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GAUGE FIELD THEORY AND CONSERVATION LAWS IN ELASTIC DISLOCATION AND DISCLINATION CONTINUUM MECHANICS*

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ABSTRACT

The geometric structure of a material manifold with dislocations and disclinations is briefly introduced using vielbein and gauge field theory. Combining this theory with Noether's theorem we present a unified equation of variational invariance and obtain different type of conservation laws for dislocation and disclination continuum under the assumption that the variations of entropy and temperature inside body can be ignored. In particular, this procedure yields a new conservation relation by means of a small gauge transformation which is called the gauge conservation law. The duality principle of conservation laws which are present in material and spatial spaces is clearly demonstrated by the variational invariance equations.

1. Introduction

Defect continuum mechanics is an important developing branch of modern continuum mechanics, which aims at establishing a sound theoretical basis for exploring the elastic and nonelastic behaviors of material with imperfections of various kinds, such as voids, inclusions, inhomogeneities, microcracks, dislocations as well as disclinations on the microscopic and macroscopic levels. Since 1952 when Kondo,[1,2] initially studied the theory of dislocation continua, followed by Krőner,[3,4] Bilby,[5,6] et al., a great deal of progress has been made in this field (for instance, see Refs. 7-9). The most significant contribution originally due to Kondo in the development of dislocation continuum theory was that the geometrical properties of plastic imperfection can be closely related to that of a nonriemannian space. In other words, nonriemannian geometry provides a mathematical basis to describe the motion and deformation of such a continuum.

For a practical application of the theory we need to deal not only with the geometrical properties of plastic imperfections but also with the dynamic governing equations. Especially, strong interest arose in the study of conservation laws for elastic continuum since Eshebly[10] introduced the concept of the force on an elastic singularity using an energy-momentum tensor. A. G. Herrman[11] gave a brief account of the history of development in the study of conservation laws for an elastic continuum. These studies[12–14] were generally done on the basis of Noether's theorem. This theorem is stated as follows: If an action integral of a certain field continuum based on a Lagrangian function satisfying with Euler equations of motion keeps infinitesimally invariant under some small transformations of independent or field variables, there must exist some conservation laws for the field corresponding to the transformations and the number of conservation laws are just equal to that of the transformations proposed. Furthermore, since the action integral can be represented in both Lagrangian description and Eulerian description, the duality principle of conservation laws for the both descriptions can be established also based on Noether's theorem.[11,14] This principle says that the conservation laws established by simultaneously applying a small transformations of the same kind to the Lagrangian and Eulerian action integrals are dual to each other since they have a different mathematical form but contain the same physical information.

On the other hand, interest arose in recent years in attempting to relate dislocation and disclination theory to gauge theory. Since 1955 when Yang-Mills field theory[15] was explored, one recognized that riemannian geometry itself belongs to a kind of gauge field theory. In quite recent years,[16] it was learned from the study of supergravity theory that nonriemannian geometry with non-vanishing torsion also belongs to a kind of nonAbelian gauge group theory. From this point of view we are convinced that Abelian and nonAbelian gauge field theory could be employed in a very natural way for the study of dislocation and disclination continuum theory. In fact, some work has been done in this regard. A. G. Herrmann[17] first used Abelian gauge theory to deal with the gauge invariances in a linear elastic dislocation continuum and to compare the similarities of the governing equations of such a continuum with electromagnetic field theory. Edelen,[18] Kadic and Edelen[19] studied the Yang-Mills type minimal coupling theory for dislocation and disclination continuum. A unified approach to deal with a defect continuum with dislocations and disclinations was suggested using vielbein and gauge field theory.[20]

The purpose of the present paper is to derive and discuss the conservation relations within the framework of nonlinear elastic dislocation and disclination continuum theory by combining gauge field theory with Noether's theorem. In section 2, we shall briefly recall some basic formulation for describing the motion and deformation of material continuum with dislocations and disclinations in using gauge field theory. In section 3, we employ Noether's theorem to dislocation and disclination continua and derive the equations of variational invariance. Based on these equations, in section 4 the conservation laws are obtained by means of small transformations of various kinds which remain the action integrals infinitesimally invariant. We have proved that in addition to the conventional conservation laws such as the energy conservation law, material and physical momentum conservation laws and material and physical momentum moment conservation laws for isotropic materials, an additional conservation law called the gauge moment conservation law can be constructed using a small rotation of local anholonomic coordinate frame introduced for the dislocated and disclinated body in the natural state, which is called the small gauge transformation in nonAbelian gauge group theory. This conservation law appears to be new.

2. General Description of Motion and Deformation By Vielbein and Gauge Field Theory

2.1 General Description

The motion and deformation of a material body with dislocations and disclinations can be described through three different states, namely the reference, the deformed and the natural states. Hereafter, we always refer them to the r-, the d- and the n-state respectively, and assume that in the reference (or undeformed) state, the material body does not contain dislocations or disclinations.

Let x^μ ($\mu = 1, 2, 3$) be the coordinate system with the basis vectors $\vec{e}^{\,0}_\mu$ and metric tensor $e^0_{\mu\nu} = \vec{e}^{\,0}_\mu \cdot \vec{e}^{\,0}_\nu$ for describing the position of a material point P in the r—state. When the material body is loaded by external forces from outside, it will move continuously from the r- state to the d-state. Based on the two-point tensor method as widely used by Eringen[21,22] we introduce a new coordinate system y^a with the basis vectors $\vec{e}^{\,0}_a$ and metric tensor $e^0_{ab} = \vec{e}^{\,0}_a \cdot \vec{e}^{\,0}_b$ to describe the position of the material point P at time t. During the motion, plastic imperfections could be possibly created inside the body. For simplicity, we assume that both the basis vectors $\vec{e}^{\,0}_\mu$ and $\vec{e}^{\,0}_a$ are holonomic but not necessarily rectilinear.[1] The motion

[1]Later we will ease this restriction.

of the material point P is assumed to be given by a relation

$$y^a = y^a(x^\mu, t) \tag{2.1}$$

or its inverse

$$x^\mu = x^\mu(y^a, t) \tag{2.2}$$

On the other hand, the n-state of the material body can be carried out by cutting a very small volume element off from its surroundings and releasing it from the constraints of the surroundings. The volume element is usually taken from the d-state as spanned either by three basis vectors $\vec{e}^{\,0}_a$ $(a = 1, 2, 3)$ or by the corresponding comoving basis vectors $\vec{e}_\mu = y^a_\mu\ \vec{e}^{\,0}_a$, where $y^a_\mu \equiv \partial y^a / \partial x^\mu$. The process of cutting can be considered as an affine transformation ϕ of the torn small material elements from the d-state to the n-state.

If we introduce a local rectangular coordinate system z_A with the local basis vector $\vec{e}_A$, each small line element $\delta \vec{R}$ linking the two neighboring points P and Q in the n-state can be expressed by

$$\delta \vec{R} = \delta z_A\ \vec{e}_A \tag{2.3}$$

where z_A is the anholonomic coordinates of the material point P. Therefore, there is no one-to-one correspondence between y^a and z_A or x^μ and z_A. To describe the motion and deformation of the dislocated and disclinated body by vielbein and gauge field theory, we present two description methods below.

2.2 Lagrangian Description

In the Lagrangian description, x^μ and t are taken as independent variables, meanwhile the motion y^a, vielbeins $\phi_{\mu A}$ (in some references, called distortion) and gauge potentials $\omega_{\mu AB}$ are dependent variables considered as three kinds of determining parameters for the dislocation and disclination continuum. In the following, we shall recall some useful formulae in the Lagrangian description.

First, the transformation to map the line element $d\vec{r}_0$ linking the point P and Q in the r-state to the line element $\delta\vec{R}$ given in (2.3) by a relation

$$\delta z_A = \phi_{\mu A}\, dx^\mu \tag{2.4}$$

Since the coordinates z_A are anholonomic, the above equations (2.4) are not integrable, therefore the quantities

$$g_{\mu\nu} = \phi_{\mu A}\, \phi_{\nu A} \tag{2.5}$$

can be considered as a metric tensor for a nonriemannian space M, where $\phi_{\mu A}$ are called vielbeins in particle physics.

To fully describe the geometric structure of the material manifold with dislocations and disclination, a gauge potential (also called gauge connections) is introduced by gauge covariant derivatives of the vielbein[20] as follows

$$D_\mu\, \phi_{\nu A} = \partial_\mu\, \phi_{\nu A} - \omega_{\mu AB}\, \phi_{\nu B} \tag{2.6}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. We notice that the vielbeins $\phi_{\mu A}$ and gauge potential $\omega_{\mu AB}$ possess two different indices μ and A, B which do not belong to a same space. We call the indices μ and A the nonriemannian index and the gauge index respectively. Thus, in dealing with the index μ in $\phi_{\mu A}$ and $\omega_{\mu AB}$, the vielbein $\phi_{\mu A}$ and gauge connection $\omega_{\mu AB}$ are treated as vectors so that the metric tensor $g_{\mu\nu}$ can be used for lowering or raising the indices $\mu, \nu, \ldots$, etc.

If we keep the index A unchanged, the conventional covariant derivatives of vielbein $\phi_{\mu A}$ are defined by an affine connection $\Gamma^{\lambda}_{\mu\nu}$ in the way

$$\nabla_{\mu}\, \phi_{\nu A} = \partial_{\mu}\, \phi_{\nu A} - \Gamma^{\lambda}_{\mu\nu}\, \phi_{\lambda A} \tag{2.7}$$

or expressed in the contravariant form

$$\nabla_{\mu}\, \phi^{\nu}_{A} = \partial_{\mu}\, \phi^{\nu}_{A} + \Gamma^{\nu}_{\mu\lambda}\, \phi^{\lambda}_{A} \tag{2.8}$$

where the affine connection $\Gamma^{\lambda}_{\mu\nu}$ possesses only three nonriemannian indices.

Using two connections $\omega_{\mu AB}$ and $\Gamma^{\lambda}_{\mu\nu}$, the total covariant derivative of any physical quantity T which possesses the nonriemannian indices μ, ν, ..., and the gauge indices $A, B, ...$, is defined by

$$\begin{aligned} D_{\lambda}\, T^{..\sigma\rho....}_{\mu\nu..AB..} =& \partial_{\lambda}\, T^{..\sigma\rho....}_{\mu\nu..AB..} - \Gamma^{\gamma}_{\lambda\mu}\, T^{..\sigma\rho....}_{\gamma\nu..AB..} \\ & - \ldots + \Gamma^{\sigma}_{\lambda\gamma}\, T^{..\gamma\rho....}_{\mu\nu..AB..} + \ldots - \omega_{\lambda AD}\, T^{..\sigma\rho....}_{\mu\nu..DB..} - \ldots \end{aligned} \tag{2.9}$$

From this definition, it is easily proved[20] that the gauge connection $\omega_{\mu AB}$ has to be antisymmetric in the indices A and B, that is,

$$\omega_{\mu AB} = -\omega_{\mu BA} \tag{2.10}$$

Within the framework of the current dislocation and disclination continuum theory, it is assumed that the total covariant derivatives of $\phi_{\mu A}$ are identically zero

$$D_{\mu}\, \phi_{\nu A} \equiv \partial_{\mu}\, \phi_{\nu A} - \Gamma^{\lambda}_{\mu\nu}\, \phi_{\lambda A} - \omega_{\mu AB}\, \phi_{\nu B} = 0 \tag{2.11}$$

From (2.6), (2.7) and (2.11), we obtain two important basic equations

$$\Gamma^{\lambda}_{\mu\nu} = \phi^{\lambda}_{A}\, D_{\mu}\, \phi_{\nu A} \quad , \quad \omega_{\mu AB} = \phi^{\nu}_{B}\, \nabla_{\mu}\, \phi_{\nu A} \tag{2.12}$$

which represent the connections among $\phi_{\mu A}$, $\Gamma^{\lambda}_{\mu\nu}$ and $\omega_{\mu AB}$.

It is known in nonriemannian geometry that torsion and gauge curvature tensors play a central role in determining the geometric structure of the nonriemannian space M, which are defined through the affine connection $\Gamma^{\lambda}_{\mu\nu}$ and gauge connection $\omega_{\mu AB}$ by

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]} = \phi^{\lambda}_{A}\, D_{[\mu}\, \phi_{\nu]A} \tag{2.13}$$

$$F_{\mu\nu AB} = 2\, \partial_{[\mu}\, \omega_{\nu]AB} - [\omega_{\mu}, \omega_{\nu}]_{AB} \tag{2.14}$$

respectively, where

$$[\omega_{\mu}, \omega_{\nu}]_{AB} \equiv \omega_{\mu AB}\, \omega_{\nu EB} - \omega_{\nu AE}\, \omega_{\mu EB} \tag{2.15}$$

Substituting $(2.12)_2$ into (2.14) and through some straightforward algebra, we obtain

$$F_{\mu\nu AB} = -R_{\mu\nu\lambda}^{\ \ \ \sigma}\, \phi_{\sigma A}\, \phi^{\lambda}_{B} \tag{2.16}$$

where

$$R_{\mu\nu\lambda}^{\ \ \ \sigma} = 2\partial_{[\mu}\, \Gamma^{\sigma}_{\nu]\lambda} - \Gamma^{\rho}_{\mu\lambda}\, \Gamma^{\sigma}_{\nu\rho} + \Gamma^{\sigma}_{\mu\rho}\, \Gamma^{\rho}_{\nu\lambda} \tag{2.17}$$

represents the conventional Riemann–Christoffel curvature tensor based on the affine connection $\Gamma^{\lambda}_{\mu\nu}$.

Since torsion tensor $T_{\mu\nu}^{\lambda}$ is antisymmetric in the nonriemannian indices μ and ν, and the gauge curvature tensor $F_{\mu\nu AB}$ is antisymmetric both in the nonriemannian indices μ, ν and in the gauge indices A, B, there are only nine independent components left for torsion and curvature tensor. In addition, it is known from the dislocation and disclination continuum theory[9] that the torsion and curvature tensors are responsible for dislocations and disclinations respectively, therefore, we may define two following second order tensors

$$\alpha^{\mu\nu} = \epsilon^{\mu\lambda\sigma} \, T_{\lambda\sigma}^{\ \ \nu} \tag{2.18}$$

and

$$\theta^{\mu\nu} = \epsilon^{\mu\lambda\sigma} \, \epsilon^{\nu\rho\gamma} \, R_{\lambda\sigma\rho\gamma} \tag{2.19}$$

to represent the dislocation density tensor and disclination density tensor respectively, where $\epsilon^{\mu\lambda\sigma}$ is the permutation symbol divided by $\sqrt{g}$ and $g = det(g_{\mu\nu})$.

Substitution of (2.13) into (2.18) leads to the decomposition of dislocation density tensor into the following form

$$\begin{aligned} \alpha^{\mu\nu} &= \epsilon^{\mu\lambda\sigma} \, \phi^{\nu}_{A}(\partial_{[\lambda} \, \phi_{\sigma]A} - \omega_{[\lambda|AB|} \, \phi_{\sigma]B}) \\ &= B^{\mu\nu} + \Omega^{\mu\nu} \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} B^{\mu\nu} &= \epsilon^{\mu\lambda\sigma} \, \phi^{\nu}_{A} \, \partial_{[\lambda} \, \phi_{\sigma]A} \\ \Omega^{\mu\nu} &= \epsilon^{\mu\lambda\sigma} \, \phi^{\nu}_{A} \, \omega_{[\lambda|BA|} \, \phi_{\sigma]B} \end{aligned} \tag{2.21}$$

are called the pure vielbein part and the gauge connection part of dislocation density, and the pure vielbein part $B^{\mu\nu}$ is responsible only for the Burgers vector density. When the space M is flat, that is, the gauge connection $\omega_{\mu AB}$ vanishes, the Burgers vector density is identically the dislocation density tensor.

2.3 Eulerian Description

In the Eulerian description, instead of x^{μ} and t, the coordinates y^{a} and t are considered as independent variables. We use the inverse motion $x^{\mu}(y, t)$, vielbein

ϕ_{aA} and gauge connection ω_{aAB} to describe the motion and deformation of the material body. The vielbein ϕ_{aA}, gauge connection ω_{aAB} are related to $\phi_{\mu A}$ and $\omega_{\mu AB}$ by

$$\phi_{aA} = x_a^\mu \, \phi_{\mu A}, \qquad \phi_{\mu A} = y_\mu^a \, \phi_{aA} \tag{2.22}$$

and

$$\omega_{aAB} = x_a^\mu \, \omega_{\mu AB}, \qquad \omega_{\mu AB} = y_\mu^a \, \omega_{aAB} \tag{2.23}$$

where

$$y_\mu^a \equiv \frac{\partial y^a}{\partial x^\mu}, \qquad x_a^\mu \equiv \frac{\partial x^\mu}{\partial y^a} \tag{2.24}$$

Following the same rule as given in the previous derivation, we list below some important formulae in the Eulerian description.

Metric tensor:

$$g_{ab} = \phi_{aA}\phi_{bA} = x_a^\mu \, x_b^\nu \, g_{\mu\nu} \tag{2.25$_1$}$$

or

$$g_{\mu\nu} = y_\mu^a \, y_\nu^b \, g_{ab} \tag{2.25$_2$}$$

Gauge Connections:

$$\omega_{aAB} = -\omega_{aBA}$$

Gauge covariant derivatives of ϕ_{aA} :

$$D_a\phi_{bA} = \partial_a\phi_{bA} - \omega_{aAB}\phi_{bB} \tag{2.26$_1$}$$

$$D_a\phi^b_A = \partial_a\phi^b_A + \omega_{aAB}\phi^b_B \tag{2.26$_2$}$$

Affine covariant derivatives of ϕ_{aA}:

$$\nabla_a\phi_{ba} = \partial_a\phi_{bA} - \Gamma^c_{ab}\phi_{cA} \tag{2.27$_1$}$$

$$\nabla_a\phi^b_A = \partial_a\phi^b_A + \Gamma^b_{ac}\phi^c_A \tag{2.27$_2$}$$

where Γ^c_{ab} are the conventional affine connections, which satisfies

$$\Gamma^c_{ab} = y^c_\lambda\ x^\mu_a\ x^\nu_b\ \Gamma^\lambda_{\mu\nu} + y^c_\mu\ x^\mu_{ab} \tag{2.28$_1$}$$

and

$$\Gamma^\lambda_{\mu\nu} = y^a_\mu\ y^b_\nu\ x^\lambda_c\ \Gamma^c_{ab} + y^a_{\mu\nu}\ x^\lambda_a \tag{2.28$_2$}$$

Total covariant derivatives of any quantity $T^{..cd....}_{ab..AB..}$

$$\begin{aligned} \mathcal{D}_e T^{..cd....}_{ab..AB..} &= \partial_e T^{..cd....}_{ab..AB..} - \Gamma^f_{ea}T^{..cd....}_{fb..AB..} \\ &- \ \ldots \ + \Gamma^c_{ef}T^{..fd....}_{ab..AB..} + \ \ldots \ - \omega_{eAD}T^{..cd....}_{ab..DB..} \end{aligned} \tag{2.29}$$

Specially

$$\begin{aligned} \mathcal{D}_a\phi_{bA} &= \nabla_a\phi_{bA} - \omega_{aAB}\phi_{bB} \\ &= D_a\phi_{bA} - \Gamma^c_{ab}\phi_{cB} \end{aligned} \tag{2.30}$$

Using (2.22), (2.23) and (2.28), we may prove from the basic assumption (2.11) that

$$\mathcal{D}_a\phi_{bA} = 0 \tag{2.31}$$

Therefore, we have

$$\omega_{aAB} = \phi^b_B \nabla_a \phi_{bA}, \qquad \Gamma^c_{ab} = \phi^c_A D_a \phi_{bA} \tag{2.32}$$

Torsion tensor:

$$T^c_{ab} = \Gamma^c_{[ab]} = x^\mu_a \; x^\nu_b \; y^c_\lambda \; T^\lambda_{\mu\nu} \tag{2.33$_1$}$$

or

$$T^\lambda_{\mu\nu} = y^a_\mu \; y^b_\nu \; x^\lambda_c \; T^c_{ab} \tag{2.33$_2$}$$

Gauge curvature tensor and Riemann–Christoffel curvature

$$\begin{aligned} F_{abAB} &= 2\partial_{[a}\omega_{b]AB} - [\omega_a, \omega_b]_{AB} \\ &= -R_{abc}{}^d \; \phi^c_{cA} \; \phi_{dB} \\ &= x^\mu_a \; x^\mu_b \; F_{\mu\nu AB} \end{aligned} \tag{2.34}$$

where the Riemann–Christoffel curvature tensor $R_{abc}{}^d$ is expressed by

$$\begin{aligned} R_{abc}{}^d &= x^\mu_a \; x^\nu_b \; x^\lambda_c \; y^d_\sigma \; R_{\mu\nu\lambda}{}^\sigma \\ &= -\phi_{ca} \; \phi^d_B \; F_{abAB} \end{aligned} \tag{2.35}$$

Dislocation density tensor in the Eulerian description

$$\alpha^{ab} = \epsilon^{acd} \; T^b_{cd} = B^{ab} + \omega^{ab} \tag{2.36}$$

and

$$B^{ab} = \epsilon^{acd} \; \phi^b_A \; \partial_{[c} \; \phi_{d]A} \tag{2.37$_1$}$$

$$\Omega^{ab} = \epsilon^{acd} \; \phi^b_A \; \omega_{[c|BA|} \; \phi_{bB]} \tag{2.37$_2$}$$

Disclination density tensor in the Eulerian description

$$\theta^{ab} = \epsilon^{acd}\, \epsilon^{bef}\, R_{cdef} = y^a_\mu\, y^b_\nu\, \theta^{\mu\nu} \tag{2.38}$$

From the above listed formulae, we may see that ϕ_{aA}, x^μ and ω_{aAB} are three kinds of basic determining parameters, based on which all other physical quantities can be evaluated.

3. Noether's Theorem and Variational Equations

3.1 Noether's Theorem

Let us take into account the following action integral

$$I(\Psi^{A_i}) = \int_{E_4} L\,(z_k; \Psi^{A_i}, \Psi^{A_i}_k, \Psi^{A_i}_{k\ell})\, d^4z \tag{3.1}$$

where L represents a Lagrangian density depending on the field variables Ψ^{A_i} and their first and second derivatives $\Psi^{A_i}_k$ and $\Psi^{A_i}_{k\ell}$ with respect to z_k in 4-dimensional Euclidean space with rectangular coordinates z_k $(k = 1,2,3)$ and $z_0 \equiv t$, t is time. The integral (3.1) is taken over a bounded or unbounded region E_4 in the space. We should notice that the dependent variables Ψ^{A_i} $(A_i = A_1, A_2, ...)$ with the generalized indices $A_1, A_2, \ldots$, might be scalar, vector or tensor–valued fields.

As we know, the variational Euler equations of motion following from $\delta I = 0$ in (3.1) for the problem with fixed boundaries are

$$E(L) \equiv \frac{\partial L}{\partial \Psi^{A_i}} - \frac{\partial}{\partial z_k}\left(\frac{\partial L}{\partial \Psi^{A_i}_k}\right) + \frac{\partial^2}{\partial z_k \partial z_\ell}\left(\frac{\partial L}{\partial \Psi^{A_i}_{k\ell}}\right) = 0 \qquad (k,\ell = 0,1,2,3) \tag{3.2}$$

In the action integral (3.1), we introduce the small transformations of dependent and indepent variables as

$$\bar{z}_k = z_k + \delta z_k \qquad (k = 0,1,2,3) \tag{3.3}$$

and

$$\bar{\Psi}^{A_i}(\bar{z}) = \Psi^{A_i}(z) + \delta\Psi^{A_i} \tag{3.4}$$

where δz_k and $\delta\Psi^{A_i}$ represent the variations of independent and dependent variables respectively. With these transformations (3.3) and (3.4), the action integral (3.1) changes into

$$\bar{I}(\bar{\Psi}^{A_i}) = \int_{\bar{E}_4} L(\bar{z}_k, \bar{\Psi}^{A_i}, \bar{\Psi}^{A_i}_k, \bar{\Psi}^{A_i}_{k\ell}) d^4\bar{z} \tag{3.5}$$

From (3.3) and (3.4), we calculate

$$\begin{aligned}
\bar{\Psi}^{A_i}(\bar{z}) &= \Psi^{A_i}(z) + \delta_*\Psi^{A_i} + \Psi^{A_i}_k \delta z_k + O(\delta z^2) \\
\bar{\Psi}^{A_i}_k(\bar{z}) &= \Psi^{A_i}_k(z) + \delta_*\Psi^{A_i}_k + \Psi^{A_i}_{k\ell}\delta z_\ell + O(\delta z^2) \qquad (k = 0, 1, 2, 3) \\
\bar{\Psi}^{A_i}_{k\ell}(\bar{z}) &= \Psi^{A_i}_{k\ell}(z) + \delta_*\Psi^{A_i}_{k\ell} + \Psi^{A_i}_{k\ell m}\delta z_m + O(\delta z^2)
\end{aligned} \tag{3.6}$$

where δ_* means a variational operator only due to the transformation of field variables Ψ^{A_i}, thus

$$\begin{aligned}
\delta_*\Psi^{A_i} &= \bar{\Psi}^{A_i}(z) - \Psi^{A_i}(z) \\
\delta_*\Psi^{A_i}_k &= \bar{\Psi}^{A_i}_k(z) - \Psi^{A_i}_k(z) \\
\delta_*\Psi^{A_i}_{k\ell} &= \bar{\Psi}^{A_i}_{k\ell}(z) - \Psi^{A_i}_{k\ell}(z)
\end{aligned} \tag{3.7}$$

In addition, the new volume element $d^4\bar{z}$ will change into

$$d^4\bar{z} = [1 + (\delta z_k)_k] d^4 z \tag{3.8}$$

Substituting (3.6) and (3.8) into (3.5) and making use of Taylor series expansion technique, we obtain

$$\bar{I}(\bar{\Psi}^{A_i}) = I(\Psi^{A_i}) + \int_{E_4} \Big(E(L)\delta_*\Psi^{A_i} + \nabla_k F_k \Big) d^4 z + O(\delta z^2) \tag{3.9}$$

where the operator E on L is given in (3.2), ∇_k means a 4-D divergence operator and F_k is expressed by

$$F_k = B_{k\ell}\delta z_\ell + \left[\frac{\partial L}{\partial \Psi_k^{A_i}} - \frac{\partial}{\partial z_\ell}\left(\frac{\partial L}{\partial \Psi_{k\ell}^{A_i}}\right)\right]\delta\Psi^{A_i}$$
$$+ \frac{\partial L}{\partial \Psi_{k\ell}^{A_i}}\,\delta\Psi_\ell^{A_i} - \frac{\partial L}{\partial \Psi_{k\ell}^{A_i}}\,\Psi_m^{A_i}(\delta x_m)_\ell \tag{3.10}$$

and

$$B_{k\ell} = L\,\delta_{k\ell} - \frac{\partial L}{\partial \Psi_k^{A_i}}\Psi_\ell^{A_i} + \nabla_m\left(\frac{\partial L}{\partial \Psi_{km}^{A_i}}\right)\Psi_\ell^{A_i} - \frac{\partial L}{\partial \Psi_{km}^{A_i}}\Psi_{m\ell}^{A_i} \tag{3.11}$$

From (3.9), we come to the conclusion that if the fields Ψ^{A_i} $(A_i = A_1, A_2, ...)$ satisfy the corresponding Euler equations (3.2), that is,

$$E(L) = 0 \tag{3.12}$$

then the functional (3.1) is infinitesimally invariant at Ψ^{A_i} under the small transformations (3.3) and (3.4) of both independent and dependent variables if and only if Ψ^{A_i} also satisfies

$$\nabla_k F_k = 0 \tag{3.13}$$

where F_k is given in (3.10). The equation (3.13) which we call the equation of variational invariance is the mathematic version of the celebrated Noether's theorem. In what follows, we shall apply this basic formalism (3.13) to derive the conservation laws for the elastic dislocation and disclination continuum by means of both Lagrangian and Eulerian representations, and discuss the duality principle of these two sets of conservation laws.

3.2 Variational Formulation For Dislocation and Disclination Continuum

Generally speaking, a material body containing a large number of moving dislocations and disclinations could not possibly be considered as a conservative system because during the motion of dislocations and disclinations, the macroscopic plastic deformation takes place as a irreversible thermo-mechanical process and the plastic work done by stresses is irreversibly converted to the thermal energy leading to the evident increase of temperature inside the body. Meanwhile, other irreversible effects, due to heat conduction and viscous dissipation are, in general, involved. In this sense, the conservation laws which are valid for the perfect elastic medium do not exist for the medium considered here. Sedov, et al.[7] gave a rather detailed description based on a general variational principle in constructing a mathematical model for dislocation continua by taking into account all the irreversible phenomena mentioned above.

However, if the deformation which occurs in the material is not large, all the irreversible effects can be ignored. In other words, the variations of entropy and temperature inside body are not significant, therefore the conservation relations can also be worked out within the framework of elastic dislocation and disclination continuum theory as was done for perfect elastic medium.

For simplicity, in the following derivation, the coordinates x^μ and y^a are always assumed to be rectilinear so that the difference between the lower and upper indexed vector- or tensor-valued quantitites disappears.

In Lagrangian representation, we assume in (3.1) that

$$z_0 = t, \qquad z_k = x_\mu \qquad (k = \mu = 1, 2, 3) \tag{3.14}$$

and the fields Ψ^{A_i} take

$$\Psi^{A_i} \equiv \begin{cases} y_a & A_i = A_1 = a \\ \phi_{\mu A} & A_i = A_2 = \mu A \\ \omega_{\mu AB} & A_i = A_3 = \mu AB \end{cases} \tag{3.15}$$

and the Lagrangian function has the form

$$\begin{aligned} L = & L(x_\mu,\ t;\ y_a,\ \dot{y}_a,\ y_{a\mu},\ \phi_{\mu A},\ \dot{\phi}_{\mu A},\ \phi_{\mu A\nu},\ \dot{\phi}_{\mu A\nu}, \\ & \omega_{\mu AB},\ \dot{\omega}_{\mu AB},\ \omega_{\mu AB\nu},\ \dot{\omega}_{\mu AB\nu}) \end{aligned} \tag{3.16}$$

which does not depend on the second position derivatives of $y_a, \phi_{\mu A}$ and $\omega_{\mu AB}$, that is,

$$\frac{\partial L}{\partial y^a_{\mu\nu}} = \frac{\partial L}{\partial \phi_{\mu A\lambda\nu}} = \frac{\partial L}{\partial \omega_{\mu AB\lambda\nu}} = 0 \tag{3.17}$$

The Euler equations of motion corresponding to (3.2) are

$$\begin{aligned} & \frac{\partial L}{\partial y_a} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial y_{a\mu}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{y}_a}\right] \\ & \frac{\partial L}{\partial \phi_{\mu A}} - \frac{\partial}{\partial x_\nu}\left[\frac{\partial L}{\partial \phi_{\mu A\nu}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}_{\mu A}} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\right)\right] \\ & \frac{\partial L}{\partial \omega_{\mu AB}} - \frac{\partial}{\partial x_\nu}\left[\frac{\partial L}{\partial \omega_{\mu AB\nu}}\right] = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\omega}_{\mu AB}} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\omega}_{\mu AB}}\right)\right] \end{aligned} \tag{3.18}$$

For the following small transformations of x_μ, t, y_a, $\phi_{\mu A}$ and $\omega_{\mu AB}$

$$\begin{aligned} \bar{x}_\mu &= x_\mu + \delta x_\mu \\ \bar{t} &= t + \delta t \\ \bar{y}_a &= y_a + \delta y_a \\ \bar{\phi}_{\mu A} &= \phi_{\mu A} + \delta\phi_{\mu A} \\ \bar{\omega}_{\mu AB} &= \omega_{\mu AB} + \delta\omega_{\mu AB} \end{aligned} \tag{3.19$_1$}$$

the equation (3.14) is specified to the form

$$\frac{\partial}{\partial t}\left\{e\delta t + b_\mu \delta x_\mu + P_a \delta y_a + S_{\mu A}\delta\phi_{\mu A} + S_{\mu AB}\delta\omega_{\mu AB} + \frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\phi_{\lambda A\nu}(\delta x_\lambda)_\mu + \frac{\partial L}{\partial \dot{\omega}_{\mu AB\nu}}\omega_{\lambda AB\nu}(\delta x_\lambda)_\mu\right\}$$

$$+\frac{\partial}{\partial x_\mu}\left\{e_\mu \delta t + b_{\mu\nu}\delta x_\nu + P_{a\mu}\delta y_a + S_{\nu A\mu}\delta\phi_{\nu A} + S_{\nu AB\mu}\delta\omega_{\nu AB} + \frac{\partial L}{\partial \dot{\phi}_{\nu A\mu}}\delta\dot{\phi}_{\nu A} + \frac{\partial L}{\partial \dot{\omega}_{\nu AB\mu}}\delta\dot{\omega}_{\nu AB}\right\} = 0 \qquad (3.19)_2$$

where the following abbreviations were introduced

$$
\begin{aligned}
e &\equiv L + \left[\frac{\partial}{\partial x_\mu}\left(\frac{\partial L}{\partial \dot{\Psi}^{A_i}_\mu}\right) - \frac{\partial L}{\partial \dot{\Psi}^{A_i}}\right] \dot{\Psi}^{A_i} - \frac{\partial L}{\partial \dot{\Psi}^{A_i}_\mu} \dot{\Psi}^{A_i}_\mu \\
e_\mu &\equiv \left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\Psi}^{A_i}_\mu}\right) - \frac{\partial L}{\partial \Psi^{A_i}_\mu}\right] \dot{\Psi}^{A_i} - \frac{\partial L}{\partial \dot{\Psi}^{A_i}_\mu} \ddot{\Psi}^{A_i} \\
b_\mu &\equiv \left[\frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\Psi}^{A_i}_\nu}\right) - \frac{\partial L}{\partial \dot{\Psi}^{A_i}_\nu}\right] \Psi^{A_i}_\mu - \frac{\partial L}{\partial \dot{\Psi}^{A_i}_\nu} \Psi^{A_i}_{\nu\mu} \\
b_{\mu\nu} &\equiv L\delta_{\mu\nu} + \left[\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\Psi}^{A_i}_\nu}\right) - \frac{\partial L}{\partial \Psi^{A_i}_\nu}\right] \Psi^{A_i}_\mu - \frac{\partial L}{\partial \dot{\Psi}^{A_i}_\nu} \dot{\Psi}^{A_i}_\mu \\
P_a &\equiv \frac{\partial L}{\partial \dot{y}_a} - \frac{\partial}{\partial x_\mu}\left[\frac{\partial L}{\partial \dot{y}_{a\mu}}\right] \\
P_{a\mu} &\equiv \frac{\partial L}{\partial y_{a\mu}} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{y}_{a\mu}}\right] \\
S_{\mu A} &\equiv \frac{\partial L}{\partial \dot{\phi}_{\mu a}} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\phi}_{\mu A\nu}}\right) \\
S_{\nu A\mu} &\equiv \frac{\partial L}{\partial \phi_{\nu A\mu}} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\phi}_{\nu A\mu}}\right] \\
S_{\mu AB} &\equiv \frac{\partial L}{\partial \dot{\omega}_{\mu AB}} - \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial \dot{\omega}_{\mu AB\nu}}\right) \\
S_{\nu AB\mu} &\equiv \frac{\partial L}{\partial \omega_{\nu AB\mu}} - \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial \dot{\omega}_{\nu AB\mu}}\right]
\end{aligned}
\tag{3.20}
$$

in $(3.20)_{1-4}$, we have made use of the notations introduced in (3.15). Similarily, in Eulerian representation, we assume in (3.1) that

$$z_0 = t, \qquad z_k = y_a \qquad (k = a = 1, 2, 3) \tag{3.21}$$

and

$$\Phi^{A_i} \equiv \begin{cases} x_\mu & A_i = A_1 = \mu \\ \phi_{aA} & A_i = A_2 = aA \\ \omega_{aAB} & A_i = A_3 = aAB \end{cases} \tag{3.22}$$

here we use Φ^{A_i} to replace Ψ^{A_i} in (3.1) to distinguish Eulerian description from Lagrangian description and the Lagrangian density takes the form

$$\begin{aligned} \mathcal{L} = & \mathcal{L}(y_a,\ t;\ x_\mu,\ \dot{x}_\mu,\ x_{\mu a},\ \phi_{aA},\ \dot{\phi}_{aA},\ \phi_{aAb},\ \dot{\phi}_{aAb}, \\ & \omega_{aAB},\ \dot{\omega}_{aAB},\ \omega_{aABb},\ \dot{\omega}_{aABb}) \end{aligned} \tag{3.23$_1$}$$

or simply

$$\mathcal{L} = \mathcal{L}(y_a,\ t;\ \Phi^{A_i},\ \Phi_a^{A_i},\ \dot{\Phi}_a^{A_i}) \tag{3.23$_2$}$$

The Euler equations of motion corresponding to (3.2) are expressed by

$$\frac{\partial \mathcal{L}}{\partial \Phi^{A_i}} - \frac{\partial}{\partial y_a}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}}\right) = \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{\Phi}^{A_i}} - 2\frac{\partial}{\partial y_a}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}}\right)\right] \tag{3.24}$$

With the small transformations of $y_a,\ t,\ x_\mu,\ \phi_{aA}$ and ω_{aAB}, as given by

$$\begin{aligned} \bar{y}_a &= y_a + \delta y_a \\ \bar{t} &= t + \delta t \\ \bar{x}_\mu &= x_\mu + \delta x_\mu \\ \bar{\phi}_{aA} &= \phi_{aA} + \delta\phi_{aA} \\ \bar{\omega}_{aAB} &= \omega_{aAB} + \delta\omega_{aAB} \end{aligned} \tag{3.25$_1$}$$

the equation (3.14) is also specified to the form

$$\begin{aligned}
&\frac{\partial}{\partial t}\Big\{E\delta t + p_a\delta y_a + B_\mu \delta x_\mu + S_{aA}\delta\phi_{aA} + S_{aAB}\delta\omega_{aAB} \\
&\qquad + \frac{\partial \mathcal{L}}{\partial\, \dot{\phi}_{aAb}}\ \phi_{cAb}(\delta y_c)_a + \frac{\partial \mathcal{L}}{\partial\, \dot{\omega}_{aABb}}\ \omega_{cABb}(\delta y_c)_a\Big\} \\
&+ \frac{\partial}{\partial y_a}\Big\{E_a\delta t + p_{ab}\delta y_b + B_{\mu a}\delta x_\mu + S_{bAa}\delta\phi_{bA} + S_{bABa}\delta\omega_{bAB} \\
&\qquad + \frac{\partial \mathcal{L}}{\partial\, \dot{\phi}_{bAa}}\delta\, \dot{\phi}_{bA} + \frac{\partial \mathcal{L}}{\partial\, \dot{\omega}_{bABa}}\delta\, \dot{\omega}_{bAB}\Big\} = 0
\end{aligned} \tag{3.25$_2$}$$

where we introduced the following abbreviations

$$
\begin{aligned}
E &\equiv \mathcal{L} + \left[\frac{\partial}{\partial y_a}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^{A_i}}\right] \dot{\Phi}^{A_i} - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}} \ddot{\Phi}_a^{A_i} \\
E_a &\equiv \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}}\right] \dot{\Phi}^{A_i} - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}} \ddot{\Phi}^{A_i} \\
p_a &\equiv \left[\frac{\partial}{\partial y_b}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_b^{A_i}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}^{A_i}}\right] \Phi_a^{A_i} - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_b^{A_i}} \dot{\Phi}_{ba}^{A_i} \\
p_{ab} &\equiv \mathcal{L}\delta_{ab} + \left[\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\Phi}_b^{A_i}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_b^{A_i}}\right] \Phi_a^{A_i} - \frac{\partial \mathcal{L}}{\partial \dot{\Phi}_a^{A_i}} \dot{\Phi}_a^{A_i} \\
B_\mu &\equiv \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} - \frac{\partial}{\partial y_a}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu a}}\right] \qquad (3.26) \\
B_{\mu a} &\equiv \frac{\partial \mathcal{L}}{\partial x_{\mu a}} - \frac{\partial}{\partial t}\left[\frac{\partial \mathcal{L}}{\partial \dot{x}_{\mu a}}\right] \\
S_{\mu a} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aA}} - \frac{\partial}{\partial y_b}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}}\right) \\
S_{aAb} &\equiv \frac{\partial \mathcal{L}}{\partial \phi_{aAb}} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{aAb}}\right) \\
S_{aAB} &\equiv \frac{\partial \mathcal{L}}{\partial \dot{\omega}_{aAB}} - \frac{\partial}{\partial y_b}\left(\frac{\partial \mathcal{L}}{\partial \dot{\omega}_{aABb}}\right) \\
S_{aABb} &\equiv \frac{\partial \mathcal{L}}{\partial \omega_{aABb}} - \frac{\partial}{\partial y_b}\left(\frac{\partial \mathcal{L}}{\partial \dot{\omega}_{aABb}}\right)
\end{aligned}
$$

in $(3.26)_{1-4}$, we have also made use of the notations given in (3.22). In comparison of (3.19) and (3.25), we may find out that these two equations (3.19) and (3.25) of variational invariance are dual in form, where the role of x_μ and y_a is merely interchanged. Using these two basic equations, the conservation laws of various kinds and their duality principles can be worked out quite simply as discussed below.

4. Conservation Laws and Their Principle of Duality

A. G. Herrmann[11,17] and Duan and G. Herrmann[14] discussed in some detail the conservation laws and their principle of duality in elastic continua in terms of an alternate and simpler procedure. The results have been extended to dislocation continuum by Duan and Duan,[18] who indicated that if the same kind of transformation of either independent or dependent variables which keeps the action integral infinitesimally invariant is applied simultaneously to the Lagrangian and Eulerian conservation equations, for instance, $(3.19)_2$ and $(3.25)_2$, we can obtain the dual conservation laws, which are expressed in different mathematical forms but contain the same physical information. Now, we intend to extend the result further to dislocation and disclination continuum by using vielbein and gauge field theory. In dealing with the conservation laws, we have to keep in mind that the small transformations of independent or dependent variables should be chosen in such a way that the corresponding action integral must remain infinitesimally invariant. Following this rule, except for the conventional laws such as energy conservation law, material and physical momentum conservation laws, material and physical angular momentum conservation laws, a new conservation law called gauge moment conservation law is derived using a small gauge transformtion.

<u>Translation of Time and Energy Conservation Law</u>

Let

$$\delta t = \epsilon_t, \qquad \delta x_\mu = 0, \qquad \delta y_a = \delta\phi_{\mu A} = \delta\omega_{\mu AB} = 0 \tag{4.1}$$

in $(3.19)_2$ and

$$\delta t = \epsilon_t, \qquad \delta y_a = 0, \qquad \delta x_\mu = \delta\phi_{aA} = \delta\omega_{aAB} = 0 \tag{4.2}$$

in $(3.25)_2$, where ϵ_t is a small time parameter, we obtain

$$\frac{\partial e}{\partial t} + \frac{\partial e_\mu}{\partial x_\mu} = 0 \tag{4.3$_a$}$$

and

$$\frac{\partial E}{\partial t} + \frac{\partial E_a}{\partial y_a} = 0 \tag{4.3$_b$}$$

respectively. These two equations, which hold true if the Lagrangians do not depend on t explicitly, represent the energy conservation law and correspond to each other.

<u>Translation of the Material Coordinate Frame and Material Momentum</u>

<u>Conservation Law</u>

Let

$$\delta t = 0, \qquad \delta x_\mu = \epsilon_\mu \qquad (\mu = 1, 2, 3) \tag{4.4}$$

in (3.19) and (3.25), where ϵ_μ $(\mu = 1, 2, 3)$ is three small parameters of same order. Since the translation of coordinate x_μ does not make any change in y_a, $\phi_{\mu A}$ and $\omega_{\mu AB}$, we have

$$\delta y_a = \delta\phi_{\mu A} = \delta\omega_{\mu AB} = 0 \tag{4.5}$$

in $(3.19)_2$ and

$$\delta y_a = \delta\phi_{aA} = \delta\omega_{aAB} = 0 \tag{4.6}$$

in $(3.25)_2$. Substituting (4.4) and (4.5) into (3.19), we obtain

$$\frac{\partial b_\mu}{\partial t} + \frac{\partial b_{\mu\nu}}{\partial x_\nu} = 0 \qquad (4.7)_a$$

Similarily

$$\frac{\partial B_\mu}{\partial t} + \frac{\partial B_{\mu a}}{\partial y_a} = 0 \qquad (4.7)_b$$

These two equations represent material momentum conservation laws and are dual to each other. As shown for elastic materials, these conservation laws hold if and only if the Lagrangians do not depend on x^μ explicitly.

Translation of the Spatial Coordinates Frame and Physical Momentum Conservation Law

In the classical physics, it is known that the translation of space coordinates y_a leads to the linear momentum conservation laws. This also holds true in dislocation and disclination continuum mechanics. To show this, let us suppose

$$\delta t = 0, \qquad \delta y_a = \epsilon_a \qquad (4.8)$$

in (3.19) and (3.25), where ϵ_a is three small parameters. By the same reason as presented above, translation of spatial coordinate frame does not make any change in x_μ, $\phi_{\mu A}$ and $\omega_{\mu AB}$ or ϕ_{aA} and ω_{aAB}, thus we have

$$\delta x_\mu = 0, \qquad \delta\phi_{\mu A} = \delta\omega_{\mu AB} = 0 \qquad (4.9)$$

and

$$\delta\phi_{aA} = 0, \qquad \delta\omega_{aAB} = 0 \qquad (4.10)$$

Substitution of (4.8) and (4.9) into (3.19) leads to the following physical momentum conservation law in Lagrangian representation:

$$\frac{\partial P_a}{\partial t} + \frac{\partial P_{a\mu}}{\partial x_\mu} = 0 \qquad (4.11)_a$$

Similarily we obtain

$$\frac{\partial p_a}{\partial t} + \frac{\partial p_{ab}}{\partial y_b} = 0 \qquad (4.11)_b$$

which represents the physical momentum conservation law in Eulerian representation. In fact, the equations $(4.11)_a$ and $(4.11)_b$ correspond to each other and hold true if and only if the Lagrangians L and $\mathcal{L}$ do not depend on y_a explicitly.

If we compare the expressions for b_μ and $b_{\mu\nu}$ in $(3.20)_{3-4}$ with the expressions for p_a and p_{ab} in $(3.26)_{3-4}$, we may observe that b_μ and $b_{\mu\nu}$ are related to x_μ, Ψ^{A_i} in the same fashion as p_a and p_{ab} are related to y_a, Φ^{A_i}. This comparison is also confirmed for B_μ, $B_{\mu a}$ and P_a, $P_{a\mu}$. We call b_μ, $b_{\mu\nu}$ (or B_μ, $B_{\mu a}$) the material momenta and p_a, p_{ab} (or P_a, $P_{a\mu}$) the physical momenta. The material momenta are independent of the physical momenta, therefore, the conservation laws $(4.7)_{a-b}$ by no means imply the conservation equations $(4.11)_a - (4.11)_b$ and vice-versa.

As mentioned in the introductory section, the above obtained conservation laws can be derived using a different procedure. In fact if we take the derivatives of Lagrangian function L in (3.1) with respect to z_k $(k = 0, 1, 2, 3)$, through some straightforward algebra, we may obtain

$$E(L)\ \Psi_k^{A_i} + \frac{\partial}{\partial z_\ell}\ B_{k\ell} = -\left(\frac{\partial L}{\partial z_k}\right)_{exp} \qquad (4.12)$$

where $(\ )_{exp}$ means the explicit derivative of the argument and

$$B_{k\ell} = -L\,\delta_{k\ell} + \frac{\partial L}{\partial \Psi_{\ell}^{A_i}}\,\Psi_{k}^{A_i} - \frac{\partial}{\partial z_m}\left(\frac{\partial L}{\partial \Psi_{m\ell}^{A_i}}\right)\Psi_{k}^{A_i} + \frac{\partial L}{\partial \Psi_{m\ell}^{A_i}}\,\Psi_{mk}^{A_i} \tag{4.13}$$

represents an energy-momentum tensor in 4-dimensional space. From (4.12), we conclude that if the field variables Ψ^{A_i} satisfy with Euler equation of motion (3.2) and the Lagrangian does not depend on $z_k (k = 0, 1, 2, 3)$ explicitly, then, the following conservation equations

$$\frac{\partial B_{k\ell}}{\partial z_{\ell}} = 0 \tag{4.14}$$

hold true.

Obviously, the conservation equations $(4.3)_{a-b}$, $(4.7)_{a-b}$ and $(4.11)_{a-b}$ are the specific forms of (4.14) when the Lagrangian function L and z_k take either (3.15) and (3.17) or (3.21) and (3.23) respectively.

In defect mechanics of elastic continua, the explicit derivative of the Lagrangian with respect to material coordinates has been termed as "a material force density" acting on the elastic singularity or inhomogeneity. Independence of the Lagrangian on material coordinates leads to conservation of material momentum. On the other hand, as known in classical physics, the explicit derivative of the Lagrangian with respect to space coordinates represents a real force acting on that element. When the Lagrangian does not depend on the space coordinates explicitly, the conservation laws of physical momentum hold true. Now, from the above derivation, we have easily extended the results to dislocation and disclination continuum.

Rotation of Material Coordinate Frame and Material Momentum Moment Conservation Law

The small rotation of a material coordinate frame can be expressed by a transformation of coordiates

$$\delta x_\mu = \epsilon_{\mu\nu\lambda}\ \alpha_\nu\ x_\lambda \tag{4.15}$$

where $\epsilon_{\mu\nu\lambda}$ are permutation symbol, and α_ν are the three small arbitrarily chosen parameters. We introduce

$$\delta x_{\mu\lambda} \equiv \frac{\partial(\delta x_\mu)}{\partial x_\lambda} = \epsilon_{\mu\nu\lambda}\ \alpha_\nu \tag{4.16}$$

to express this transformation, by which any physical quantity $f_{\mu\nu\ldots}$ having indices $\mu, \nu, \ldots$ is transformed to

$$\delta f_{\mu\nu\ldots} = \delta x_{\mu\lambda}\ f_{\lambda\nu\ldots} + \delta x_{\nu\lambda}\ f_{\mu\lambda\ldots} + \cdots \tag{4.17}$$

Applying the rule (4.17) to the vielbein and gauge connection transformations $\delta\phi_{\mu A}$ and $\delta\omega_{\mu AB}$, ... etc in $(3.19)_2$, we obtain

$$\epsilon_{\mu\nu\lambda}\left(\frac{\partial m_{\mu\nu}}{\partial t} + \frac{\partial m_{\sigma\mu\nu}}{\partial x_\sigma}\right) = 0 \tag{4.18$_a$}$$

where we define $m_{\mu\nu}$ and $m_{\sigma\mu\nu}$ as

$$\begin{aligned} m_{\mu\nu} &= b_\mu\ x_\nu + s_{\mu A}\ \phi_{\nu A} + s_{\mu AB}\ \omega_{\nu AB} \\ &+ \frac{\partial L}{\partial\,\dot\phi_{\nu A\lambda}}\ \phi_{\mu A\lambda} + \frac{\partial L}{\partial\,\dot\omega_{\nu AB\lambda}}\ \omega_{\mu AB\lambda} \end{aligned} \tag{4.19}$$

$$\begin{aligned} m_{\sigma\mu\nu} &= b_{\sigma\mu}\ x_\nu + s_{\mu A\sigma}\ \phi_{\nu A} + s_{\mu AB\sigma}\ \omega_{\nu AB} \\ &+ \frac{\partial L}{\partial\,\dot\phi_{\mu A\sigma}}\ \dot\phi_{\nu A} + \frac{\partial L}{\partial\,\dot\omega_{\mu AB\sigma}}\ \dot\omega_{\nu AB} \end{aligned} \tag{4.20}$$

represent the material momentum momenta.

In Eulerian representation, x_μ ($\mu = 1,2,3$) are treated as dependent variables, therefore, the rotation of coordinates x_μ cannot make any change to the veilbein and gauge conection. Substitution of (4.15) into (3.25) leads to

$$\epsilon_{\mu\nu\lambda}\Big(\frac{\partial M_{\mu\nu}}{\partial t}+\frac{\partial M_{\mu\nu a}}{\partial y_a}\Big)=0 \qquad (4.18)_b$$

where

$$M_{\mu\nu}=B_\mu\,x_\nu, \qquad M_{\mu\nu a}=B_{\mu a}\,x_\nu \qquad (4.21)$$

The equations $(4.18)_a$ and $(4.18)_b$ represent the same physical law – material momentum moment conservation law. The equations $(4.18)_a$ take a very complicated form, but, $(4.18)_b$ is given in a rather simple form. Usually, we make use of $(4.18)_b$ to replace $(4.18)_a$ in solving practical problems.

<u>Rotation of Space Coordinate Frame and Conservation Laws of Physical Momentum Moment</u>

The physical momentum moment conservation law can be derived by applying the small rotation transformation of space coordinates y_a

$$\delta y_a=\epsilon_{abc}\,\alpha_b\,y_c \qquad (4.22)$$

to the equations $(3.19)_2$ and $(3.25)_2$ in the same fashion as we did for $(4.18)_a$ – $(4.18)_b$. This conservation law is expressed either by

$$\epsilon_{abc}\Big(\frac{\partial m_{ab}}{\partial t}+\frac{\partial m_{\mu ab}}{\partial x_\mu}\Big)=0 \qquad (4.23)_a$$

in its Lagrangian representation or by

$$\epsilon_{abc}\left(\frac{\partial M_{ab}}{\partial t}+\frac{\partial M_{dab}}{\partial y_d}\right)=0 \tag{4.23$_b$}$$

in its Eulerian representation, where

$$m_{ab}=P_a\ y_b, \qquad m_{\mu ab}=P_{a\mu}\ y_b \tag{4.24}$$

and

$$\begin{aligned} M_{ab} &= p_a\ y_b + S_{aA}\ \phi_{bA} + S_{aAB}\ \omega_{bAB} \\ &+ \frac{\partial \mathcal{L}}{\partial\,\dot{\phi}_{bAe}}\ \phi_{aAe} + \frac{\partial \mathcal{L}}{\partial\,\dot{\omega}_{bABe}}\ \omega_{aABe} \end{aligned} \tag{4.25}$$

$$\begin{aligned} M_{dab} &= p_{da}\ y_b + S_{aAd}\ \phi_{bA} + S_{aABd}\ \omega_{bAB} \\ &+ \frac{\partial \mathcal{L}}{\partial\,\dot{\phi}_{aAd}}\ \dot{\phi}_{bA} + \frac{\partial \mathcal{L}}{\partial\,\dot{\omega}_{aABd}}\ \dot{\omega}_{bAB} \end{aligned} \tag{4.26}$$

<u>Gauge Transformation and Gauge Moment Conservation Law</u>

In dealing with the conservation laws for dislocation and disclination continuum, a question arises: except for the conservation laws derived above, does there exist another kind of conservation law which is related to gauge transformation? To answer the question, let us introduce a small rotation transformation for local anholonomic coordinate frame as

$$\bar{\delta}\, z_A = \epsilon_{ABC}\ \alpha_B\ \delta z_C \tag{4.27}$$

where α_C are small parameters. With the transformation (4.27) we call gauge one and using the same rule as given in (4.17), the variations of vielbein and gauge connection and their derivatives are given by

$$
\begin{aligned}
\bar{\delta}\, \phi_{\mu A} &= \epsilon_{ABC}\, \alpha_B\, \phi_{\mu C} \\
\bar{\delta}\, \phi_{\mu AB} &= \alpha_E\, (\epsilon_{AEC}\, \omega_{\mu CB} + \epsilon_{BEC}\, \omega_{\mu AC}) \\
&= 2\alpha_E\, \epsilon_{[A|EC}\, \omega_{\mu C|B]} \\
\bar{\delta}\, \dot{\phi}_{\mu A} &= \epsilon_{ABC}\, \alpha_B\, \dot{\phi}_{\mu C} \\
\bar{\delta}\, \dot{\omega}_{\mu AB} &= 2\alpha_E\, \epsilon_{[A|EC}\, \omega_{\mu C|B]} \\
\bar{\delta}\, \phi_{\mu A\nu} &= \epsilon_{ABC}\, \alpha_B\, \phi_{\mu C\nu} \\
\bar{\delta}\, \omega_{\mu AB\nu} &= 2\alpha_E\, \epsilon_{[A|EC}\, \omega_{\mu C|B]\nu} \\
\bar{\delta}\, \dot{\phi}_{\mu A\nu} &= \epsilon_{ABC}\, \alpha_B\, \dot{\phi}_{\mu C\nu} \\
\bar{\delta}\, \dot{\omega}_{\mu AB\nu} &= 2\alpha_E\, \epsilon_{[A|EC}\, \dot{\omega}_{\mu C|B]\nu}
\end{aligned}
\tag{4.28}
$$

Since the rotation of local anholonomic coordinate system is independent of the coordinate x_μ or y_a and does not make any change in the indices μ and a, therefore under this transformation (4.27), we have

$$\delta t = \delta x_\mu = \delta y_a = 0 \tag{4.29}$$

Substituting (4.28) and (4.29) into (3.19), we find the following conservation equations

$$\frac{\partial g_E}{\partial t} + \frac{\partial g_{E\mu}}{\partial x_\mu} = 0 \qquad (4.30)_a$$

where

$$
\begin{aligned}
g_E &= \epsilon_{AEC}\, S_{\mu a}\, \phi_{\mu C} + 2\, \epsilon_{[A|EC}\, \omega_{\mu C|B]}\, S_{\mu AB} \\
g_{E\mu} &= \epsilon_{AEC}\, S_{\nu a\mu}\, \phi_{\nu C} + 2\, S_{\nu AB\mu}\, \epsilon_{[A|EC}\, \omega_{\nu C|B]} \\
&\quad + \epsilon_{AEC}\, \frac{\partial L}{\partial\, \dot{\phi}_{\nu A\mu}}\, \dot{\phi}_{\nu C} + 2\, \frac{\partial L}{\partial\, \dot{\omega}_{\nu AB\mu}}\, \epsilon_{[A|EC}\, \dot{\omega}_{\nu C|B]}
\end{aligned}
\tag{4.31}
$$

are called gauge moment tensor in Lagrangian representation. When disclinations disappear, that is, the gauge connection vanishes, the equations $(4.30)_a$ becomes

$$\epsilon_{AEC}\Big[\frac{\partial}{\partial t}(S_{\mu A}\,\phi_{\mu C})+\frac{\partial}{\partial x_\nu}\Big(S_{\mu A\nu}\,\phi_{\mu C}+\frac{\partial L}{\partial\,\dot{\phi}_{\mu A\nu}}\,\dot{\phi}_{\mu C}\Big)\Big]=0 \tag{4.32}$$

Especially, for the static problem the above equations are simplified to

$$\epsilon_{AEC}\frac{\partial}{\partial x_\mu}\Big(\frac{\partial W}{\partial\phi_{\nu A\mu}}\,\phi_{\nu C}\Big)=0 \tag{4.33_1}$$

or

$$\epsilon_{AEC}\frac{\partial}{\partial x_\nu}(\sigma_{\mu\nu A}\,\phi_{\mu C})=0 \tag{4.33_2}$$

where W is the internal energy of the dislocation continuum per unit volume before deformation, and

$$\sigma_{\mu\nu A}=\frac{\partial W}{\partial\phi_{\mu A\nu}} \tag{4.34}$$

represents the hyperstress tensor in Lagrangian representation.

In a very similar way, the gauge moment conservation equations $(4.30)_a$ can be written as

$$\frac{\partial G_E}{\partial t}+\frac{\partial G_{Ea}}{\partial y_a}=0 \tag{4.30_b}$$

in Eulerian representation, where

$$\begin{aligned} G_E &= \epsilon_{AEC}\,S_{aA}\,\phi_{aC}+2\,\epsilon_{[A|EC}\,\omega_{aC|B]}\,S_{aAB}\\ G_{Ea} &= \epsilon_{AEC}\,S_{bAa}\,\phi_{bC}+2\,\epsilon_{[A|EC}\,\omega_{bC|B]}\,S_{bABa}\\ &\quad+\epsilon_{AEC}\,\frac{\partial L}{\partial\,\dot{\phi}_{bAa}}\,\dot{\phi}_{bC}+2\,\frac{\partial L}{\partial\,\dot{\omega}_{bABa}}\,\epsilon_{[A|EC}\,\omega_{bC|B]} \end{aligned} \tag{4.35}$$

are also called gauge moment tensors in Eulerian representation. When the disclinations disappear inside material body, the equations $(4.30)_b$ reduces to

$$\epsilon_{AEC}\left[\frac{\partial}{\partial t}(S_{aA}\ \phi_{aC})+\frac{\partial}{\partial y_a}\left(S_{bAa}\ \phi_{bC}+\frac{\partial \mathcal{L}}{\partial\ \dot{\phi}_{bAa}}\ \dot{\phi}_{bC}\right)\right]=0 \qquad (4.36)$$

Furthermore, for the static problem, this equation is simplified to

$$\epsilon_{AEC}\frac{\partial}{\partial y_a}\left(\frac{\partial W}{\partial \phi_{bAa}}\ \phi_{bC}\right)=0 \qquad (4.37)_1$$

or

$$\epsilon_{AEC}\frac{\partial}{\partial y_a}(\sigma_{baA}\ \phi_{bC})=0 \qquad (4.37)_2$$

and

$$\sigma_{baA}=\frac{\partial W}{\partial \phi_{bAa}}$$

represents the hyperstress tensor in Eulerian representation.

Finally, we should notice that since gauge transformation (4.27) has nothing to do with time and point coordinates, whether or not the Lagrangian function depends on time or the coordinates explicitly, the gauge moment conservation law always holds true. On the other hand, we can also prove from (4.33) that the dislocation density tensor $\alpha^{\mu\nu}$ defined in (2.20) is symmetrical. (Further study of the gauge moment conservation laws and its application will be given in a separate paper.)

5. Concluding Remarks

Combining the vielbein and gauge field theory of dislocation and disclination continua with Noether's theorm, an effective method is presented to deal with conservation laws and their duality principles in such media. Besides the conventional conservation laws derived from the conventional small transformation of time, material and spatial coordinates, the procedure yields an additional conservation law termed as the gauge moment conservation law by employing a small gauge transformation to the variational invariance equations. This law corresponds to the isotropic characteristics of the gauge field. When dislocation and disclination vanish, all conservation laws reduce to those studied extensively in elastic (linear or non-linear) continua[11,13,14] therefore, the results given in the paper can be considered as a natural extension from elastic continuum theory to dislocation and disclination continuum theory.

We notice that all conservation laws are expressed in 4-dimensional divergence-free forms. For the static problem, the conservation laws can be represented through so-called path-independent integral forms which are of major importance in the study of defect and fracture mechanics. We would like to mention here that any physical quantities appearing in the Lagrangian must remain not only covariant with respect to coordinate transformations but invariant with respect to gauge transformations as well. Following this principle, the elastic strain tensor, dislocation density and disclination density tensors and their time differentials are suggested to be such proper quantities. In a separate paper, we shall discuss this issue in some detail where special attention will be given to the problem of determining the dependence of the Lagrangian on its determining parameters and the practical application of path-independent integrals due to conservation laws.

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