qquad \delta\vec{A} = \epsilon\{\nabla \cdot E\,, \vec{A}\} = \vec{\nabla}\epsilon \tag{2.2}$$

In this case, the degeneracy of paths may be factored out using the Faddeev-Popov method[9]; with gauge-fixing condition $F[\pi,\phi]=0$ the amplitude becomes

$$\begin{aligned} Z \equiv \langle e^{iHT}\rangle = \int \mathcal{D}\pi^i \mathcal{D}\phi_i \mathcal{D}\lambda_a \quad & \det\{\chi^a, F^b\}\,\delta(F^b) \\ & \times \exp i\int_0^T (\pi\dot{\phi} - H - \lambda_a\chi^a) \end{aligned} \tag{2.3}$$

In our gauge theory example, $H = E^2 + B^2$, $\chi = \nabla\cdot E$, and $\lambda = A^0$.

Integration over λ enforces the constraint $\chi = 0$; the determinant

$$\det\{\chi, F\} = \det\left(\frac{\delta(\chi,F)}{\delta(\pi,\phi)}\right) \tag{2.4}$$

is precisely the Faddeev-Popov determinant – the Jacobian that allows the elimination of $\delta(\chi)$ and $\delta(F)$, leaving only the "physical" degrees of freedom orthogonal to the gauge direction generated by χ and the gauge constraint F.

In gravity, the symmetries are local space and time translations generated by the local momentum ($i = 1, \ldots, d$ where d is the spatial dimension) and the local Hamiltonian $\mathcal{H}_0(x)$, respectively. Being symmetry generators, they are constrained to vanish:

$$\mathcal{H}_\mu = 0 \qquad \mu = 0, \ldots, d \tag{2.5}$$

If we write the spacetime metric as

$$g_{\mu\nu} = \left(\begin{array}{c|c} \eta^k \eta^\ell g_{k\ell} - \eta^{\circ 2} & \eta^\ell g_{\ell i} \\ \hline \eta^k g_{kj} & g_{ij} \end{array} \right) \tag{2.6}$$

then the Einstein - Hilbert action $\int \sqrt{g^{(d+1)}} R^{(d+1)}$ may be cast in the form[10]

$$S = \int \left(\pi^{ij} \dot{g}_{ij} - \eta^\mu \mathcal{H}_\mu \right) \tag{2.7}$$

where

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{\kappa\sqrt{g}} \left(\pi^j_i \pi^i_j - \frac{1}{(d-1)} \pi^2 \right) + \kappa\sqrt{g} R + \lambda\sqrt{g} \\ \mathcal{H}_i &= - \nabla_j \pi^j_i \end{aligned} \tag{2.8}$$

Here R and ∇ are the curvature and covariant derivative intrinsic to the spacelike hypersurface defined by the g_{ij}, and π^j_i is related to the extrinsic curvature of that hypersurface in spacetime (also $\pi = \pi^i_i$). The Lagrange multipliers η^μ are simple functions of the $g^{0\mu}$ and play the same role as A^0 in gauge theories (note also the similarity of the momentum constraint to $\nabla \cdot E = 0$).

The Hamiltonian path integral for gravity may now be written as

$$Z = \int \mathcal{D}\eta^{\mu}\mathcal{D}\pi^{ij}\mathcal{D}g_{ij}\,\delta[F^{\nu}(\pi,g)]\ \det\{\mathcal{H}_{\mu},F^{\nu}\}\ e^{iS} \tag{2.9}$$

In 1+1 dimensions, the Einstein action is a topological invariant, so the spatial metric has no conjugate momentum and the canonical formalism breaks down. There are, however, quantum effects which give rise to a non-trivial effective action. Polyakov has shown that in the gauge $g_{\mu\nu} = e^{2\phi}\delta_{\mu\nu}$, the functional measure has an anomaly[11] leading to an effective action

$$S_{eff} = \int \left[\frac{26}{48\pi^2}(\partial_{\mu}\phi)^2 + \lambda\, e^{2\phi}\right] \tag{2.10}$$

The Hamiltonian generators which reproduce this result in the conformal gauge are[12]

$$\begin{aligned} \mathcal{H}_0 &= \frac{1}{2\kappa}e^{-\phi}\pi^2 + \kappa e^{-\phi}\left(\frac{1}{2}\phi'^2 - \phi''\right) + \lambda\, e^{\phi} \\ \mathcal{H}_1 &= -\nabla\pi = \pi\phi' - \pi' \end{aligned} \tag{2.11}$$

where $g_{11} = e^{2\phi}$ and $\kappa = \frac{26}{48\pi^2}$ in the conformal gauge. Simailar results are also found in the gauge specified by $F^{\mu} = (\eta^{\mu} - a\delta^{\mu 0})$[13]

Classically, these generators form a closed algebra

$$\begin{aligned} \{\mathcal{H}_0(x),\mathcal{H}_0(y)\} &= e^{-2\phi}[\mathcal{H}_1(x) + \mathcal{H}_1(y)]\ \delta'(x,y) \\ \{\mathcal{H}_1(x),\mathcal{H}_1(y)\} &= [\mathcal{H}_1(x) + \mathcal{H}_1(y)]\ \delta'(x,y) \\ \{\mathcal{H}_1(x),\mathcal{H}_0(y)\} &= [\mathcal{H}_0(x) + \mathcal{H}_0(y)]\ \delta'(x,y) - \kappa e^{-\phi}\delta'''(x,y) \end{aligned} \tag{2.12}$$

up to the anomalous term proportional to κ. Apart from this term, this algebra is identical to the algebra of the Einstein generators (2.8). This implies that the classical evolution of spacelike hypersurfaces is invariant under local space and time translations![12] Indeed, the classical equations of motion

$$\dot{\phi} = \{\eta \cdot \mathcal{H}, \phi\} \qquad \dot{\pi} = \{\eta \cdot \mathcal{H}, \pi\} \tag{2.13}$$

are equivalent to

$$R^{(2)} = -2\lambda \tag{2.14}$$

We would now like to quantize the system (2.11) by choosing a gauge and evaluating the path integral (2.9). In order to consistently implement the constraints $\mathcal{H}_\mu = 0$, we must quantize in such a way that there is no central charge the algebra (2.12). Also, we must find a regularization which preserves the coordinate invariance of the theory. Finding such regularization is not trivial – in the conformal gauge, the regulator must preserve conformal invariance. In fact, the central charge is related to the regularization; the renormalization of the functional determinants associated with gauge fixing and matter fields produces contributions to the central charge, and quantization of π and ϕ will yield additional contributions.

Our assumption is supported by the fact that there does exist a theory of 1+1 gravity with no central charge which is found by taking the formal limit $\kappa \to 0$ in (2.12), so that

$$\mathcal{H}_0 = \frac{1}{2\kappa}\pi e^{-\phi}\pi - \lambda e^{\phi} \quad . \tag{2.15}$$

The constraint algebra becomes

$$\begin{aligned}
&\{\mathcal{H}_0(x), \mathcal{H}_0(y)\} = 0 \\
&\{\mathcal{H}_1(x), \mathcal{H}_1(y)\} = [\mathcal{H}_1(x) + \mathcal{H}_1(y)]\,\delta'(x,y) \qquad (2.16) \\
&\{\mathcal{H}_1(x), \mathcal{H}_0(y)\} = [\mathcal{H}_0(x) + \mathcal{H}_0(y)]\,\delta'(x,y)
\end{aligned}$$

which may be consistently quantized. This theory, which has been studied by Banks and Susskind,[14] is just the strong coupling limit of 1+1 gravity.[15] When matter fields are added, κ can be nonzero such that the gravitational central charge is cancelled by the charge of the quantized matter fields.

3. 1+1 Dimensional Gravity

In what follows, we will consider space to be a circle. In order to fix the freedom of spatial reparametrization, let us choose the gauge

$$\phi' = 0 \qquad (3.1)$$

The momentum constraint $\mathcal{H}_1 = 0$ then implies

$$\pi' = 0 \qquad (3.2)$$

and we have eliminated all the canonical variable (except for global degrees of freedom; these cannot be fixed because $V = \int e^{\phi}\,dx$ is a geometric invariant). There now remain no canonical variables to be fixed by $\mathcal{H}_0$. This curious situation arises because there are two constraints but only one pair of canonical variables - 1+1 gravity has -1 degrees of freedom in the sense that upon adding one matter field, all local degrees of freedom are eliminated in a canonical gauge (see Section

4). Thus, in order to fix local time translations, we must choose a non-canonical gauge such as

$$\eta^0 = \text{const} \tag{3.3}$$

With the gauge choices (3.1) and (3.3) the Faddeev-Popov determinant is

$$\det\left[\frac{\delta F^\nu}{\delta(\text{gauge transf})}\right] = \det\begin{bmatrix}\partial_0 & \{\mathcal{H}_0,\phi'\} \\ & \\ 0 & \{\mathcal{H}_1,\phi'\}\end{bmatrix} \tag{3.4}$$

$$= \det(\partial_0) \times \det\Big(\{\mathcal{H}_1,\phi'\}\Big)$$

As before, the determinant of canonical variables just serves as the Jacobian needed to eliminate the constraints. The determinant $\det(\partial_0)$ yields formally just the determinant of Δ_0 which may be shown to give simply a renormalization of the Hamiltonian density (2.11).[13]

All that remains in the theory are the global variables Π and Φ defined by

$$\Pi(t) = \int dx\ \pi(x,t) \quad , \quad e^{\Phi(t)} = \int dx\ e^{\phi(x,t)} \tag{3.5}$$

Note that in the elimination of η^0, we must separately integrate out the part of η^0 which is a constant in both space and time since this mode of η^0 can be absorbed into the definition of T:

$$\int_0^T [\Pi\dot{\Phi} - \eta^0 H_0]dt = \int^{\eta^0 T} \Big(\Pi\frac{d\Phi}{d(\eta^0 t)} - H_0\Big)d(\eta^0 t) \tag{3.6}$$

The transition amplitude reduces to

$$Z = \int dT\, D\Pi\, D\Phi\ \exp\Big\{i\int_0^T \Big[\Pi\dot{\Phi} - \Big(\frac{1}{2k}e^{-\Phi}\Pi^2 - \lambda e^{\Phi}\Big)\Big]dt\Big\} \tag{3.7}$$

i.e., quantum mechanics with the Hamiltonian

$$H = \frac{1}{2\kappa}\Pi\, e^{-\Phi}\, \Pi \;-\; \lambda\, e^{\Phi} \tag{3.8}$$

where we have taken the simplest hermitian ordering for the kinetic term.

Additional justification for this ordering is provided by an analysis of the strong-coupling ($\kappa \to 0$) limit,[14] in which this is the only ordering for which the wavefunction $\psi[\phi]$ solves the local Schroedinger equation

$$\left[\frac{1}{2k}\pi(x)\, e^{-\phi(x)}\, \pi(x) - \lambda e^{\phi(x)}\right]\psi[\phi] = 0 \tag{3.9}$$

without ambiguous terms proportional to $\delta(0)$.

The integration over T enforces the constraint $H = 0$:

$$\int dT\, e^{i\hat{H}T} = \delta(\hat{H}) \tag{3.10}$$

so that the wavefunction ψ solves

$$\left[\frac{1}{2\kappa}\, \frac{\delta}{\delta\Phi}\, e^{-\Phi}\, \frac{\delta}{\delta\Phi} \;+\; \lambda e^{\Phi}\right]\psi[\Phi] = 0 \tag{3.11}$$

or, changing variables to $V = e^{\Phi} = \int dx\, e^{\phi}$,

$$\left[\frac{1}{2\kappa}\, \frac{\delta^2}{\delta V^2} + \lambda\right]\psi[V] = 0 \quad . \tag{3.12}$$

The solution $\psi[V]$ is

$$\psi[V] = a e^{i\sqrt{2\kappa\lambda}\, V} + \; b\, e^{-i\sqrt{2\kappa\lambda}\, V} \tag{3.13}$$

Expectation values of quantum operators O are calculated using the integration measure $d\Phi = \frac{dV}{V}$

$$\langle O\rangle = \int \frac{dV}{V}\, \psi^* O \psi \tag{3.14}$$

It seems meaningful in our gauge to talk of the expansion rate of the "world", even though this is not a coordinate invariant object; the two branches $a = 0$ and $b = 0$ are eigenstates of the operator

$$\Pi_V \equiv e^{-\Phi}\, \Pi \sim \dot{V} \tag{3.15}$$

with eigenvalues $\pm\sqrt{2\kappa\lambda}$, corresponding to uniformly expanding or contracting universes. From the equations of motion (2.13) we find

$$R = \dot{\Pi} = -\left[\frac{\Pi_V^2}{2} + \lambda\right] \quad , \tag{3.16}$$

which yields the expectation values

$$\frac{\langle \int eR^n\rangle}{\langle \int e\rangle} = (-2\lambda)^n \tag{3.17}$$

Thus the expectation values of covariant quantities, with the exception of the volume, have no dispersion. The volume, however, fluctuates wildly since all volume have unit probability for $\lambda > 0$ (the situation for $\lambda < 0$ is somewhat better behaved – large volumes are exponentially damped). In calculating (3.14) we must impose an ultraviolet cutoff to control the logarithmic integral $\int \frac{dV}{V}$; when the cutoff is removed the result is finite so long as we factor out the volume dependence of any quantity of interest.

4. Adding Matter Fields

When a scalar field is coupled to the geometry, we find that the constraints are sufficient to eliminate all canonical variables using the methods of Section 2, save for the global degrees of freedom. The Hamiltonian and momentum densities are

$$\begin{aligned}\mathcal{H}_0 =& \frac{1}{2\kappa}\pi\, e^{-\phi}\pi + \kappa e^{-\phi}(\phi'^2/2 - \phi'') - \lambda e^{\phi} + \frac{e^{-\phi}}{2}(P^2 + X'^2) + e^{\phi}U(X) \\ \mathcal{H}_1 =& \pi\phi' - \pi' + PX'\end{aligned} \tag{4.1}$$

where X and P are the field and its conjugate momentum. A convenient gauge is

$$\phi' = X' = 0 \tag{4.2}$$

Proceeding as before, we arrive at a quantum mechanics problem for the wave-function; the Hamiltonian is

$$H = \frac{1}{2\kappa}\Pi_V^2 - \lambda V + \frac{1}{2V}P^2 + V \cdot U(X) \tag{4.3}$$

In the special case where $U = 0$, we find that the solution to $H\psi = 0$ is a Bessel function

$$\psi(V, P) = (\frac{\pi}{2}\sqrt{2\kappa\lambda}V)^{1/2}\ J_\alpha(\sqrt{2\kappa\lambda}V) \tag{4.4}$$

with $\alpha = (\kappa P^2 + 1/4)^{1/2}$. The asymptotic forms

$$J_\alpha(z) \sim \frac{(\frac{1}{2}z)^\alpha}{\Gamma(\alpha+1)} \qquad z \to 0 \tag{4.5}$$

$$\sim (\frac{2}{\pi z})^{1/2}\ \cos\left(z - \frac{\pi}{2}\alpha - \frac{\pi}{4}\right) \qquad z \to \infty \tag{4.6}$$

show that the solution behaves likes the sum of the solutions for an expanding and a contracting universe for large volumes, and that the small-volume behavior will be regular, the probability for finding the universe at small volumes no longer diverges logarithmically.

In order to interpret the result (4.4), it is helpful to explore the classical physics of the Hamiltonian (4.3). The classical equation of motion

$$\kappa\frac{\ddot{V}}{V} = \frac{\kappa}{2}\left(\frac{\dot{V}}{V}\right)^2 + \lambda + \frac{P^2}{2V^2} \tag{4.7}$$

and Hamiltonian constraint

$$\frac{1}{V}H = \frac{\kappa}{2}\left(\frac{\dot{V}}{V}\right)^2 - \lambda + \frac{P^2}{2V^2} = 0 \tag{4.8}$$

together yield the solution

$$V = \frac{P}{\sqrt{2\lambda}}\cosh\left(\sqrt{\frac{2\lambda}{\kappa}}\,t\right) \tag{4.9}$$

describing a universe which contracts from large volumes down to a minimum $V_{min} = \frac{P}{\sqrt{2\lambda}}$ and then "bounces" back into an expansion phase. In deriving (4.4), though, we have integrated away all references to an external time parameter. How then, can we compare the two? In the classical system, the probability of finding the universe in the range $(V, V+\Delta V)$ is proportional to the time spent in that interval

$$\int dt = \int_{V}^{V+\Delta V} \frac{\sqrt{\kappa}}{P}\left[2\lambda\left(\frac{V'}{P}\right)^2 - 1\right]^{-1/2} dV' \tag{4.10}$$

which is asymptotically proportional to $\frac{\Delta V}{V}$ for $V \gg P/\sqrt{2\lambda}$ and falls to zero sharply at $V_{min} = P/\sqrt{2\lambda}$. For the quantum system, the probability is

$$\begin{aligned}\text{Prob}(V, V+dV) &= \int_V^{V+\Delta V} \frac{dV'}{V'}\, \psi^*(V')\psi(v') \\ &\sim \frac{\Delta V}{V} \quad , \quad V \gg \frac{P}{\sqrt{2\lambda}}\end{aligned} \tag{4.11}$$

and decays like a power law (c.f. eq. (4.5)) for volumes less than about V_{min}. Thus the classical regime is the region of large volumes, and quantum mechanics causes a smearing of the wavefunction into the classically forbidden region $V < V_{min}$. For a plot of the classical and quantum probability amplitudes for $\kappa P^2 = 2$ as a function of volume, see Fig. 1.

Our approach runs into trouble if we consider 1+1 gravity coupled to more than one scalar field, since in this case it is no longer possible to find an explicit solution to the constraints. Unfortunately, this is one of the most interesting cases, because 1+1 gravity coupled to D+1 massless free scalar fields is precisely the theory of vibrating strings in D+1 dimensions. We can use the methods developed here to shed some light on the difficulties of quantized strings. We have

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{2\kappa}\pi e^{-\phi}\pi + \kappa e^{-\phi}\Big(\frac{1}{2}\phi'^2 - \phi''\Big) - \lambda e^{\phi} + \frac{1}{2}e^{-\phi}(P_a^2 + X_a'^2) \\ \mathcal{H}_1 &= \pi\phi' - \pi' + P^a \cdot X_a' \qquad , \quad a = 0, \ldots, D\end{aligned} \tag{4.12}$$

In the conformed gauge $g_{\mu\nu} = e^{2\phi}\,\delta_{\mu\nu}$, we have precisely Polyakov's result in Hamiltonian form.[12] Another popular gauge is the light-cone gauge

$$X^+ \ \alpha\ t \qquad , \qquad P^+ = \text{const.} \tag{4.13}$$

where $X^{\pm} = X^0 \pm X^D$. Because of the Minkowski signature of the embedding space, the constraint $\mathcal{H}_0 = 0$ can be solved to give

$$P^- = \frac{1}{2\kappa}\pi e^{-\phi}\pi + \kappa e^{-\phi}\Big(\frac{1}{2}\phi'^2 - \phi''\Big) - \lambda e^{\phi} + \frac{1}{2}e^{-\phi}(P_i^2 + X_i'^2) \qquad (4.14)$$

where $i = 1, \dots, D-1$. Substituting this solution into the action we find

$$\begin{aligned} S &= \int (\pi\dot{\phi} + P_a\dot{X}_a)\, dx\, dt \\ &= \int \Big\{\pi\dot{\phi} + P_i\dot{X}_i - \frac{1}{2}e^{-\phi}(P_i{}^2 + X_i{}'^2) - \\ &\quad -\Big[\frac{1}{2\kappa}e^{-\phi}\pi^2 + \kappa e^{-\phi}(\phi'^2/2 - \phi'') - \lambda e^{\phi}\Big]\Big\} dx\, dt \end{aligned} \qquad (4.15)$$

Thus the gravitational field ϕ acts as a sort of longitudinal oscillation of the string, exponentially coupled to the transverse fields X_i. Even though the constraints have been solved, the effective theory appears quite formidable.

Finally, a third gauge choice is also interesting to consider. We may, as in the soluble examples previously considered, eliminate the gravitational field with the gauge

$$\phi' = \pi' = 0 \qquad (4.16)$$

Here the Faddeev-Popov determinant is not quite a Jacobian, because we are fixing both elements of a conjugate pair; rather, we find

$$\begin{aligned} \det\begin{bmatrix} \{\mathcal{H}_0, \pi'\} & \{\mathcal{H}_1, \pi'\} \\ & \\ \{\mathcal{H}_0, \phi'\} & \{\mathcal{H}_1, \phi'\} \end{bmatrix} &= \det\Big[\Big\{\mathcal{H}_0^{grav}(x)\,,\ \mathcal{H}_1^{grav}(y)\Big\}\Big] \times \text{const.} \\ &= \prod_x \mathcal{H}_0^{grav}(x) \quad \times \text{const.} \end{aligned} \qquad (4.17)$$

where the superscript *grav* indicates the purely gravitational part of the full Hamiltonian density. Thus, the path integral measure depends on the energy in the volume fluctuations. The Hamiltonian becomes

$$H = \frac{1}{2\kappa} V\, \Pi_V^2 - \lambda V + \int \frac{1}{2V} (P_a^2 + X_a'^2)\, dx \tag{4.18}$$

subject to the quantum mechanical constraints

$$0 = \mathcal{H}_0 \pm \mathcal{H}_1 = \left(\frac{1}{2\kappa} V\, \Pi_V^2 - \lambda V\right) + \frac{1}{2} V\, (P_a \pm X_a')^2 \tag{4.19}$$

But these constraints, apart from the single additional degree of freedom, are just those of the covariantly quantized string[16]; they are notoriously difficult to satisfy without destroying either unitarity or Lorentz invariance. It would appear that fundamental progress in quantizing strings is still lacking, and will require a better understanding of the quantized vacuum since the difficulties with central charges, unitarity, etc., can be traced directly to the divergent zero-point fluctuations of the fields.

5. 2+1 Dimensional Gravity

In 2+1 dimensions, the Einstein action (2.7) exists, although the classical equations of motion allow only flat spacetime as a solution[5] (there is no gravitational radiation in 2+1 dimensions).

There are only a finite number of physical degrees of freedom, even though the theory is perturbatively nonrenormalizable. Again, if we assume the existence of a coordinate invariant regulator, we may apply the canonical formalism. If we choose a gauge where the constraints can be solved explicitly, we won't have to confront the difficult question of regularization. In what follows, we consider space to be closed with a toroidal topology. A convenient gauge is specified by choosing the metric to be spatially constant

$$g_{ij} = g_{ij}(t) \tag{5.1}$$

(For other spatial topologies it is possible to choose a metric with constant curvature described by a finite number of parameters known as moduli.[7,5]) The constraints $\mathcal{H}_\mu = 0$ imply that π^{ij} is spatially constant

$$\pi^{ij} = \pi^{ij}(t) \tag{5.2}$$

The Hamiltonian density (2.8) may be rewritten

$$\mathcal{H}_0 = \frac{1}{\kappa}\Big(\frac{1}{\sqrt{g}}\tilde{g}_{ij}\tilde{g}_{k\ell}\,\tilde{\pi}^{ik}\tilde{\pi}^{j\ell} - \frac{1}{2}\,\pi\,\frac{1}{\sqrt{g}}\,\pi\Big) + \kappa\sqrt{g}R + \lambda\sqrt{g} \tag{5.3}$$

where

$$\begin{aligned} \tilde{g}_{ij} &= g^{-1/2}\,g_{ij} \qquad , \\ \tilde{\pi}^{ij} &= g^{1/2}\Big(\pi^{ij} - \frac{1}{2}\,g^{ij}\pi\Big) \end{aligned} \tag{5.4}$$

are the degrees of freedom orthogonal to the local volume $\sqrt{g}$ and its conjugate π.[17] We again choose the ordering such that $\mathcal{H}_0$ is hermitian in the measure[18]

$$d\mu(g) = \prod_{i\leq j} \mathcal{D}g_{ij} \cdot g^{-\frac{(d+1)}{2}} \tag{5.5}$$

In the gauge (5.1), and using the formalism of Section 2, we see that the wave-function of the world satisfies

$$H\psi = \left[\frac{1}{\kappa}\left(\frac{1}{V}\,\tilde{g}_{ij}\tilde{g}_{k\ell}\tilde{\pi}^{ik}\tilde{\pi}^{j\ell} - \frac{1}{2}\,V\,\Pi_V^2\right) + \lambda V\right]\psi[V,\tilde{g}] = 0 \tag{5.6}$$

with

$$\Pi_V = \frac{\delta}{\delta V} \quad , \quad \tilde{\pi}^{ij} = \frac{\delta}{\delta \tilde{g}_{ij}} \tag{5.7}$$

Since the $\tilde{\pi}^{ij}$ are the generators of symmetric, traceless deformations of the metric, it is natural that those deformations are elements of symmetric space $SL(2,R)/SO(2)$ and that

$$\Delta \equiv \tilde{\pi}^j_i\tilde{\pi}^i_j \tag{5.8}$$

is the covariant Laplace operator on that space.[4] The eigenfunctions $e_{b,p}\,[\tilde{g}]$ of Δ are "plane waves" on $SL(2,R)/SO(2)$ satisfying

$$\Delta_{b,p} \;=\; (p^2 \,+\, 1/8)\; e_{b,p} \tag{5.9}$$

where p is a "wave number" and b is its "direction vector" (see the appendix). Inserting (5.9) into (5.6), this equation takes just the form of (4.3), and so the solution to the Schroedinger problem for the volume may be read off from (4.4). We find the wavefunction of 2+1 gravity in the gauge (5.1) to be

$$\psi = \left(\frac{\pi}{2}\sqrt{2\kappa\lambda}\;V\right)^{\frac{1}{2}}\; J_{\pm i\sqrt{2}p}\;(\sqrt{2\kappa\lambda}\,V)\;e_{b,p}\,(\tilde{g}) \tag{5.10}$$

It is again instructive to contrast this result with the set of classical solutions. The most general classical solution in the gauge (5.1) is

$$g_{ij} = A^2_{ij}\, e^{\sqrt{\lambda t}} + B^2_{ij}\, e^{-\sqrt{\lambda t}} + (AB + BA)_{ij} \tag{5.11}$$

where A and B are symmetric matrices satisfying the constraint

$$\begin{aligned} \mathrm{tr}\{A^{-1}B\} &= 0 \quad , \quad \text{for } A \text{ nondegenerate} \\ \mathrm{tr}\{B^{-1}A\} &= 0 \quad , \quad \text{for } B \text{ nondegenerate} \end{aligned} \tag{5.12}$$

The formula (5.11) is easily verified in the vierbein formalism. The constraint implies that the volume of space shrinks to zero at a finite time. Physically, this is clear because the Hamiltonian is just like the 1+1 gravity theory with a scalar field except that the conformal degrees of freedom $\tilde{g}_{ij}$ contribute with the opposite sign compared to the scalar field. Thus the volume feels an attractive $1/V^2$ potential rather than repulsive – the world is drawn towards zero volume, which is reached at finite time. The wave function (5.10)reflects this feature in that the probability amplitude $\psi^*\psi$ does not approach zero as V approaches zero (cf. the asymptotic form Eq. (4.6)). The classical and quantum probabilities are shown in Fig. 2. Note that the quantum amplitude follows the classically expected value even more closely than in 1+1 dimensions, and also has no zeroes as may be seen from the asymptotic form (4.5).

Finally let us examine qualitatively how these results are modified when we consider different spatial topologies. When space is topologically a sphere, we can fix a gauge where the metric has constant positive curvature and the volume is the only dynamical variable – the conformal metric $\tilde{g}_{ij}$ has no dynamics. The $\sqrt{g}R$ term in the Hamiltonian contributes a repulsive $1/V$ potential, and the universe will have a smooth classical bounce solution that does not reach zero volume – much like the 1+1 case with scalar field. When space is a closed surface with $n \geq 2$ handles, we may choose a metric which has constant negative curvature. In addition to the volume, there will be $6n-6$ real parameters (known to mathematicians as the moduli of the space) describing the global geometry[7] In fact, the two degrees of freedom in $\tilde{g}_{ij}$ in our torus example are an example of these moduli. The parameters will all enter into the Hamiltonian (5.3) with a kinetic energy opposite in the sign to the volume kinetic energy; these energies and the $\sqrt{g}R$ potential energy will push the volume towards zero. Thus the more involved the topology is, the more singular the dynamics becomes at small volumes.

6. Discussion

We have considered quantized gravity in 1+1 and 2+1 dimensions, as well as matter fields coupled to gravity in 1+1 dimensions. The Hamiltonian version of the path integral has proved useful in isolating the physical degrees of freedom in those cases where the gauge constraints allow an explicit solution. Such instances typically reduce the problem to a finite number of degrees of freedom, quite similar to the minisuperspace models of DeWitt[4] and others but less suspect in that no approximations are involved beyond the (admittedly delicate) assumption of a regulator which preserves the algebra of $\mathcal{H}_0$ and $\mathcal{H}_i$. There are no great surprises – the wavefunctions correspond quite closely to what one would expect from an analysis of the classical equations of motion, together with the smearing of probabilities mandated by the uncertainty principle. There is no need for a modification of the framework of quantum theory in order to fit geometrodynamics into it, at least at this level. In addition, we now have a stepping stone from which we may proceed to consider, e.g., a non-trivial matter field (i.e. massive or self-interacting) in 1+1 dimensions, or explore the possibility of topological metamorphosis. A parallel analysis should be possible for low-dimensional supergravity.

Of course, it may be that qualitatively different effects occur when there are an infinity of physical modes in the system. Then an explicit regularization is necessary, a question we have carefully avoided here. It seems that herein lies the major difficulty of quantum gravity.

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APPENDIX SL(2,R)/SO(2)

We describe here some of the elegant mathematics associated with space $M = SL(2,R)/SO(2)$ of the $\tilde{\pi}_i^j$. The exposition is a direct transcription of the beautiful exposition of Ref. 15 to 2+1 dimensions. Any element $\tilde{g} \epsilon M$ may be written

$$\tilde{g} = N A A^t N^t \tag{6.1}$$

where the matrices A and N are of the form

$$A = \begin{bmatrix} e^{+r/\sqrt{8}} & 0 \\ 0 & e^{-r/\sqrt{8}} \end{bmatrix} \quad , \quad N = \begin{bmatrix} 1 & \frac{n}{\sqrt{2}} \\ 0 & 1 \end{bmatrix} \tag{6.2}$$

the natural metric on M is

$$\begin{aligned} G(d\tilde{g}, d\tilde{g}) &= tr\ \{\tilde{g}^{-1}\, d\tilde{g}\, \tilde{g}^{-1}\, d\tilde{g}\} \\ &= dr^2 + e^{-\sqrt{2}r}\, dn^2 \end{aligned} \tag{6.3}$$

The Laplace operator on M is

$$\begin{aligned} \Delta = &\ \frac{1}{\sqrt{G}} \frac{\partial}{\partial \tilde{g}_{ij}} \sqrt{G}\, G^{ijk\ell} \frac{\partial}{\partial \tilde{g}_{k\ell}} \\ \text{with } G^{ijk\ell} &= \frac{1}{2}\left[\tilde{g}^{ik}\,\tilde{g}^{j\ell} + \tilde{g}^{i\ell}\,\tilde{g}^{jk} - \tilde{g}^{ij}\tilde{g}^{k\ell}\right] \end{aligned} \tag{6.4}$$

In terms of the line element (6.3) this operator becomes

$$\Delta = \frac{\partial^2}{\partial r^2} - \frac{1}{\sqrt{2}} \frac{\partial}{\partial r} + e^{\sqrt{2}r} \frac{\partial^2}{\partial n^2} \tag{6.5}$$

To find the eigenfunctions of Δ, let us first solve for those that are independent of n:

$$e_p(r) = e^{(ip + \frac{1}{\sqrt{8}})r} \quad ; \quad \Delta e_p = -\left(p^2 + \frac{1}{8}\right) e_p \tag{6.6}$$

Then, just as we can generate all two-dimensional plane waves by rotating a plane wave travelling along the x-axis (the y-independent solution), we can generate all the "plane waves" on M through the action of $SO(2)$ on the n-independent solution $e_p(r)$

$$e_{p,b}\,(\tilde{g}) \;=\; e_p\,(B^t\,\tilde{g}\,B) \tag{6.7}$$

where $B \in SO(2)$ and r is determined from $B^t\,\tilde{g}\,B$ through the decomposition

$$B^t\,\tilde{g}\,B \;=\; N\,A\,A^t\,N^t \tag{6.8}$$

The integration measure for inner products is deduced from (6.3)

$$d\tilde{g} \;=\; e^{-r/\sqrt{2}}\,dr\,dn \tag{6.9}$$

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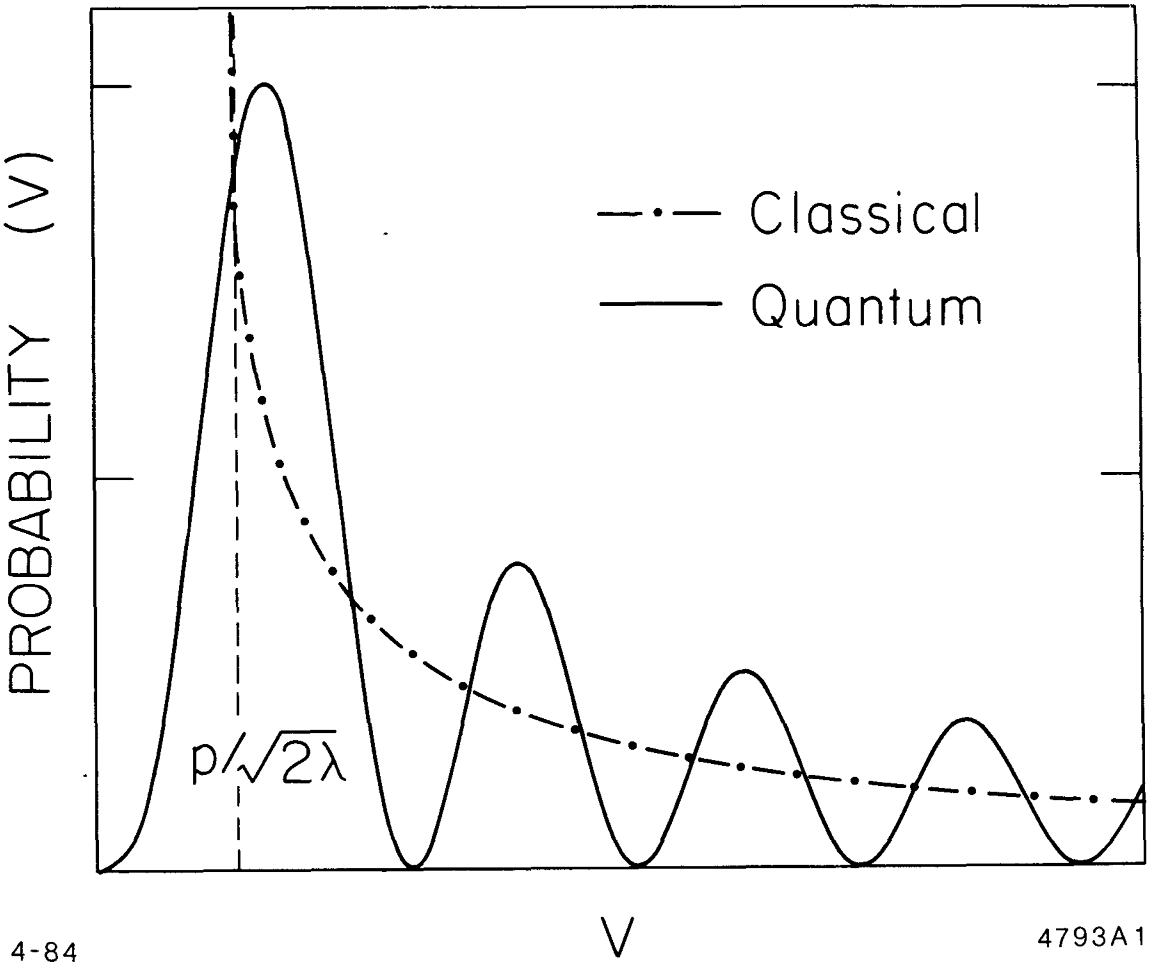

PROBABILITY (V)
Classical
Quantum
$p/\sqrt{2\lambda}$
V
4-84
4793A1

Fig. 1

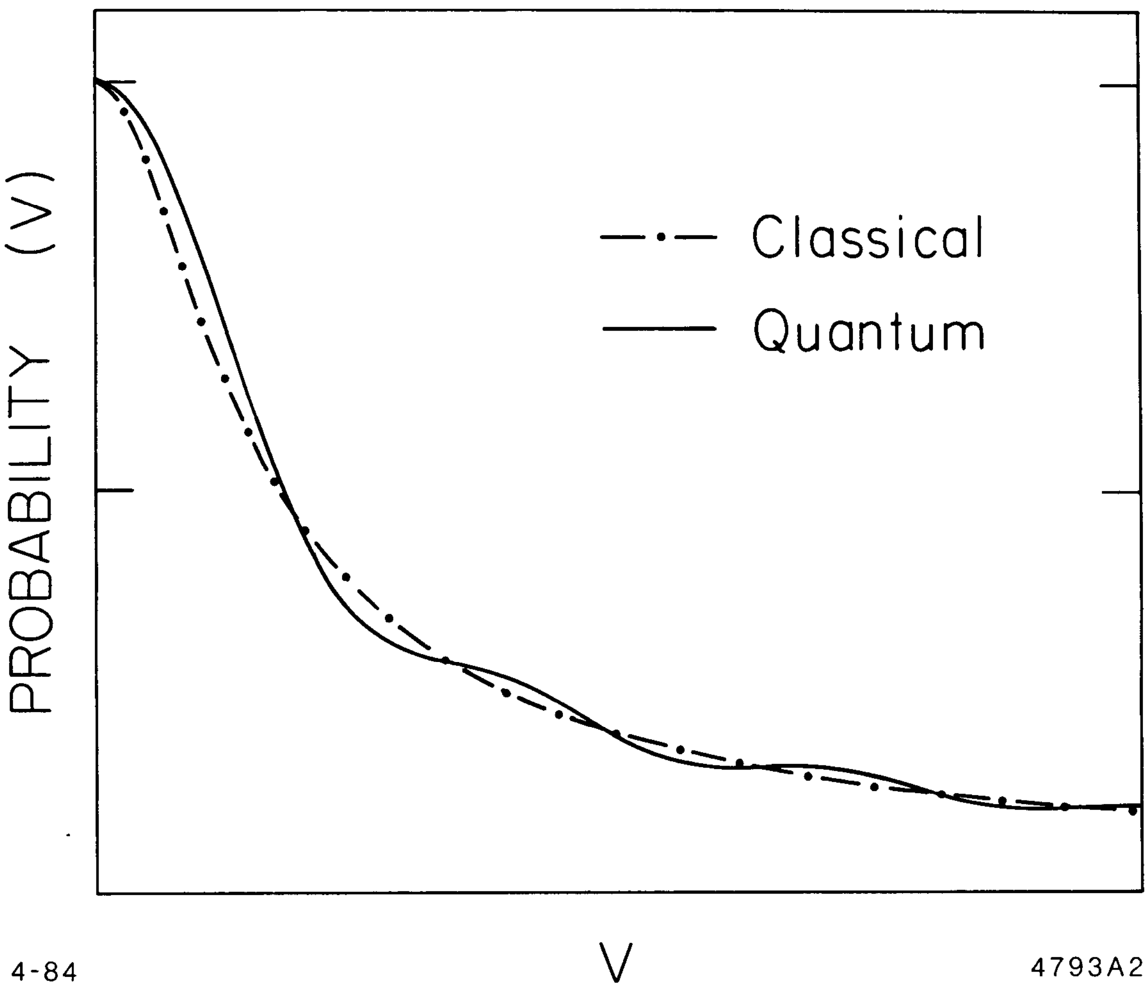

Fig. 2