ON MEASURING EMITTANCES AND SIGMA MATRICES*

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1. INTRODUCTION

The method depicted below has been much in use lately at SLAC for measuring emittances in the linac and the SLC damping ring. To carry out the measurement, the strength of the lens is varied and the width and height of the beam profile on the screen is observed. The input beam is kept unaltered in all aspects during the measurement. When the square of the width or height of the profile is plotted against the lens strength k, a quadratic (parabolic) curve results. The three coefficients of the quadratic are sufficient to determine the state of the input beam, including its emittance.
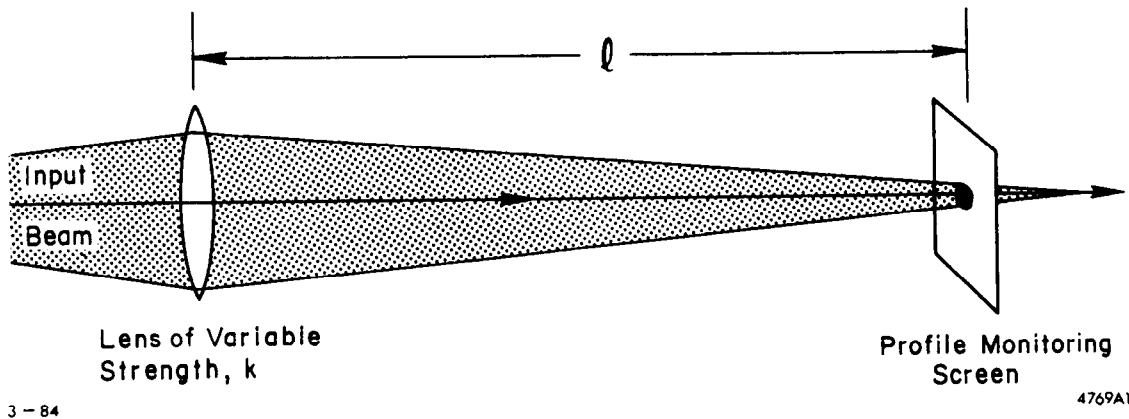

Figure 1

The basis of this method is readily derived using one two-by-two matrix to specify the state of the input beam (σ-matrix) and another to describe the lens-drift transport system (R-matrix). The process can be applied to both of the usual degrees of freedom, (x, x') and (y, y'), and in each case, three σ-matrix elements are determined. These three elements, in turn, fix the the corresponding emittances, i.e., the determinant of the corresponding σ-matrix.

The formalism can be dealt with quite compactly in a four-by-four matrix notation in which the σ-matrix is

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 & 0 \\ \sigma_{12} & \sigma_{22} & 0 & 0 \\ 0 & 0 & \sigma_{33} & \sigma_{34} \\ 0 & 0 & \sigma_{34} & \sigma_{44} \end{pmatrix} \tag{1}$$

It is always symmetric.

This matrix describes what we usually call an "uncoupled" beam—a distribution in which there are no correlations between horizontal and vertical motions.

In general, such correlations may exist and are prescribed by the missing elements of the matrix above. The general σ-matrix—still symmetric—is

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_{44} \end{pmatrix} \tag{2}$$

In the present note, we shall show that the method described above always measures the matrix elements $\sigma_{11}, \sigma_{12}, \sigma_{22}$ when carried out with the horizontal profile and $\sigma_{33}, \sigma_{34}, \sigma_{44}$ when carried out with the vertical profile, and that it does so independent of the "coupling elements" $\sigma_{13}, \sigma_{23}, \sigma_{14}, \sigma_{24}$. Having proven the foregoing assertion, we shall describe a new method for determining the four coupling elements using a skewed quadrupole and a drift. These four elements, together with the six elements obtained by the original method comprise **all ten** of the independent elements of the σ-matrix and therefore suffice to determine the four-dimensional emittance of the beam despite correlations.

2. THE NORMAL (UPRIGHT) QUADRUPOLE CASE

The evolution of the beam ellipsoid (σ-matrix) through the measuring system can be described by the following matrix equation

$$\sigma^S = \mathbf{R}\sigma\tilde{\mathbf{R}} \tag{3}$$

where σ^S is the beam matrix at the screen, $\mathbf{R}$ represents the transfer matrix of the measuring system (quad and drift) and $\tilde{\mathbf{R}}$ is the transpose of $\mathbf{R}$. We are using here the formalism developed for the program TRANSPORT.

The transfer matrix describing the normal quadrupole of strength k in the thin lens approximation is

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ k & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -k & 1 \end{pmatrix}$$

and the drift of length l is represented by

$$\mathbf{L} = \begin{pmatrix} 1 & l & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The transfer matrix $\mathbf{R}$ is then just the product

$$\mathbf{R} = \mathbf{LQ} \tag{4}$$

and the two elements of the beam matrix that are measured on the screen are

$$\sigma_{11}^S = (1 + kl)^2 \sigma_{11} + 2l(1 + kl)\sigma_{12} + l^2 \sigma_{22}$$

$$\sigma_{33}^S = (1 - kl)^2 \sigma_{33} + 2l(1 - kl)\sigma_{34} + l^2 \sigma_{44}$$

For this simple measuring setup, the square of the beam width is in fact a quadratic function in the variable $(1 + kl)$ and the square of the beam height is quadratic in $(1 - kl)$. For reference all ten independent elements of the beam matrix at the screen as a function of the lens strength k are presented in the Appendix A.

For the more general case of the measuring setup, where the transport line between the quadrupole lens and the screen can be represented by the following $\mathbf{R_{tr}}$ matrix

$$\mathbf{R_{tr}} = \begin{pmatrix} r_{11} & r_{12} & 0 & 0 \\ r_{21} & r_{22} & 0 & 0 \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & r_{43} & r_{44} \end{pmatrix}$$

which itself creates no x-y coupling, the answer is

$$\sigma_{11}^S = (r_{11} + kr_{12})^2 \sigma_{11} + 2r_{12}(r_{11} + kr_{12})\sigma_{12} + r_{12}^2 \sigma_{22}$$

$$\sigma_{33}^S = (r_{33} + kr_{34})^2\sigma_{33} + 2r_{34}(r_{33} + kr_{34})\sigma_{34} + r_{34}^2\sigma_{44}$$

The horizontal and vertical sizes of the beam on the measuring screen ($\sqrt{\sigma_{11}^S}$ and $\sqrt{\sigma_{33}^S}$) do not depend on the "coupling elements" of the σ-matrix, $\sigma_{13}, \sigma_{23}, \sigma_{14}, \sigma_{24}$. This demonstrates the first assertion above.

To this proof, we wish to add the following comment. Let us rewrite the general coupled four-by-four beam matrix, Eq.(2), in terms of two-by-two submatrices in the following way:

$$\sigma = \begin{pmatrix} U & C \\ \tilde{C} & V \end{pmatrix}$$

Measurements with the normal quadrupole completely determine the submatrices U and V. If we start with the uncoupled beam, i.e., all the elements of the C submatrix are zero, the product of the determinants of U and V gives us the four-dimensional transverse emittance of the beam.

$$\det U \cdot \det V = \det\sigma = \epsilon_4^2$$

In the case of a coupled beam, the product of the determinants does not equal to the full determinant of the σ-matrix and the following inequality is probably true

$$\det U \cdot \det V > \det\sigma = \epsilon_4^2$$

3. THE SKEWED QUADRUPOLE CASE

Let us see now what happens if we replace the normal quadrupole by a skewed quadrupole. The transformation matrix for the skewed quadrupole of strength k in the thin lens approximation is

$$S_{45} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & 0 \\ k & 0 & 0 & 1 \end{pmatrix}$$

Now we replace Q in Eq.(4) with S_{45} so that

$$R = LS_{45}$$

For this simple setup of the lens and a drift l

$$\sigma_{11}^S = k^2 l^2 \sigma_{33} + 2kl(l\sigma_{23} + \sigma_{13}) + l^2\sigma_{22} + 2l\sigma_{12} + \sigma_{11}$$

$$\sigma_{33}^S = k^2 l^2 \sigma_{11} + 2kl(l\sigma_{14} + \sigma_{13}) + l^2\sigma_{44} + 2l\sigma_{34} + \sigma_{33}$$

$$\sigma_{13}^S = k^2 l^2 \sigma_{13} + kl(l\sigma_{34} + l\sigma_{12} + \sigma_{33} + \sigma_{11}) + l^2\sigma_{24} + l(\sigma_{23} + \sigma_{14}) + \sigma_{13}$$

and for the more general "uncoupled" transport line

$$\sigma_{11}^S = k^2 r_{12}^2 \sigma_{33} + 2kr_{12}(r_{11}\sigma_{13} + r_{12}\sigma_{23}) + r_{11}^2\sigma_{11} + 2r_{11}r_{12}\sigma_{12} + r_{12}^2\sigma_{22}$$

$$\sigma_{33}^S = k^2 r_{34}^2 \sigma_{11} + 2kr_{34}(r_{34}\sigma_{14} + r_{33}\sigma_{13}) + r_{34}^2\sigma_{44} + 2r_{34}r_{33}\sigma_{34} + r_{33}^2\sigma_{33}$$

$$\sigma_{13}^S = k^2 r_{34}r_{12}\sigma_{13} + k[r_{11}r_{34}\sigma_{11} + r_{12}r_{34}(\sigma_{34} + \sigma_{12}) + r_{12}r_{33}\sigma_{33}]$$
$$+ r_{11}(r_{34}\sigma_{14} + r_{33}\sigma_{13}) + r_{12}(r_{34}\sigma_{24} + r_{33}\sigma_{23})$$

Now the measuring setup mixes the elements of the original beam matrix, and the width, the height and the tilt of the beam on the screen provide us with some information about the elements of the $\mathbf{C}$ submatrix.

4. THE PROPOSED METHOD

In order to determine all ten independent elements of the general beam matrix Eq.(2) we propose to use **both** normal and skewed quadrupoles. In addition to the width and the height of the beam on the screen, the tilt angle of the beam spot (which is simply related to the σ_{13} element of the beam matrix) would be measured too, and a parabola fitted to the squares of the measured quantities.

Using the normal quadrupole we can determine the following beam matrix elements and combinations

$$\sigma_{11}, \quad \sigma_{12}, \quad \sigma_{22} \qquad\qquad \text{See } (A1)$$

$$\sigma_{33}, \quad \sigma_{34}, \quad \sigma_{44} \qquad\qquad \text{See } (A8)$$

$$\sigma_{13}, \quad \sigma_{14} - \sigma_{23}, \quad l^2\sigma_{24} + l(\sigma_{23} + \sigma_{14}) + \sigma_{13} \qquad \text{See } (A3)$$

and using the skewed quadrupole we can determine

$$\sigma_{33}, \quad l\sigma_{23} + \sigma_{13}, \quad l^2\sigma_{22} + 2l\sigma_{12} + \sigma_{11} \qquad \text{See } (B1)$$

$$\sigma_{11}, \quad l\sigma_{14} + \sigma_{13}, \quad l^2\sigma_{44} + 2l\sigma_{34} + \sigma_{33} \qquad \text{See } (B8)$$

$$\sigma_{13}, \quad l\sigma_{34} + l\sigma_{12} + \sigma_{33} + \sigma_{11}, \quad l^2\sigma_{24} + l(\sigma_{23} + \sigma_{14}) + \sigma_{13} \qquad \text{See } (B3)$$

These eighteen equations overdetermine the ten independent elements of the σ-matrix. Although there is not enough information to determine the full matrix using either the normal quad or the skewed quad by itself, using both quads, we determine some of the elements more than once. This provides not only the matrix elements, but a measure of goodness of fit.

An alternative setup can include **one** skewed quadrupole and **two** screens, separated by a drift. Measurements of the height, the width and the tilt of the beam on **both** screens will again provide eighteen equations for the determination of ten independent elements of the σ-matrix.

APPENDIX A. THE NORMAL QUADRUPOLE CASE

Starting with the most general four-by-four symmetric σ-matrix in front of the quadrupole, the ten independent elements of the beam matrix at the screen σ^S, as a function of the quadrupole strength are:

$$\sigma_{11}^S = (1 + kl)^2\sigma_{11} + 2l(1 + kl)\sigma_{12} + l^2\sigma_{22} \tag{A1}$$

$$\sigma_{12}^S = k(1 + kl)\sigma_{11} + (1 + 2kl)\sigma_{12} + l\sigma_{22} \tag{A2}$$

$$\sigma_{13}^S = -k^2l^2\sigma_{13} + kl^2(\sigma_{14} - \sigma_{23}) + l^2\sigma_{24} + l\sigma_{23} + l\sigma_{14} + \sigma_{13} \tag{A3}$$

$$\sigma_{14}^S = -k^2l\sigma_{13} + kl(\sigma_{14} - \sigma_{23}) - k\sigma_{13} + l\sigma_{24} + \sigma_{14} \tag{A4}$$

$$\sigma_{22}^S = k^2\sigma_{11} + 2k\sigma_{12} + \sigma_{22} \tag{A5}$$

$$\sigma_{23}^S = -k^2l\sigma_{13} + k(l\sigma_{14} - l\sigma_{23} + \sigma_{13}) + l\sigma_{24} + \sigma_{23} \tag{A6}$$

$$\sigma_{24}^S = -k^2\sigma_{13} + k(\sigma_{14} - \sigma_{23}) + \sigma_{24} \tag{A7}$$

$$\sigma_{33}^S = (1 - kl)^2\sigma_{33} + 2l(1 - kl)\sigma_{34} + l^2\sigma_{44} \tag{A8}$$

$$\sigma_{34}^S = k(1 - kl)\sigma_{33} + (1 - 2kl)\sigma_{34} + l\sigma_{44} \tag{A9}$$

$$\sigma_{44}^S = k^2\sigma_{33} - 2k\sigma_{34} + \sigma_{44} \tag{A10}$$

APPENDIX B. THE SKEWED QUADRUPOLE CASE

The elements of the σ^S matrix are modified in the following way if we replace the normal quadrupole with a skewed one

$$\sigma_{11}^S = k^2l^2\sigma_{33} + 2kl(l\sigma_{23} + \sigma_{13}) + l^2\sigma_{22} + 2l\sigma_{12} + \sigma_{11} \tag{B1}$$

$$\sigma_{12}^S = k^2l\sigma_{33} + k(2l\sigma_{23} + \sigma_{13}) + l\sigma_{22} + \sigma_{12} \tag{B2}$$

$$\sigma_{13}^S = k^2l^2\sigma_{13} + kl(l\sigma_{34} + l\sigma_{12} + \sigma_{33} + \sigma_{11}) + l^2\sigma_{24} + l\sigma_{23} + l\sigma_{14} + \sigma_{13} \tag{B3}$$

$$\sigma_{14}^S = k^2l\sigma_{13} + kl(\sigma_{34} + \sigma_{12}) + l\sigma_{24} + \sigma_{14} \tag{B4}$$

$$\sigma_{22}^S = k^2\sigma_{33} + 2k\sigma_{23} + \sigma_{22} \tag{B5}$$

$$\sigma_{23}^S = k^2l\sigma_{13} + k(l\sigma_{34} + l\sigma_{12} + \sigma_{33}) + l\sigma_{24} + \sigma_{23} \tag{B6}$$

$$\sigma_{24}^S = k^2\sigma_{13} + k(\sigma_{34} + \sigma_{12}) + \sigma_{24} \tag{B7}$$

$$\sigma_{33}^S = k^2l^2\sigma_{11} + 2kl(l\sigma_{14} + \sigma_{13}) + l^2\sigma_{44} + 2l\sigma_{34} + \sigma_{33} \tag{B8}$$

$$\sigma_{34}^S = k^2l\sigma_{11} + k(2l\sigma_{14} + \sigma_{13}) + l\sigma_{44} + \sigma_{34} \tag{B9}$$

$$\sigma_{44}^S = k^2\sigma_{11} + 2k\sigma_{14} + \sigma_{44} \tag{B10}$$