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THE STRUCTURE OF THE TOPOLOGICAL CURRENT*

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ABSTRACT

The topological current corresponding to a system of moving point-like particles is introduced. The baryon number current, its related field equation and $U(1)$ gauge potential corresponding to the baryon number are discussed in a general way. The 't Hooft static magnetic monopole theory is generalized to the case of a system of moving monopoles.

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INTRODUCTION

In recent years, topological currents have been found to play a significant role in particle physics. The interesting well known examples are the magnetic charge current of 't Hooft monopoles,[1,2] the Wess-Zumino anomaly[3] in nonlinear σ-models and the Skyrmions current,[4] the charge of which has been discovered to possess many properties in common with baryon number.[5,6] In general, the topological current need not be deduced from the Noether's theorem, a conserved topological current is always identically conserved and the charge corresponding to which is determined only by the topological property of the current (topological number) which does not depend on the concrete model.

In this paper we study the inner structure of the conserved topological currents in $SU(2)$ theory. In Sec. 1 we introduce the topological current corresponding to a system fo point like particles,[6] which is a foundation to discuss other kinds of topological currents. In Sec. 2 we study the structure of baryon number current in a more general way. It contains the discussion of the inner motion of the constituents in baryon, the field equation related to this current and the $U(1)$ gauge potential corresponding to the baryon number. In Sec. 3 we apply the theory of Sec. 1 to study the topological structure of the magnetic charge current of a system of moving monopoles and show that in topology the motion of magnetic monopoles may be interpreted as the motion of the zeros of the Higgs field. It is in fact a generalization of the topological theory of magnetic monopoles in Ref. 2.

1. TOPOLOGICAL CURRENTS CORRESPONDING TO POINT LIKE PARTICLES

We start by studying the topological current corresponding to the point like particles, which is of importance to discuss other kinds of topological currents in our paper. Let us consider a current with charge g_0 in the form[7]

$$j^{\lambda}=\frac{1}{4\pi}g_0\frac{1}{2}\epsilon^{\lambda\mu\nu\rho}\epsilon_{abc}\partial_\mu n^a\partial_\nu n^b\partial_\rho n^c \tag{1}$$

where $\mu=0,1,2,3$ and $a=1,2,3$. n^a are normalized functions in iso-space which are defined in terms of three basic field functions ϕ^a as follows

$$n^a = \frac{\phi^a}{\phi} \quad , \quad \phi = |\vec{\phi}| \quad , \tag{2}$$

and

$$n^a n^a = 1 \quad , \quad n^a \partial_\mu n^a = 0 \quad . \tag{3}$$

It is evident that j^λ is identically conserved:

$$\partial_\lambda j^\lambda = 0 \quad . \tag{4}$$

Using

$$\partial_\mu n^a = \frac{\delta^{a\ell}\phi^2 - \phi^a \phi^\ell}{\phi^2} \partial_\mu \phi^\ell$$

and

$$\frac{\partial}{\partial \phi^\ell}\left(\frac{1}{\phi}\right) = -\frac{\phi^\ell}{\phi^3} \quad , \quad \frac{\partial}{\partial \phi^\ell}\left(\frac{1}{\phi}\right) = -\frac{\phi^\ell}{\phi^3} \quad . \tag{5}$$

(1) can be written as

$$\begin{aligned} j^\lambda &= \frac{1}{4\pi} g_0 \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} \epsilon_{abc} \partial_\mu (n^a \partial_\nu n^b \partial_\rho n^c) \\ &= -\frac{1}{4\pi} g_0 \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} \epsilon_{abc} \frac{\partial}{\partial \phi^\ell} \frac{\partial}{\partial \phi^a} \left(\frac{1}{\phi}\right) \partial_\mu \phi^\ell \partial_\nu \phi^b \partial_\rho \phi^c \quad . \end{aligned} \tag{6}$$

If we define four Jacobians:

$$\epsilon_{\ell bc} J^\lambda\left(\frac{\phi}{x}\right) = \epsilon^{\lambda\mu\nu\rho} \partial_\mu \phi^\ell \partial_\lambda \phi^b \partial_\rho \phi^c, \qquad \lambda = 0, 1, 2, 3 \tag{7}$$

in which the usual 3 - dimensional Jacobian

$$J\left(\frac{\phi}{x}\right) = J^0\left(\frac{\phi}{x}\right) \quad , \quad J\left(\frac{\phi}{x}\right) = \frac{\partial(\phi^1, \phi^2, \phi^3)}{\partial(x^1, x^2, x^3)} \tag{8}$$

and make use of the Laplacian relation in iso-space

$$\partial_a \partial_a \left(\frac{1}{\phi}\right) = -4\pi \delta(\vec{\phi}) \quad , \quad \partial_a = \frac{\partial}{\partial \phi^a} \tag{9}$$

we obtain the δ-like current

$$j^{\lambda} = g_0 \delta(\vec{\phi}) J^{\lambda}\left(\frac{\phi}{x}\right) \quad . \tag{10}$$

Suppose that the fundamental field functions $\phi^a(\vec{x}, t)$ $(a = 1, 2, 3)$ possess k isolated zeros and let the i^{th} zero be $\vec{x} = \vec{Z}_i$, we have

$$\phi^a(\vec{Z}_i(t), t) = 0 \quad , \quad i = 1 \cdots k \quad , \quad a = 1, 2, 3 \tag{11}$$

where

$$\vec{Z}_i = \vec{Z}_i(t) \quad , \quad i = 1 \cdots k \tag{12}$$

is the trajectory of the motion of the i^{th} zero. It can be proved from (11) that the velocity of the i^{th} zero is determined by

$$\frac{dZ_i^{\mu}}{dt} = \left[J^{\mu}\left(\frac{\phi}{x}\right) \Big/ J\left(\frac{\phi}{x}\right) \right]_{x = Z_i} \tag{13}$$

and it is well known from the ordinary theory of δ-function that

$$\delta(\vec{\phi}) = \sum_{i=1}^{k} \frac{1}{|J(\frac{\phi}{x})|_{x = Z_i}} \, \delta(\vec{x} - \vec{Z}_i(t)) \quad . \tag{14}$$

However, if we further consider the case that while the point $\vec{x}$ covers the region neighboring the zero $\vec{x} = \vec{Z}_i(t)$ once, the function $\vec{\phi}$ covers the corresponding region β_i times, then we have

$$\delta(\vec{\phi}) J\left(\frac{\phi}{x}\right) = \sum_{i=1}^{k} \beta_i \eta_i \, \delta(\vec{x} - \vec{Z}_i(t)) \tag{15}$$

where β_i is a positive integer (the Hopf index) and

$$\eta_i = \left[J\left(\frac{\phi}{x}\right) \Big/ \left| J\left(\frac{\phi}{x}\right) \right| \right]_{x = Z_i} = \pm 1 \tag{16}$$

is called the Brouwer index.

Therefore the current (10) can be further written in the form

$$j^{\lambda}=g_0\sum_{i=1}^{k} n_i\delta(\vec{x}-\vec{Z}_i(t))\frac{dZ_i^{\lambda}}{dt} \quad , \quad \lambda=1,2,3 \tag{17}$$

and the charge density

$$\rho=g_0\sum_{i=1}^{k} n_i\delta(\vec{x}-\vec{Z}_i(t)) \quad , \quad n_i=\beta_i\eta_i \quad . \tag{18}$$

We find that (17) and (18) are exactly the current and density of a system of k classical point particles with charge $g_0 n_i$ moving in 3 - dimensional space.

From (13) we have the total charge of the system

$$G=\int_R \rho\, d^3x=g_0\sum_{i=1}^{k} n_i \quad . \tag{19}$$

But on the other hand from (10) the total charge is

$$G=\int_R \rho\, d^3x=g_0\int_T \delta(\vec{\phi})\, d^3\phi \quad . \tag{20}$$

If we write

$$\sum_{i=1}^{k} n_i=N_+-N_-$$

from (19) and (20) we see that while the vector $\vec{x}$ covers R once, the vector must cover T, N_+ times with $\eta=+1$ and N_- times with $\eta=-1$, where η is defined by (16). Therefore the particle or anti-particle character is distinquished by the sign of the Jacobian at the point $\vec{x}=\vec{Z}_i$. The topological number $n=\sum_{i=1}^{k} n_i$ is called Kronecker index.

The current defined by (1) and its topological characters we mentioned above is of importance to study the theory of monopoles, which we will discuss in Sec. 3. We would like here to introduce another interesting example of (1). Let us consider the topological current

$$J^{\lambda}=\frac{1}{16\pi}g_0\epsilon^{\lambda\mu\nu\rho}\, Tr[u^{-1}\partial_\mu u\, u^{-1}\partial_\nu u\, u^{-1}\partial_\rho u] \tag{21}$$

where u is a unitary matrix with constant angle α

$$u = e^{in\alpha} = \cos\alpha + in\sin\alpha \tag{22}$$

$$n = n^a\tau_a \quad , \quad n^a = \frac{\phi^a}{\phi} \quad , \tag{23}$$

and τ_a $(a = 1, 2, 3)$ are Panli matrices. Current (21) is different from the Skyrmion Current in normalization constant and the matrix u.

Substituting (22) into (21) we find

$$j^\lambda = \frac{1}{4\pi} g_0 \sin^3\alpha \cos\alpha \frac{1}{2} \epsilon^{\lambda\mu\nu\rho} \epsilon_{abc} \partial_\mu n^a \partial_\nu n^b \partial_\rho n^c \tag{24}$$

which is of a form similar to current (1), but the charge

$$g = g_0 \sin^3\alpha \cos\alpha \tag{25}$$

depends on angle α. Therefore current (21) with (22) can be expressed as

$$j^\lambda = g_0 \sin^3\alpha \cos\alpha\, \delta(\vec{\phi}) J^\lambda\left(\frac{\phi}{x}\right)$$

and is corresponding to the current of a system of point like particles with charge dependent on α. When $\alpha = \frac{\pi}{3}$, g has its maximal value

$$g_{max} = \frac{3\sqrt{3}}{8} g_0 < g_0 \quad .$$

The physical meaning and the application of this topological current will be discussed elsewhile.[8]

2. THE BARYON NUMBER CURRENT

Over 20 years ago, Skyrme[4] showed that the meson-field configuration carries a topological charge which should be interpreted as baryon number. Witten[5,6] revived this idea and proposed that baryons may be described phenomenologically as the solitons of nonlinear sigma models. These solitons by now are called Skyrmions.

The baryon number current is defined by

$$j^{\lambda} = \frac{1}{24\pi^2}\epsilon^{\lambda\mu\nu\rho}\, Tr[U^{-}\partial_\mu U\, U^{-1}\partial_\nu U\, U^{-1}\partial_\rho U] \tag{26}$$

which is identically conserved. It's associated charge is the winding number of the map U

$$B(U) = \frac{1}{24\pi^2}\epsilon^{0\mu\nu\rho}\, Tr[U^{-1}\partial_\mu U\, U^{-1}\partial_\nu U\, U^{-1}\partial_\rho U] \quad . \tag{27}$$

$B(U)$ defined by (27) is additive

$$B(U_1 U_2) = B(U_1) + B(U_2)$$

and is conjectured to interpret it as baryon number. The elementary Skyrmion solution is static and spherically symmetric

$$U = e^{i\hat{x}^a \tau^a F(r)} \quad , \quad \hat{x}^a = \frac{x^a}{r} \tag{28}$$

with boundary condition $F(0) = 0$ and $F(\infty) = \pi$, which corresponds to unit winding number.

In order to study the topological structure of baryon number current (26), we write the field U in $SU(2)$ space in a more general form

$$U = e^{in\, F(\phi)} = \cos F + in \sin F \tag{29}$$

where

$$n = n^a \tau^a \quad , \quad n^a = \frac{\phi^a}{\phi} \quad , \quad \phi = |\vec{\phi}| \tag{30}$$

$\phi^a\, (a = 1, 2, 3)$ are three fundamental functions of space-time coordinates x^μ. When $\phi^a = x^a$, the field U defined by (29) reduces to (28). If we take $\phi^a = \pi^a$ and $F(\phi) = \frac{1}{F_\pi}\phi$, (29) is agrees with that of Ref. 5.

Substituting (29) into (26), taking notice of the relations

$$n^2 = 1 \quad , \quad n\partial_\mu n + \partial_\mu n\, n = 0$$

we find j^λ is composed of two parts:

$$j^\lambda = j_i + j_2 \tag{31}$$

where

$$j_1^\lambda = \frac{1}{4\pi^2}\epsilon^{\lambda\mu\nu\rho}\epsilon_{abc}\sin^2 F\,\partial_\mu F\, n^a\partial_\nu n^b\partial_\rho n^c \tag{32}$$

and

$$j_2 = \frac{1}{12\pi^2}\epsilon^{\lambda\mu\nu\rho}\epsilon_{abc}\sin^3 F\,\cos F\,\partial_\mu n^a\partial_\nu n^b\partial_\rho n^c \quad . \tag{33}$$

If we make use of (5), (7) and (9), j_1^λ and J_2^λ can be also expressed as

$$j_1^\lambda = \frac{1}{2\pi^2}\frac{\sin^2 F}{\phi^2}\frac{dF}{d\phi}\, J^\lambda\Big(\frac{\phi}{x}\Big) \tag{34}$$

$$j_2^\lambda = \frac{2}{3\pi}\sin^3 F\,\cos F\,\delta(\vec{\phi})J^\lambda\Big(\frac{\phi}{x}\Big) \quad . \tag{35}$$

We shall see later that the current j_1 denoted by (34) is identical with the usual baryon current. While the current j_2 denoted by (35) is analogous to the current in the last Section, which corresponds to the current of a system of point like particles with charge

$$g = \frac{2}{3\pi}\sin^3 F(0)\,\cos F(0) \quad . \tag{36}$$

From (36) we see that to guarantee the baryon number to be an integer, the function $F(\phi)$ at the origin $\vec{\phi} = 0$ in iso-space must satisfy the condition

$$F(\phi)\,|_{\phi=0} = \frac{1}{2}m\pi \quad , \quad m = \text{integer} \tag{37}$$

which leads to $j_2^\lambda = 0$. However we may conjecture that in baryons there really exist some new kind of point like particles corresponding to j_2^λ, but the number of particles and anti-particles must be equal, it can also lead to $j_2^\lambda = 0$. Therefore in both cases we must have

$$j^\lambda = j_1^\lambda \quad . \tag{38}$$

From (34) and (38) we find the baryon number density

$$\rho(x) = \rho(\phi) J\left(\frac{\phi}{x}\right)$$

where

$$\rho(\phi) = \frac{1}{2\pi^2} \frac{\sin^2 F}{\phi^2} \frac{dF}{d\phi} \tag{39}$$

and the baryon number

$$B = \int_V \rho(x)\, d^3x = \int_T \rho(\phi)\, d^3\phi \quad . \tag{40}$$

The above integral shows that the fundamental field $\phi^a = \phi^a(x)$ maps V into T in iso-space. If we consider the case that while the point vector $\vec{x}$ covers the region V once, function $\vec{\phi}$ covers the corresponding elementary region T_0 N times (i.e. $T = NT_0$) and choose the boundary conditions:

$$\begin{array}{lllll}
(a) & r \to \infty, & \phi = 0, & F(\phi)\,|_{\phi\to 0} & = \pi \\
(b) & r \to 0, & \phi = a, & F(\phi)\,|_{\phi\to a} & = 0 \\
(or & r \to 0, & \phi \to \infty, & F(\phi)\,|_{\phi\to\infty} & = 0)
\end{array}$$

then from (39) and (40) we have the baryon number

$$B = \frac{2}{\pi} N \int_0^\pi \sin^2 F\, dF = N \tag{41}$$

where

$$N = \text{integer} \quad .$$

We notice that the boundary condition (a) satisfies the condition (37) which guarantees the unexpected current $j_2 = 0$ and the integral does not depend on the concrete form of the function $F(\phi)$.

Using (34), (38) and (39) the baryon number current j^λ can be expressed as

$$j^\lambda = \rho(x)\, u^\lambda \tag{42}$$

where

$$u^\lambda = J^\lambda\left(\frac{\phi}{x}\right) \Big/ J\left(\frac{\phi}{x}\right) \quad , \tag{43}$$

u^λ may be interpreted as 4 - velocity of the constituents of baryon (nucleon). It can be proved from (7) that

$$\partial_\lambda \phi\, J^\lambda\left(\frac{\phi}{x}\right) = \frac{\phi^a}{\phi}\, \partial_\lambda \phi^a\, J^\lambda\left(\frac{\phi}{x}\right) = 0$$

then we have

$$\partial_\lambda \phi\, u^\lambda = 0$$

which can be expressed as

$$\nabla\phi \cdot \vec{u} = -\frac{\partial\phi}{\partial t} \tag{44}$$

where $\vec{u}$ is the three dimensional velocity of the constituents.

In stationary case:

$$\phi = f(x, y, z)$$

$$\phi^a = f^a(x, y, z, t) \quad , \quad a = 1, 2, 3 \quad .$$

$\rho(\phi)$ does not depend on t, but the constituents are in moving state. In this case

$$\nabla\phi \cdot \vec{u} = 0 \quad , \quad (\vec{u} \perp \nabla\phi)$$

therefore velocity $\vec{u}$ is always tangent to the surface

$$\phi = f(x, y, z) = c$$

This means the moving constituents will always lie on the family of surfaces (shells) $\phi = c$. For one stationary soliton $\phi = r$, and $\phi = c$ is a closed surface (sphere). Therefore if the Skyrme-soliton is a baryon and the soliton has a finite size, the constituents in baryon should be confined.

We continue to study the field equations related to the baryon number current j^λ. Let K^1, K^2, K^3 be functions of ϕ^1, ϕ^2, ϕ^3 such that[8]

$$J\left(\frac{K}{\phi}\right) = \frac{\partial(K^1, K^2, K^3)}{\partial(\phi^1, \phi^2, \phi^3)} = \frac{\sin^2 F(\phi)}{\phi^2}\frac{dF}{d\phi} \tag{45}$$

then from (7), (34), (38) and

$$\epsilon_{abc}\, J\left(\frac{K}{\phi}\right) = \epsilon_{ijk}\,\partial_a K^i \partial_b K^j\, \partial_c K^k \ , \ \partial_a = \frac{\partial}{\partial \phi^a}$$

j^λ and ρ can be written in the form

$$\begin{aligned} j^\lambda &= \frac{1}{2\pi^2}\, J\left(\frac{K}{\phi}\right) J^\lambda\left(\frac{\phi}{x}\right) \\ &= \frac{1}{2\pi^2}\frac{1}{6}\, \epsilon^{\lambda\mu\nu\rho}\, \epsilon_{ijk} \partial_\mu K^i \partial_\nu K^j \partial_\rho K^k \\ &= \frac{1}{2\pi^2}\, J^\lambda\left(\frac{K}{x}\right) \end{aligned} \tag{46}$$

$$\rho = \frac{1}{2\pi^2}\, J\left(\frac{K}{x}\right) \ . \tag{47}$$

Since the preceding current can be also expressed as

$$j^\lambda = \frac{1}{12\pi^2}\, \epsilon^{\lambda\mu\nu\rho} \epsilon_{ijk}\, \partial_\mu [K^i \partial_\nu K^j \partial_\rho K^k] \quad , \tag{48}$$

if we define a field tensor

$$f_{\nu\rho} = \frac{2}{3\pi}\, \epsilon_{ijk}\, K^i\, \partial_\nu\, K^j\, \partial_\rho\, K^k \tag{49}$$

and its dual tensor

$$\tilde{f}^{\lambda\mu} = \frac{1}{2} \epsilon^{\lambda\mu\nu\rho}\, f_{\nu\rho} = \frac{1}{3\pi}\, \epsilon^{\lambda\mu\nu\rho} \epsilon_{ijk}\, K^i \partial_\nu K^j \partial_\rho\, K^k \tag{50}$$

then from (48) we have the field equation for the baryon number current

$$\partial_\mu \tilde{f}^{\lambda\mu} = 4\pi j^\lambda \quad . \tag{51}$$

We could also define a dual current

$$\tilde{j}^\nu = \frac{1}{6\pi^2} \epsilon_{ijk} \partial_\rho [K^i \partial_\nu K^j \partial_\rho K^k] \tag{52}$$

and from (49) we have

$$\partial_\rho f^{\nu\rho} = 4\pi \tilde{j}^\nu \quad . \tag{53}$$

We notice that field equations (51) and (53) are analogous to Maxwell's equations with magnetic charge and the baryon number current j^λ is the dual current defined by (52).

The field tensor (49) can be also written in the form

$$f_{\mu\nu} = \frac{2}{3\pi} \vec{K} \cdot \partial_\mu \vec{K} \times \partial_\nu \vec{K} \quad . \tag{54}$$

To study its inner structure, we denote the vector $\vec{K}$ by

$$\vec{K} = K \, \vec{e} \tag{55}$$

where K and $\vec{e}$ are magnitude and direction of $\vec{K}$, respectively. Let $\vec{e}_1$ and $\vec{e}_2$ be two arbitrary perpendicular unit vectors, and

$$\vec{e} = \vec{e}_1 \times \vec{e}_2$$

and define a vector potential function

$$a_\mu = \frac{1}{2} \epsilon^{ab} (\vec{e}_a \cdot \partial_\mu \vec{e}_b) \quad , \quad \mu = 1, 2, 3, 4 \quad , \tag{56}$$

then $f_{\mu\nu}$ can be expressed in term of a_μ:

$$f_{\mu\nu} = \frac{2}{3\pi} K^3 [\partial_\mu a_\nu - \partial_\nu a_\mu] \quad . \tag{57}$$

This means a_μ is a $U(1)$ gauge potential and which is relevant to the baryon number B. $f_{\mu\nu}$ is invariant under the $U(1)$ gauge transformation

$$a'_\mu = a_\mu + \partial_\mu \theta$$

and θ is related to the angle that $\vec{e}_1$ and $\vec{e}_2$ are rotating about the axis in direction $\vec{e}$:

$$\vec{e}'_1 = \cos\theta\, \vec{e}_1 - \sin\theta\, \vec{e}_2$$

$$\vec{e}'_2 = \sin\theta\, \vec{e}_1 + \cos\theta\, \vec{e}_2 \quad .$$

It must be pointed out that the field tensor $f_{\mu\nu}$ expressed as (54) is different from the usual electromagnetic field tensor by a factor K^3, which is a function of space-time coordinates x^μ. This factor guarantees that the baryon number current $j^\lambda = \frac{1}{4\pi}\partial_\nu \tilde{f}^{\lambda\nu}$ does not vanish identically.

To conclude this section, we give an example of the solutions of the partial differential Eq. (45):

$$\begin{vmatrix} \frac{\partial K^1}{\partial \phi^1}, \frac{\partial K^1}{\partial \phi^2}, \frac{\partial K^1}{\partial \phi^3} \\ \frac{\partial K^2}{\partial \phi^1}, \frac{\partial K^2}{\partial \phi^2}, \frac{\partial K^2}{\partial \phi^3} \\ \frac{\partial K^3}{\partial \phi^1}, \frac{\partial K^3}{\partial \phi^2}, \frac{\partial K^3}{\partial \phi^3} \end{vmatrix} = \frac{\sin^2 F(\phi)}{\phi^2}\frac{dF}{d\phi} \quad . \tag{58}$$

We take the Ansatz

$$K^1 = \frac{F}{2} - \frac{1}{4}\sin 2F$$

then we have

$$\frac{\partial K^1}{\partial \phi^a} = \sin^2 F \frac{dF}{d\phi}\frac{\phi^a}{\phi}$$

and Eq. (58) reduces to

$$\begin{vmatrix} \phi^1, & \phi^2, & \phi^3 \\ \frac{\partial K^2}{\partial \phi^1}, & \frac{\partial K^2}{\partial \phi^2}, & \frac{\partial K^2}{\partial \phi^3} \\ \frac{\partial K^3}{\partial \phi^1}, & \frac{\partial K^3}{\partial \phi^2}, & \frac{\partial K^3}{\partial \phi^3} \end{vmatrix} = \frac{1}{\phi} \quad . \tag{59}$$

It can be proved that in spherical coordinates in ϕ-space

$$\begin{aligned}\phi^1 &= \phi \sin\alpha \cos\beta \\ \phi^2 &= \phi \sin\alpha \sin\beta \\ \phi^3 &= \phi \cos\beta\end{aligned}$$

Eq. (59) becomes

$$\frac{\partial K_2}{\partial \alpha}\frac{\partial K_3}{\partial \beta} - \frac{\partial K_2}{\partial \beta}\frac{\partial K_3}{\partial \alpha} = \sin\alpha \quad .$$

A solution of the above equation is

$$K_3 = \beta \quad \text{and} \quad K_2 = -\cos\alpha \quad .$$

Substituting this solution into (46) we have

$$j^\lambda = -\frac{1}{2\pi^2}\,\epsilon^{\lambda\mu\nu\rho}\partial_\mu \eta\, \partial_\nu \cos\alpha\, \partial_\rho \beta$$

and the corresponding baryon number density is

$$\rho = \frac{1}{2\pi^2}\sin\alpha\, \nabla\eta \cdot \nabla\alpha \times \nabla\beta$$

where

$$\eta = \frac{1}{2}F(\phi) - \frac{1}{4}\sin^2 F(\phi) \quad .$$

3. THE TOPOLOGICAL CURRENT OF MAGNETIC MONOPOLES

In $SU(2)$ gauge theory the electromagnetic field is defined by 't Hooft[1]:

$$f_{\mu\nu} = F^a_{\mu\nu} n^a - \frac{1}{e}\epsilon_{abc} n^a D_\mu n^b D_\nu n^c \tag{60}$$

where e is the electromagnetic coupling constant, $F^a_{\mu\nu}$ is the gauge field tensor

$$F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu + e\,\epsilon_{abc} B^b_\mu B^c_\nu \quad . \tag{61}$$

B^a_μ is the gauge potential and

$$D_\mu n^a = \partial_\mu n^a + e\,\epsilon_{abc} B^b_\mu n^c \tag{62}$$

is the covariant derivative of the unit vector n^a in iso-space which is defined by

$$n^a = \frac{\phi^a}{\phi} \quad , \quad \phi = |\vec{\phi}| \quad .$$

Here the fundamental field ϕ^a $(a = 1, 2, 3)$ in 't Hooft's theory is identified with the Higgs field.

Using (61) and (62) the electromagnetic field tensor $f_{\mu\nu}$ can be further written in the form

$$f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - K_{\mu\nu} \tag{63}$$

in which

$$A_\mu = B^a_\mu n^a$$

is the electromagnetic potential and

$$K_{\mu\nu} = \frac{1}{e}\,\epsilon_{abc}\, n^a \partial_\mu n^b \partial_\nu n^c \quad . \tag{64}$$

Substituting $f_{\mu\nu}$ denoted by (63) into first pair of Maxwell's equation

$$\begin{aligned} \partial_\mu \tilde{f}^{\lambda\mu} &= -4\pi\, \tilde{j}^\lambda \\ \tilde{f}^{\lambda\mu} &= \frac{1}{2}\epsilon^{\lambda\mu\nu\rho} f_{\nu\rho} \end{aligned} \tag{65}$$

we find

$$\frac{1}{4\pi}\frac{1}{e}\frac{1}{2}\,\epsilon^{\lambda\mu\nu\rho}\epsilon_{abc}\,\partial_\mu n^a \partial_\nu n^b \partial_\rho n^c = \tilde{j}^\lambda \tag{66}$$

which is just of the same form as the topological current defined by (1).

For a system of k classical monopoles with arbitrary magnetic charge g_i ($i = 1 \cdots k$), the magnetic charge current and density in general can be expressed as

$$\tilde{j}^{\lambda} = \sum_{i=1}^{k} g_i \delta(\vec{x} - \vec{Z}_i(t)) \frac{dZ_i^{\lambda}}{dt} \quad , \quad \lambda = 1,2,3 \tag{67}$$

$$\rho = \sum_{i=1}^{k} g_i \delta(\vec{x} - \vec{Z}_i(t)) \quad . \tag{68}$$

We notice that comparing (17), (18) with (67) and (68) through (66) gives a general proof of the well-known conclusion that if Maxwell's Eq. (65) has solution, the magnetic charge of a magnetic monopole must satisfy the quantized condition

$$g_i = n_i \, g_0 \quad , \quad i = 1 \cdots k$$

where

$$n_i = \eta_i \beta_i = \text{integer}$$

is a topological number 1^{st} Chern class and the unit magnetic charge of a magnetic monopole is

$$g_0 = \frac{1}{e} \quad .$$

The magnetic monopole and anti-magnetic monopole character is distinguished by the Brouwer index

$$\eta_i = \left[j\left(\frac{\phi}{x}\right) \Big/ \left| J\left(\frac{\sigma}{x}\right) \right| \right]_{x=Z_i} = \pm 1 \quad .$$

A magnetic monopole with multiple unit magnetic charge is determined by the Hopf index β_i. In topology the motion of a system of monopoles may be interpreted as the motion of the zeros of the Higgs field.

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