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GAUGE FIELD THEORY OF DISLOCATION AND DISCLINATION CONTINUUM*

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ABSTRACT

The geometrical structure of a material manifold with dislocations and disclinations is built by applying vielbein theory and gauge field theory. Two kinds of connections, namely the affine and gauge connection are introduced for describing the characteristics of plastic imperfections. As a result, the torsion tensor can be decomposed into a pure vielbein part and a gauge potential part. The gauge field tensor constructed based on the gauge connection is shown to be equivalent to the curvature tensor, and is directly related to disclination. Taking displacement field, vielbein and gauge connection as three basic determining parameters, the constitutive equations and governing equations are obtained based on a variational principle for the stationary problem. It is emphasized that the physical quantities appearing in Lagrangian function must be covariant not only in the coordinate transformation but in the gauge transformation as well. Furthermore, when the elastic strain tensor, the dislocation and disclination densities are regarded as such quantities, the governing equations are shown to be covariant for the combined transformation. For practical applications, two specific forms of the internal energy density are proposed, especially, only nine material constants are needed for fully description of macroscopically isotropic material with dislocations and disclinations.

1. Introduction

Defect continuum physics is an important phase of current development in modern continuum physics and aims at establishing a sound theoretical basis by which the elastic and nonelastic behavior of a material body with dislocations and disclinations can be explored. This field, initially studied by Kondo,[1,2] and then developed by Kroñer,[3,4] Bilby[5,6] et al. is closely related to the theory of the nonriemannian geometry. Since then, a great deal progress has been made through many researchers' efforts.[7,8,9,10] It has been discovered that the geometric constructs of nonriemannian space such as metric, torsion and curvature tensors naturally describe the main characteristics of plastic imperfections (dislocations and disclinations), there the affine connection plays an important role in building up the relation between the macroscopic motion and deformation of a material body and the moving dislocations and disclinations.

Nonriemannian geometry is known as a classic branch in pure mathematics, however, it is only in recent decades has the connection been made to modern continuum physics, especially with dislocation continuum theory. On the other hand, since 1955 when Yong-Mills field theory[11] was discovered, one recognized that riemannian geometry itself essentially belongs to a kind of gauge field theory.[12,13] Furthermore, in quite recent years, from the study of supergravity theory, it was learned that the geometry of nonriemannian space with non-vanishing torsion also belongs to a kind of nonAbelian gauge theory. We should notice that introducing a local basis by means of the vielbein is a necessary step to describe the nonriemannian geometry with gauge field theory. We are convinced that the non-Abelian gauge theory can be naturally applied to any field in theoretical physics provided that it is related to riemannian and nonriemannian geometry theory. Based on this point of view, some work has been done in using the gauge theory to the study of dislocation continua. A. G. Herrmann[14,15] first used Abelian gauge theory to discuss the gauge invariances of the governing equations with electromagnetic field theory. Her work lead to the further study by Edelen[16] and Kadic and Edelen[17] in the Yong-Mills type minimal coupling theory for materials with dislocation and disclinations.

The paper is intended to present a unified approach to the study of defect mechanics by applying gauge field theory to dislocation and disclination continuum. If both dislocations and disclinations are taken into account, the vielbein theory appears to be necessary to describe the material manifold in transforming from the deformed stated to the natural state or from the reference state to the natural state. In using the vielbein theory, two kinds of connection, namely the affine and gauge connection are introduced, the former having its usual meaning in nonriemannian geometry theory. Thus, what we have considered in dealing with the geometric structure of the material manifold is different from that done in the previous studies, there only the affine connection is treated. Because torsion and curvature are responsible, respectively, for the dislocations and disclinations, the advantage of introducing two conections is that the torsion can be directly decomposed into the pure vielbein part and the gauge potential part, the former responsible for Burgers vector density. When the gauge potential vanishes, the nonriemannian space becomes a flat one and the dislocation tensor is completely due to the vielbein. The vielbein is also shown to be responsible for the plastic strain tensor. In the vielbein theory, the gauge field tensor constructed by the gauge connection is equivalent to the curvature tensor so that the gauge potential plays an important role in the description of disclination.

For a complete theory of defect mechanics, we have to deal with not only the geometric aspects of material manifold but also the dynamic governing equations which are established usually based on variational principle. It should be emphasized that the real physical quantities which appear in Lagrangian function must keep their invariance not only for the coordinate transformation but for the gauge transformation as well. Keeping this in mind and using the displacement field, vielbein and gauge connection as three determining variables, the governing equations has been obtained in Lagrangian description method without the assumption of small deformation. For practical purposes, elastic strain tensor, dislocation and disclination density which are taken to be involved in the fundamental governing equations are proved to be covariant with the combined transformation too. Within the framework of small deformation theory, a quadratic form for the internal energy is proposed for enisotropic and isotropic dislocated materials. Only nine material constants are needed for the fully description of isotropic medium.

2. Vielbein Theory and Descriptions of Motion and Deformation of Materials with Plastic Imperfections

The motion and deformation of a material body with plastic imperfections (dislocations and disclinations) can be described by three different states, namely the reference, the deformed and the natural states respectively. Hereafter, we always refer these three states to the r-state, the d-state and the n-state. Considering the r-state, let x^{μ} be the coordinate system with the base vectors $\vec{e}_{\mu}$ in an Euclidian space E_3, in which the material body is immersed. The metric tensor in the r-state is

$$h^{o}_{\mu\nu} = \vec{e}_{\mu} \cdot \vec{e}_{\nu} \tag{2.1}$$

When the material body is loaded by external forces from outside, it moves and deforms from the r-state to the d-state. During the continuous motion and deformation, new plastic imperfectons can be created inside body. The deformed body can be described by a new coordinate system y^{a} with the base vectors $\vec{e}_{a}$ and metric tensor

$$h_{ab} = \vec{e}_{a} \cdot \vec{e}_{b} \tag{2.2}$$

Therefore, we define a relation

$$y^a = y^a(x,t) \qquad (a = 1,2,3) \tag{2.3}$$

or its inverse

$$x^\mu = x^\mu(y,t) \qquad (\mu = 1,2,3) \tag{2.4}$$

to describe the motion of the body from the r-state to the d-state or from the d-state to the r-state respectively. The increment vector $d\vec{r}$ in the d-state between two material points with the coordinates y^a and $y^a + dy^a$ is given by

$$d\vec{r} = dy^a \vec{e}_a = J^a_\mu \; dx^\mu \vec{e}_a = \vec{h}_\mu \; dx^\mu \tag{2.5}$$

where

$$J^a_\mu = \partial_\mu y^a \quad , \quad \vec{h}_\mu = J^a_\mu \vec{e}_a \quad , \quad h_{\mu\nu} = \vec{h}_\mu \cdot \vec{h}_\nu \tag{2.6}$$

express, respectively, the "displacement gradient" and comoving coordinate base vectors and its corresponding metric tensor. Thus, the Lagrangian strain tensor is defined by

$$E_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} - h^o_{\mu\nu}) \tag{2.7}$$

as indicated in the conventional theory of nonlinear elasticity.

It is known from the dislocation continuum theory[3,4] that the n-state of the material body can be obtained by cutting a very small volume element spanned by three base vectors $\vec{h}_\mu$ in the d-state off from its surroundings and releasing it from the constraints of the surroundings. The process of the cutting is described by using an affine transformation A of the torn small material element. We may express each small line element $\delta\vec{R}$ in the n-state which is from $d\vec{r}$ in the d-state by

$$\delta\vec{R} = \delta z_A \; \vec{e}_A = A \cdot d\vec{r} = \phi_{aA} \; dy^a \; \vec{e}_A \tag{2.8}$$

or in component form

$$\delta z_A = \phi_{aA}\, dy^a = \phi_{aA}\, J^a_\mu\, dx^\mu \tag{2.9}$$

where z_A expresses the local coordinate with the local rectangular coordinate base vectors $\vec{e}_A$ and ϕ_{aA} is usually called the distortion tensor. If we consider the corresponding increment vector $d\vec{r}_0$ of $d\vec{r}$ in the r-state, (2.9) can be written as

$$\delta z_A = \phi_{\mu A}\, dx^\mu \quad , \quad \phi_{\mu A} = \phi_{aA}\, J^a_\mu \tag{2.10}$$

From (2.9) and (2.10), we can introduce two metric tensors as

$$g_{ab} = \phi_{aA}\, \phi_{bA} \quad , \quad g_{\mu\nu} = \phi_{\mu A}\, \phi_{\nu A} \tag{2.11a}$$

which are related to each other by

$$g_{ab} = J^\mu_a\, J^\nu_b\, g_{\mu\nu} \quad , \quad g_{\mu\nu} = J^a_\mu\, J^b_\nu\, g_{ab} \tag{2.11b}$$

or in contravariant form

$$g^{ab} = J^a_\mu\, J^b_\nu\, g^{\mu\nu} \quad , \quad g^{\mu\nu} = J^\mu_a\, J^\nu_b\, g^{ab} \tag{2.11c}$$

where g_{ab} and $g_{\mu\nu}$ are two basic metric tensors for the nonriemannian space, and $\phi_{\mu A}$ is called the vielbein having two indices μ and A, which do not belong to the same space.

To study the geometric structure of the nonriemannian space, we here introduce two different kinds of transformation for the vielbein $\phi_{\mu A}$. The first is a general coordinate transformation

$$x^{\mu'} = f^{\mu'}(x) \tag{2.12}$$

or its inverse

$$x^{\mu} = f^{\mu}(x^{\mu'}) \tag{2.13}$$

where the transformation functions $f^{\mu'}$ and f^{μ} are differentiable up to all orders of interest. With these transformations, the vielbein $\phi_{\mu A}$ is transformed to

$$\phi_{\mu' A}(x', t) = x^{\mu}_{\mu'}\ \phi_{\mu A}(x, t) \quad , \quad \phi^{\mu'}_{A}(x', t) = x^{\mu'}_{\mu}\ \phi^{\mu}_{A}(x, t) \tag{2.14}$$

where

$$x^{\mu}_{\mu'} = \partial_{\mu'}\ x^{\mu} \quad , \quad x^{\mu'}_{\mu} = \partial_{\mu}\ x^{\mu'}$$

Thus, the metric tensor $g_{\mu\nu}$ is tranformed to

$$g_{\mu'\nu'}(x', t) = x^{\mu}_{\mu'}\ x^{\nu}_{\nu'}\ g_{\mu\nu}(x, t) \tag{2.15}$$

or in the same procedure, we have

$$\begin{aligned}
J^{a}_{\mu'} &= \partial_{\mu'}\ y^{a} = x^{\mu}_{\mu'}\ J^{a}_{\mu} \\
\vec{h}_{\mu'} &= J^{a}_{\mu}\ x^{\mu}_{\mu'}\ \vec{e}_{a} = x^{\mu}_{\mu'}\ \vec{h}_{\mu} \\
h_{\mu'\nu'} &= \vec{h}_{\mu'} \cdot \vec{h}_{\nu'} = x^{\mu}_{\mu'}\ x^{\nu}_{\nu'}\ \vec{h}_{\mu} \cdot \vec{h}_{\nu} = x^{\mu}_{\mu'}\ x^{\nu}_{\nu'}\ h_{\mu\nu} \\
\phi_{\mu' A} &= J^{a}_{\mu'}\ \phi_{aA} = x^{\mu}_{\mu'}\ J^{a}_{\mu}\ \phi_{aA} = x^{\mu}_{\mu'}\ \phi_{\mu} A
\end{aligned} \tag{2.16}$$

Thus, we can see that all the above quantities behave as a tensor, especially, for the vielbein, which is treated as a vector for the index μ while the index A is fixed. We define the elastic and plastic strain tensor as

$$E^{e}_{\mu\nu} = \frac{1}{2}(h_{\mu\nu} - g_{\mu\nu}) \quad , \quad E^{p}_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - h^{o}_{\mu\nu}) \tag{2.17}$$

respectively. With the coordinate transformation (2.12), these two tensors are transformed by

$$E^e_{\mu'\nu'} = x^{\mu}_{\mu'}\, x^{\nu}_{\nu'}\, E^e_{\mu\nu} \quad , \quad E^p_{\mu'\nu'} = x^{\mu}_{\mu'}\, x^{\nu}_{\nu'}\, E^p_{\mu\nu} \tag{2.18}$$

Since the plastic strain tensor is expressed from (2.17) as

$$E^p_{\mu\nu} = \frac{1}{2}(\phi_{\mu A}\, \phi_{\nu A} - h^o_{\mu\nu}) \tag{2.19}$$

we may see that the vielbein is directly related to the macroscopic plastic deformation.

The second transformation S is a local one defined as

$$\delta z'_A = S_{AB}(x)\, \delta z_B \tag{2.20}$$

expressed in the component form, which is different from the coordinate transformation and represents a local rotation group. It will be discussed in detail in the next section. We may clearly see that each element after releasing from the d-state can translate and rotate freely in the n-state. This kind of translation and rotation does not influence the geometric behavior of the nonriemannian space. We call the transformation (2.20) a gauge transformation representing such a local rotation, therefore the gauge transformation is orthogonal

$$S^{-1}(x) = S^T(x) \tag{2.20a}$$

or in component form

$$S^{-1}_{AB} = S_{BA} \quad , \quad S_{AB}\, S_{CB} = \delta_{AC} \tag{2.20b}$$

where S^T is the transpose of S having two indices associated with local coordinates only. Under the gauge transformation, the vielbein $\phi_{\mu A}$ is transformed to

$$\phi'_{\mu A}(x,t) = S_{BA}(x)\, \phi_{\mu B}(x,t) \tag{2.21}$$

or expressed in its contravariant form

$$\phi'^{\mu}_{A}(x,t) = \phi^{\mu}_{B}(x,t)\; S^{-1}_{BA}(x) = S_{AB}(x)\; \phi^{\mu}_{B}(x,t) \tag{2.22}$$

It is easily seen that with the local orthogonal tranformation, the metric tensor $g_{\mu\nu}$ is invariant, i.e.,

$$\begin{aligned} g'_{\mu\nu}(x,t) &= \phi'_{\mu A}(x,t)\; \phi'_{\nu A}(x,t) = \phi_{\mu B}\; S_{BA}\; \phi_{\nu C}\; S_{CA} \\ &= \delta_{BC}\; \phi_{\mu B}\; \phi_{\mu C} = \phi_{\mu B}\; \phi_{\nu B} = g_{\mu\nu} \end{aligned} \tag{2.23}$$

The combined transformation is defined as a combination of the coordinate transformation (2.12) and the gauge transformation (2.20), with which the vielbein obeys from (2.1) and (2.21) the following tranformation rule

$$\phi'_{\mu' A} = x^{\mu}_{\mu'}\; S_{AB}\; \phi_{\mu B} \tag{2.24}$$

or in its contravariant form

$$\phi'^{\mu'}_{A} = x^{\mu'}_{\nu}\; S_{AB}\; \phi^{\nu}_{B} \tag{2.25}$$

From the above basic relation, we can show that with the combined transformation, the metric tensor $g_{\mu\nu}$, the elastic and plastic strain tensor should be transformed as given in (2.15) and (2.18) respectively, that is,

$$g'_{\mu'\nu'} = g_{\mu'\nu'} \quad , \quad E'^{e}_{\mu'\nu'} = E^{e}_{\mu'\nu'} \quad , \quad E'^{p}_{\mu'\nu'} = E^{p}_{\mu'\nu'} \tag{2.26}$$

In general, for the material body in the n-state, a global coordinate system which is holonomic with respect to those of the r-state and the d-state, fails to exist. In this sense, the local transformation S is essentially different from the coordinate transformation, and its properties will be discussed below in some detail by gauge group theory.

3. Gauge Theory and Two Kinds of Connection

As mentioned in section 1, the gauge field theory initiated by Yong and Mills[11] plays an important role in particle physics for describing the interaction among the fundamental particles of various kinds and in general relativity theory. In order to apply this theory to the development of defect continuum physics, we shall briefly recall some fundamental aspects of gauge theory.

Let G be a Lie group with parameters $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ and $S(\alpha)$ be a irreducible representation of G. Assume $\phi(x)$ is a vector in Lie algebraic space. If α is a local parameter with components $\alpha_1, \alpha_2, \ldots, \alpha_n$ being functions of the coordinates x^μ, then, the transformation S for $\phi(x)$

$$\phi'(x) = S(x)\ \phi(x) \tag{3.1}$$

with

$$S(x) = S(\alpha(x)) \tag{3.2}$$

is called the gauge transformation of $\phi(x)$. Since $S(x)$ is a function of x^μ, the ordinary derivatives

$$\partial_\mu\ \phi(x) \equiv \frac{\partial \phi}{\partial x^\mu} \tag{3.3}$$

are not covariant with respect to the transformation (3.1). However, according to gauge theory, we can find gauge derivative $D_\mu \phi$ which posses the covariant properties as follows

$$D_\mu\ \phi(x) = \partial_\mu\ \phi(x) - B_\mu(x)\ \phi(x) \tag{3.4}$$

where B_μ ($\mu = 1, 2, 3$) are called the gauge potential or gauge connection.

Let us define a new potential B'_μ associated with the gauge tranformation S such that

$$B'_\mu(x) = S(x)\, B_\mu\, S^{-1}(x) + \partial_\mu\, S(x)\, S^{-1}(x) \tag{3.5}$$

where S^{-1} is the inverse of S

According to the definition of the gauge derivatives D_μ in (3.4), we verify from (3.5) that

$$D'_\mu\, \phi'(x) = S(x)\, D_\mu\, \phi(x) \tag{3.6}$$

where

$$D'_\mu = \partial_\mu - B'_\mu \quad , \quad D_\mu = \partial_\mu - B_\mu$$

Thus, the covariance of the gauge derivatives has been confirmed.

Using the definition of D_μ in (3.4), one can show that there exists a commutative relation between D_μ and D_ν in the form

$$(D_\mu\, D_\nu - D_\nu\, D_\mu)\, \phi(x) = -F_{\mu\nu}(x)\, \phi(x) \tag{3.7}$$

where $F_{\mu\nu}$ are expressed by the gauge potential B_μ as follows

$$F_{\mu\nu} = \partial_\mu\, B_\nu - \partial_\nu\, B_\mu - [B_\mu, B_\nu] \tag{3.8}$$

and

$$[B_\mu, B_\nu] \equiv B_\mu\, B_\nu - B_\nu\, B_\mu \tag{3.9}$$

Eqs. (3.8) with (3.9) are a fundamental relation in gauge theory.

It is not difficult to prove from (3.5) that with the gauge transformation (3.1), $F_{\mu\nu}$ is transformed by

$$F'_{\mu\nu} = \partial_\mu\, B'_\nu - \partial_\nu\, B' - [B'_\mu, B'_\nu] = S\, F_{\mu\nu}\, S^{-1} \tag{3.10}$$

which means that $F_{\mu\nu}$ behaves as a tensor in Lie algebraic space. We call $F_{\mu\nu}$ the gauge field tensor or the curvature tensor.

When the vielbein $\phi_{\mu A}$ defined in (2.10) is taken as ϕ in (3.1), the Lie group $\{S\}$ corresponding to the orthogonal transformation (2.14) represents a local rotation group $SO(3)$. Since the vielbein $\phi_{\mu A}$ has two different indices μ and A, which correspond, respectively, to the coordinate and gauge transformation, we may introduce two kinds of connection in the following way.

According to (3.4), the gauge covariant derivatives of ϕ in the $SO(3)$ group are defined as

$$D_\mu \, \phi = \partial_\mu \, \phi - \omega_\mu \, \phi \quad , \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{3.11}$$

where ω_μ is called the gauge connection. According to (3.11), if ϕ_A is taken as the vielbein $\phi_{\mu A}$ with the index μ of coordinate system x^μ, we obtain

$$D_\mu \, \phi_{\nu A} = \partial_\mu \, \phi_{\nu A} - \omega_{\mu AB} \, \phi_{\nu B} \tag{3.12}$$

and

$$D_\mu \, \phi^\nu_A = \partial_\mu \, \phi^\nu_A - \omega_{\mu AB} \, \phi^\nu_B \tag{3.13}$$

Here the gauge potential components $\omega_{\mu AB}$ are independent of the vielbein provided that the torsion tensor does exist, which can be seen later.

On the other hand, in dealing with the index μ in $\phi_{\mu A}$, the vielbein $\phi_{\mu A}$ can be treated as a vector in nonriemannian space M, therefore, keeping the index A unchanged, an affine connection $\Gamma^\lambda_{\mu\nu}$ can be defined by the conventional covariant derivatives of $\phi_{\mu A}$ with respect to the index μ as

$$\nabla_\mu \, \phi_{\nu A} = \partial_\mu \, \phi_{\nu A} - \Gamma^\lambda_{\mu\nu} \, \phi_{\lambda A} \tag{3.14}$$

or expressed in their contravariant form

$$\nabla_\mu \phi^\nu_A = \partial_\mu \phi^\nu_A + \Gamma^\nu_{\mu\lambda} \phi^\lambda_A \tag{3.15}$$

By using the two connections $\omega_{\mu AB}$ and $\Gamma^\lambda_{\mu\nu}$, the total covariant derivatives of any quantity T, which has the nonriemannian index μ and the vielbein index A can be defined by

$$\begin{aligned} \mathcal{D}_\lambda T^{..\sigma\rho....}_{\mu\nu..AB..} = \partial_\lambda T^{..\sigma\rho....}_{\mu\nu..AB..} - \Gamma^\delta_{\lambda\mu} T^{..\sigma\rho....}_{\delta\nu..AB..} \\ - \ldots + \Gamma^\sigma_{\lambda\delta} T^{..\delta\rho....}_{\mu\nu..AB..} + \cdots - \omega_{\lambda AD} T^{..\sigma\rho....}_{\mu\nu..DB..} - \cdots \end{aligned} \tag{3.16}$$

According to this definition, the total covariant derivatives of $\phi_{\mu A}$ are given by

$$\mathcal{D}_\mu \phi_{\nu A} = \partial_\mu \phi_{\nu A} - \Gamma^\lambda_{\mu\nu} \phi_{\lambda A} - \omega_{\mu AB} \phi_{\nu B} \tag{3.17}$$

or

$$\mathcal{D}_\mu \phi^\nu_A = \partial_\mu \phi^\nu_A + \Gamma^\nu_{\mu\lambda} \phi^\lambda_A - \omega_{\mu AB} \phi^\nu_B \tag{3.18}$$

According to the definition of the total covariant derivatives in (3.17), the gauge connection $\omega_{\mu AB}$ must be antisymmetric in their latter two indices A and B, i.e.

$$\omega_{\mu AB} = -\omega_{\mu BA} \tag{3.19}$$

In fact, from (2.16), we have

$$\mathcal{D}_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \Gamma^\delta_{\lambda\mu} g_{\delta\nu} - \Gamma^\delta_{\lambda\nu} g_{\mu\delta} \tag{3.20}$$

On the other hand, it is assumed that the total covariant derivatives $\mathcal{D}_\mu$ obey the familiar sum and product roles as the ordinary differentials. Therefore, we may obtain

$$\mathcal{D}_\lambda g_{\mu\nu} = \mathcal{D}_\lambda(\phi_{\mu A} \phi_{\nu A}) = (\mathcal{D}_\lambda \phi_{\mu A}) \phi_{\nu A} + \phi_{\mu A} \mathcal{D}_\lambda \phi_{\nu A} \tag{3.21}$$

Substituting (3.17) into (3.21), it follows that

$$\mathcal{D}_\lambda \; g_{\mu\nu} = \partial_\lambda \; g_{\mu\nu} - \Gamma^\delta_{\lambda\mu} \; g_{\delta\nu} - \Gamma^\delta_{\lambda\nu} \; g_{\mu\delta} \omega_{\lambda AB} (\phi_{\mu A} \; \phi_{\nu B} + \phi_{\nu A} \; \phi_{\mu B}) \tag{3.22}$$

Comparing (3.22) to (3.20), we get

$$\omega_{\lambda AB} (\phi_{\mu A} \; \phi_{\nu B} + \phi_{\nu A} \; \phi_{\mu B}) = 0 \tag{3.23}$$

Multiplying the both sides of (3.23) by $\phi^\mu_F \; \phi^\nu_E$, we can immediately obtain

$$\omega_{\lambda EF} + \omega_{\lambda FE} = 0 \tag{3.24}$$

that is a proof of (3.19). It is usually assumed in nonriemannian geometry that the total covariant derivatives of vielbein $\phi_{\mu A}$ are identically zero, i.e.,

$$\mathcal{D}_\mu \; \phi_{\nu A} = 0 \tag{3.25}$$

Using this basic assumption, from (3.20) and (3.21), we have

$$\nabla_\lambda \; g_{\mu\nu} = \mathcal{D}_\lambda \; g_{\mu\nu} = 0 \tag{3.26}$$

and also

$$\nabla_\lambda \; g^{\mu\nu} = \mathcal{D}_\lambda \; g^{\mu\nu} = 0 \tag{3.27}$$

Based on the definition of $\mathcal{D}_\mu \; \phi_{\nu A}$, from (3.25), we may also obtain two important relations among $\phi_{\mu A}, \omega_{\mu AB}$ and $\Gamma^\lambda_{\mu\mu}$ as

$$\Gamma^\lambda_{\mu\nu} = \phi^\lambda_A \; D_\mu \; \phi_{\nu A} \quad , \quad \omega_{\mu AB} = \phi^\nu_B \; \nabla_\mu \; \phi_{\nu A} \tag{3.28}$$

which shows that there are only two independent quantities among $\phi_{\mu A}$, $\omega_{\mu AB}$ and $\Gamma^\lambda_{\mu\nu}$, each of them can be determined from the other two.

Since vielbein $\phi_{\mu A}$ has two different indices μ and A which do not belong to same space, we may use vielbein ϕ^{μ}_{A} or $\phi_{\mu A}$ and metric tensor $g_{\mu\nu}$ or $g^{\mu\nu}$ to raise or lower the indices $A, B, \ldots$ and $\mu, \nu, \ldots$ respectively. By this procedure, for instance, we may define

$$\Gamma_{\mu\nu A} = \Gamma^{\lambda}_{\mu\nu}\, \phi_{\lambda A} \quad , \quad \omega^{\lambda}_{\mu\nu} = \omega_{\mu AB}\, \phi^{\lambda}_{A}\, \phi_{\nu B} \tag{3.29}$$

From (3.28), it follows that

$$\Gamma_{\mu\nu A} = D_{\mu}\, \phi_{\nu A} \quad , \quad \omega^{\lambda}_{\mu\nu} = \phi^{\lambda}_{A}\, \nabla_{\mu}\, \phi_{\nu A} \tag{3.30}$$

where $\Gamma_{\mu\nu A}$ and $\omega^{\lambda}_{\mu\nu}$ are also called the affine and gauge connections respectively.

As indicated in (3.5), with the gauge transformation the gauge connection obeys the transformation law

$$\omega'_{\mu AB} = S_{AC}\, S_{BD}\, \omega_{\mu CD} + (\partial_{\mu}\, S_{AC}) S_{CB} \tag{3.31}$$

However, under the coordinate transformation (2.14), $\omega_{\mu AB}$ is transformed as a vector

$$\omega_{\mu' AB} = x^{\mu}_{\mu'}\, \omega_{\mu AB} \tag{3.32}$$

Therefore, from (3.31) and 93.32) we see that under the combined transformation (2.25), the gauge connection $\omega_{\mu AB}$ is transformed as

$$\omega'_{\mu' AB} = x^{\mu}_{\mu'} [S_{AC}\, S_{BD}\, \omega_{\mu CD} + \partial_{\mu}\, S_{AC}\, S_{BC}] \tag{3.33}$$

Since all three indices of $\Gamma^{\lambda}_{\mu\nu}$ have no dependence on A, it is not difficult to prove from (2.28) and (3.31) that under the gauge transformation (2.20) the affine connection $\Gamma^{\lambda}_{\mu\nu}$ remains unchanged, i.e.,

$$\Gamma'^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} \tag{3.34}$$

But with the coordinate transformation (2.14), the affine connection is transformed by

$$\Gamma^{\lambda'}_{\mu'\nu'} = x^{\lambda'}_{\lambda}\, x^{\mu}_{\mu'}\, x^{\nu}_{\nu'}\, \Gamma^{\lambda}_{\mu\nu} + x^{\lambda}_{\mu'\nu'}\, x^{\lambda'}_{\lambda} \tag{3.35}$$

where $x^{\lambda}_{\mu'\nu'}$ are the second derivatives of x^{λ} with respect to $x^{\mu'}$ and $x^{\nu'}$. Combining (3.34) and (3.35), we obtain

$$\Gamma'^{\lambda'}_{\mu'\nu'} = \Gamma^{\lambda'}_{\mu'\nu'} \tag{3.36}$$

which means that with the combined transformation (2.25), the transformation of the affinity Γ is the same as given in (3.35).

4. Torsion, Curvature and Their Relation with Plastic Imperfections

In nonriemannian geometry, torsion and curvature play a central role in describing structure of the space. It was a essential discovery that these two tensors can be directly related to dislocations and disclinations so that defect continuum physics was laid on a sound mathematical basis. In the section, we shall apply the gauge theory to define these two tensors by means of the two connections introduced in the previous section.

From (3.28), we may see that the affinity $\Gamma^{\lambda}_{\mu\nu}$ is, in general, not symmetric in the indices μ and ν. However, recalling Eq. (3.35), we can use the affine connection to form a tensor as

$$T^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]} = \frac{1}{2}(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}) \tag{4.1}$$

in the nonriemannian geometry. This tensor is called torsion. From (3.35) and (3.36), we conclude that with the combined transformation, torsion is transformed as

$$T'^{\lambda'}_{\mu'\nu'} = x^{\lambda'}_{\lambda}\, x^{\mu}_{\mu'}\, x^{\nu}_{\nu'}\, T^{\lambda}_{\mu\nu} \tag{4.2}$$

Using (3.28) and (3.12), the torsion T can be decomposed into

$$T^{\lambda}_{\mu\nu} = B^{\lambda}_{\mu\nu} - \Omega^{\lambda}_{\mu\nu}$$

where

$$B^{\lambda}_{\mu\nu} = \phi^{\lambda}_{A}\,\partial_{[\mu}\,\phi_{\nu]A} \quad , \quad \Omega^{\lambda}_{\mu\nu} = \omega^{\lambda}_{[\mu\nu]} \tag{4.3}$$

$$\omega^{\lambda}_{\mu\nu} = \phi^{\lambda}_{A}\,\omega_{\mu AB}\,\phi_{\nu B} \tag{4.4}$$

We call $B^{\lambda}_{\mu\nu}$ and $\Omega^{\lambda}_{\mu\nu}$ the pure vielbein part and gauge part of torsion $T^{\lambda}_{\mu\nu}$ respectively. By using the rule of raising or lowering indices A, B or μ, ν in the $SO(3)$ group space, the mixed torsion can be obtained in terms of $T^{\lambda}_{\mu\nu}$, for instance, we have

$$T_{\mu\nu A} = T^{\lambda}_{\mu\nu}\,\phi_{\lambda A} = B_{\mu\nu A} - \Omega_{\mu\nu A} \tag{4.5a}$$

$$T_{\mu\nu\lambda} = T^{\alpha}_{\mu\nu}\,g_{\alpha\lambda} = B_{\mu\nu\lambda} - \Omega_{\mu\nu\lambda} \tag{4.5b}$$

where

$$\begin{aligned} B_{\mu\nu A} &= \partial_{[\mu}\,\phi_{\nu]A} \quad , \quad \Omega_{\mu\nu A} = \omega_{[\mu|BA|}\,\phi_{\nu]B} \\ B_{\mu\nu\lambda} &= B^{\sigma}_{\mu\nu}\,g_{\sigma\lambda} \quad , \quad \Omega_{\mu\nu\lambda} = \Omega^{\sigma}_{\mu\nu}\,g_{\sigma\lambda} \end{aligned} \tag{4.6}$$

With the combined transformation (2.24), the mixed torsion is transformed as

$$T'_{\mu'\nu' A} = x^{\mu}_{\mu'}\,x^{\nu}_{\nu'}\,S_{AB}\,T_{\mu\nu B} \tag{4.7}$$

In dislocation continuum theory, the torsion tensor is responsible for dislocation. Since $T^{\lambda}_{\mu\nu}$ is antisymmetric in its lower indices μ and ν, therefore, there are only nine independent components for the torsion tensor T, and the dislocation density $\alpha^{\mu\nu}$ of the second order in the Lagrangian form can be defined by T as

$$\alpha^{\mu\nu} = {}^{(g)}\varepsilon^{\mu\lambda\sigma}\,T^{\nu}_{\lambda\sigma} \tag{4.8}$$

where ${}^{(g)}\varepsilon^{\lambda\mu\nu}$ is the permutation symbol divided by $\sqrt{g}$, and $g = det(g_{\mu\nu})$. Using the rule of raising or lowering indices, the second order mixed dislocation density α^{μ}_{A} is defined as

$$\alpha^{\mu}_{A} = \alpha^{\mu\nu}\phi_{\nu A} = {}^{(g)}\varepsilon^{\mu\lambda\sigma}\Big(\partial_{[\lambda}\phi_{\sigma]A} - \Omega_{\lambda\sigma A}\Big) \tag{4.9}$$

Obviously, the dislocation density $\alpha^{\mu\nu}$ keeps its covariance under the combined transformation, i.e.,

$$\begin{aligned} \alpha'^{\,\mu'\nu'} &= x^{\mu'}_{\mu}\, x^{\nu'}_{\nu}\, \alpha^{\mu\nu} \\ \alpha'^{\,\mu\nu} &= \alpha^{\mu\nu} \end{aligned} \tag{4.10}$$

According to (2.10), the vielbein $\phi_{\mu A}$ is considered as mapping which carries an element from the r-state to the n-state as

$$\delta z_A = \phi_{\mu A}\, dx^{\mu} \tag{4.11}$$

where z_A are anholonomic coordinates, thus, Eq. (4.11) is not integrable. The small volume elements taken from the d-state do not fit together to form a continuous body after releasing. In this situation, the true Burgers vector of all dislocations enclosed by a small circuit c in the d-state is given by

$$\Delta B_A = \oint_c \delta z_A = \oint_c \phi_{aA}\, dy^{a} \tag{4.12a}$$

If c is the canterpart of the circuit C in the r-state, (4.12a) can be also expressed by

$$\Delta B_A = \oint_C \phi_{\mu A}\, dx^{\mu} \tag{4.12b}$$

By using stokes' theorem, from (4.12) we obtain

$$\Delta B_A = \iint\limits_S \partial_{[\mu}\phi_{\nu]A}\, dS^{\mu\nu} \tag{4.13}$$

If the closed contour C is sufficiently small and we let the small area ΔS of the surface bounded by C tend to zero, from (4.13), we get

$$\partial_{[\mu}\phi_{\nu]A} = \lim_{\Delta S\to 0}\frac{\Delta B_A}{\Delta S^{\mu\nu}} \tag{4.14}$$

Comparing (4.14) with (4.6), we find that pure vielbein part $B_{\mu\nu A}$ of the torsion tensor $T_{\mu\nu A}$ simply represents the Burgers vector density. Using (4.9), we have

$$\alpha^{\mu}_{A} = {}^{(g)}\varepsilon^{\mu\lambda\sigma}\left(B_{\lambda\sigma A} - \Omega_{\lambda\sigma A}\right) \tag{4.15}$$

Thus, the dislocation density α^{μ}_{A} is decomposed into two parts, of which the first part called Burgers vector density is fully due to the vielbein, and the second part is due to gauge potential. If the space is flat, the dislocation density is indentical to Burgers vector density. It should be noted that such decomposition could not be obtained by introducing the affine connection only.

Let us turn to the discussion of curvature tensor and its relation with disclinations. The mixed curvature tensor F can be obtained by substituting $B_\mu = \omega_\mu$ into (3.8), and then written in their component form

$$F_{\mu\nu AB} = \partial_\mu\omega_{\nu AB} - \partial_\nu\omega_{\mu AB} - \omega_{\mu AC}\omega_{\nu CB} + \omega_{\nu AC}\omega_{\mu CB} \tag{4.16}$$

From this expression and (2.19), it is easily seen that the curvature $F_{\mu\nu AB}$ is antisymmetric in both the former indices μ, ν and in the latter indices A, B, i.e.,

$$F_{\mu\nu AB} = -F_{\nu\mu AB} = -F_{\mu\nu BA} = F_{\nu\mu BA} \tag{4.17}$$

Substituting the second expression of (3.28) into (4.16) and through some straight forward algebra, we obtain

$$F_{\mu\nu AB} = -R^{\sigma}_{\mu\nu\lambda}\,\phi_{\sigma A}\,\phi^{\lambda}_{B} \tag{4.18}$$

where

$$R^{\sigma}_{\mu\nu\lambda} = \partial_\mu \Gamma^{\sigma}_{\nu\lambda} - \partial_\nu \Gamma^{\sigma}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\sigma}_{\nu\rho} \tag{4.19}$$

is just the Riemann-Christoffel curvature tensor based on the affine connection $\Gamma^{\lambda}_{\mu\nu}$. Equation (4.18) can also be written as

$$R^{\sigma}_{\mu\nu\lambda} = -F_{\mu\nu AB}\,\phi_{\lambda B}\,\phi^{\sigma}_{A} \tag{4.20}$$

By making use of (3.31), we may prove that with the gauge transformation (2.21), $F_{\mu\nu AB}$ is transformed to

$$F'_{\mu\nu AB} = S_{AC}\,S_{BD}\,F_{\mu\nu CD} \tag{4.21}$$

therefore, $F_{\mu\nu AB}$ is a tensor in the Lie algebra space with an $SO(3)$ group associated with the latter indices A and B. Substitution of (4.21) into (4.20) leads to

$$R'^{\sigma}_{\mu\nu\lambda} = -F'_{\mu\nu AB}\,\phi'_{\lambda B}\,\phi'^{\sigma}_{A} = R^{\sigma}_{\mu\nu\lambda} \tag{4.22}$$

which means that the curvature R is invariant for the gauge transformation. Furthermore, with the combined transformation (2.25), and (4.21) becomes

$$F'_{\mu'\nu' AB} = S_{AC}\,S_{BD}\,x^{\mu}_{\mu'}\,x^{\nu}_{\nu'}\,F_{\mu\nu CD} \tag{4.23}$$

Substituting (4.23) into (4.20) again and using (2.21) and (2.22) we have

$$\begin{aligned} R'^{\sigma'}_{\mu'\nu'\lambda'} &= -F'_{\mu'\nu' AB}\,\phi'_{\lambda' B}\,\phi'^{\sigma'}_{A} \\ &= x^{\sigma'}_{\sigma}\,x^{\mu}_{\mu'}\,x^{\nu}_{\nu'}\,x^{\lambda}_{\lambda'}\,R^{\sigma}_{\mu\nu\lambda} \end{aligned} \tag{4.24}$$

It asserts that the curvature R is transformed as a tensor in the nonriemannian geometry with the combined transformation. Therefore, we can use $\phi_{\mu A}$ or $g_{\mu\nu}$ to raise or lower the indices of R or F to obtain the mixed curvature as we did for the torsion tensor, for instance, we have

$$\begin{aligned} R_{\mu\nu\lambda\sigma} &= R^{\alpha}_{\mu\nu\lambda}\, g_{\alpha\sigma} \\ &= -F_{\mu\nu AB}\, \phi_{\lambda B}\, \phi_{\sigma A} \end{aligned} \tag{4.25}$$

From (4.17), the above equation can be rewritten as

$$R_{\mu\nu\lambda\sigma} = F_{\mu\nu AB}\, \phi_{\lambda A}\, \phi_{\sigma B} \tag{4.26}$$

and

$$R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} = R_{\nu\mu\sigma\lambda} \tag{4.27}$$

This means that the curvature tensor $R_{\mu\nu\lambda\sigma}$ is antisymmetric in both former two and the latter two indices μ, ν and λ, σ. Therefore, among $R_{\mu\nu\lambda\sigma}$, only nine non-vanishing independent components are left, they are

$$R_{1212}\,,\, R_{1213}\,,\, R_{1312}\,,\, R_{1313}\,,\, R_{1223}\,,\, R_{1323}\,,\, R_{2312}\,,\, R_{2313}\,,\, R_{2323}$$

To replace the curvature tensor, we may define a second order tensor as

$$\theta^{\mu\nu} = {}^{(g)}\varepsilon^{\mu\lambda\sigma}\, {}^{(g)}\varepsilon^{\nu\alpha\beta}\, R_{\lambda\sigma\alpha\beta} \tag{4.28}$$

where $\theta^{\mu\nu}$ is called the disclination density tensor. Substituting (4.26) into (4.28), we can express $\theta^{\mu\nu}$ in terms of the vielbein $\phi_{\mu A}$ and gauge potential $\omega_{\mu AB}$ as

$$\theta^{\mu\nu} = {}^{(g)}\varepsilon^{\mu\lambda\sigma}\, {}^{(g)}\varepsilon^{\nu\alpha\beta}\, F_{\lambda\sigma AB}\, \phi_{\alpha A}\, \phi_{\beta B} \tag{4.29}$$

We may prove that with the combined transformation (2.24), $\theta^{\mu\nu}$ is transformed to

$$\theta'^{\mu'\nu'} = x^{\mu'}_{\mu}\, x^{\nu'}_{\nu}\, \theta^{\mu\nu} \tag{4.30}$$

On the other hand, the curvature tensor F or R is responsible for disclination, that is, it can be directly related to the disclination density. As known in nonriemannian geometry, the geometric significance of the curvature tensor is seen by a transplantation of a vector V^{μ} along a small closed curve in the material manifold until it returns to the starting point. The disclination density defined in (4.28) is equivalent to the difference between the final value and the initial value of the vector V^{μ} considered. In a separate paper, we shall to present its geometrical interpretation in some detail when the closed curve is not small.

Following the method introduced by Kröner and Seeger[18], the generalized equations of strain imcompability can be derived from (4.28). In fact, when the curvature does not vanish, through some straightforward algebra, Eq. (4.28) can be written as

$$\pi^{\mu\nu} = \eta^{\mu\nu} + q^{\mu\nu} + \theta^{\mu\nu} \tag{4.31}$$

in using the Lagrangian coordinate system x^{μ}. In (4.31), $\pi^{\mu\nu}$ is called the Lagrangian incompatibility tensor. $\eta^{\mu\nu}$, which depends only on the dislocation density, represents the symmetric tensor of the incompatibility. $q^{\mu\nu}$ stands for the nonlinear terms consisting of the plastic strain tensor $E^{(p)}_{\mu\nu}$ and the dislocation density. They are expressed by

$$\begin{aligned}
\pi^{\mu\nu} &= -\,\mathcal{E}_0^{\mu\lambda\sigma}\,\mathcal{E}_0^{\nu\rho\gamma}\,\nabla^{(0)}_{\lambda}\nabla^{(0)}_{\rho}\,E^{p}_{\sigma\gamma} \\
\eta^{\mu\nu} &= -\,\mathcal{E}_0^{\mu\lambda\sigma}\,\nabla^{(0)}_{\lambda}\,\alpha^{\nu}_{\sigma} \\
q^{\mu\nu} &= \frac{1}{2}\,\mathcal{E}_0^{\mu\lambda\sigma}\,\mathcal{E}_0^{\nu\rho\gamma}\,g^{\alpha\beta}\,(K_{\lambda\gamma\beta} - 2\,E^{p}_{\lambda\gamma\beta})(K_{\sigma\rho\alpha} - 2\,E^{p}_{\sigma\rho\alpha})
\end{aligned} \tag{4.32}$$

where $\nabla^{(0)}_{\lambda}$ means a covariant derivatives based on the metric $h_{\mu\nu}$, and $\mathcal{E}_0^{\mu\lambda\sigma}$ is the permutation symbol divided by $\sqrt{g_0}$, where $g_0 = det(h^{o}_{\mu\nu})$, $E^{(p)}_{\mu\nu}$ is the plastic strain tensor defined by

$$E^{p}_{\mu\nu} = E_{\mu\nu} - E^{(e)}_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - h_{\mu\nu}) \tag{4.33}$$

and

$$E^p_{\lambda\mu\nu} = \frac{1}{2}\left[\nabla^{(0)}_\lambda E^p_{\mu\nu} + \nabla^{(0)}_\mu E^p_{\nu\lambda} - \nabla^{(0)}_\nu E^p_{\lambda\mu}\right]$$
$$K_{\lambda\mu\nu} = T_{\lambda\mu\nu} + T_{\mu\lambda\nu} - T_{\nu\lambda\mu} \tag{4.34}$$

where $T_{\lambda\mu\nu}$ is given in (4.6).

5. Basic Governing Equations For y^a, $\phi_{\mu A}$ and $\omega_{\mu AB}$

Generally speaking, a material body containing a large number of moving dislocations and disclinations could not possibly be considered as a conservative system. When macroscopic plastic deformation due to the motion of dislocations and disclinations exists inside the body, the irreversible effects not only due to the plastic deformation but also due to heat conduction and other viscous dissipation should be involved in the problem. Sedov and Berditchevski[7] gave a detail description in constructing the basic equations for dislocation continuum based on a general variational principle by taking into account all the irreversible phenomena just mentioned. However if the plastic deformation is not large so that all the irreversible effects can be ignored, and the material body can be treated as a elastic dislocation and disclination continuum. Based on this assumption, we construct the governing equations for the fields considered.

For a complete description of motion and deformation of materials with dislocations and disclinations, we have to deal with twenty one determining parameters: y^a, $\phi_{\mu A}$ as well as $\omega_{\mu AB}$, which are the unknown functions of x^μ and t. The basic governing equations for them are usually constructed based on a variational principle. In the Lagrangian description method, the corresponding action integral is supposed to take the following form

$$I = \int_{t_1}^{t_2}\left(\int_{E_3} L\,\sqrt{g_0}\,d^3x\right)dt \tag{5.1}$$

where L represents the Lagrangian function given in the r-state and $g_0 = det(h^o_{\mu\nu})$. In order to obtain the corresponding Euler equation, it is necessary to choose the explicit form of L, that is, to decide how it depends on these determining parameters and their derivatives with respect to t and x^μ. For simplicity, let us consider the stationary problem in which the internal energy no longer depends on time. Therefore, the Lagrangian L can be written as

$$L = \frac{\rho_r}{2} e_{ab}\, \dot{y}^a\, \dot{y}^b - W(x^\mu;\, y^a,\, J^a_\mu,\, \phi_{\mu A},\, \phi_{\mu A\nu},\, \omega_{\mu AB},\, \omega_{\mu AB\nu}) \tag{5.2}$$

with the notations

$$\phi_{\mu A\nu} \equiv \partial_\nu\, \phi_{\mu A} \quad , \quad \omega_{\mu AB\nu} \equiv \partial_\nu \omega_{\mu AB}$$

where ρ_r is the mass density of material in the r-state, $\dot{y}^a = \partial y^a / \partial t$ is velocity and W is the internal energy per unit volume. Substituting (5.2) into (5.3), the action integral (5.1) can be written as

$$I = \int_{t_1}^{t_2} dt \int_{E_3} \left[\frac{\rho_r}{2} e_{ab}\, \dot{y}^a\, \dot{y}^b - W(x^\mu;\, y^a,\, J^a_\mu,\, \phi_{\mu A},\, \phi_{\mu A\nu},\, \omega_{\mu AB},\, \omega_{\mu AB\nu}) \right] \sqrt{g_0}\; d^3x \tag{5.3}$$

The Euler equations which follow from $\delta I = 0$ with the fixed boundary conditions are

$$\begin{aligned} \nabla^{(0)}_\mu \sigma^\mu_a \; - \; f_a \; &= \; \rho_r\, \ddot{y}_a \\ \nabla^{(0)}_\nu \sigma^{\mu\nu}_A \; - \; f^\mu_A \; &= \; 0 \\ \nabla^{(0)}_\nu \sigma^{\mu\nu}_{AB} \; - \; f^\mu_{AB} \; &= \; 0 \end{aligned} \tag{5.4}$$

where $\nabla^{(0)}_\mu$ represents the covariant derivatives based on the metric $h^o_{\mu\nu}$ which performs only on the indices $\mu, \nu, \ldots$, and the stress tensors and force densities are expressed by

$$\sigma^\mu_a = \frac{\partial W}{\partial J^a_\mu} \; , \; \sigma^{\mu\nu}_A = \frac{\partial W}{\partial \phi_{\mu A\nu}} \; , \; \sigma^{\mu\nu}_{AB} \; = \; \frac{\partial W}{\partial \omega_{\mu AB\nu}} \tag{5.5}$$

and

$$f_a = \frac{\partial W}{\partial y^a}\ ,\ f^\mu_A = \frac{\partial W}{\partial \phi_{\mu A}}\ ,\ f^\mu_{AB} = \frac{\partial W}{\partial \omega_{\mu AB}} \tag{5.6}$$

respectively. This system of Eq. (5.4) represents the linear momentum equations, the "dislocation balance" equations and the "disclination balance" equations respectively. To determine the specific form of W, we have kept it in mind that the real physical quantities which appear in the Lagrangian function must keep their covariance for the combined transformation. This is, of course, of major importance to establish the specific forms of the basic governing equations. In the previous sections, it is proved that $E^e_{\mu\nu}$, $\alpha^{\mu\nu}$ and $\theta^{\mu\nu}$ are three proper quantities of this sort, because all of them are covariant with the combined transformation. Thus, the internal energy W is given by

$$W = W(x^\mu, E^e_{\mu\nu}, \alpha^{\mu\nu}, \theta^{\mu\nu}) \tag{5.7}$$

According to (2.17), (4.9) and (4.29), we write

$$\begin{aligned} E^e_{\mu\nu} &= \frac{1}{2}(e_{ab} J^a_\mu J^b_\nu - \phi_{\mu A}\phi_{\nu A}) \\ \alpha^{\mu\nu} &= {}^{(g)}\varepsilon^{\mu\lambda\sigma} \phi^\nu_A (\partial_{[\lambda}\phi_{\sigma]A} - \omega_{[\lambda\sigma]A}) \\ \theta^{\mu\nu} &= {}^{(g)}\varepsilon^{\mu\lambda\sigma}\, {}^{(g)}\varepsilon^{\nu\alpha\beta} \phi_{\alpha A}\phi_{\beta B} (\partial_{[\lambda}\omega_{\sigma]AB} - [\omega_\lambda, \omega_\sigma]_{AB}) \end{aligned} \tag{5.8}$$

Thus, we calculate

$$
\begin{aligned}
&\frac{\partial E^{(e)}_{\alpha\beta}}{\partial J^a_\mu} = e_{ab}\,\delta^\mu_{(\alpha}\,J^b_{\beta)} \quad , \quad \frac{\partial E^{(e)}_{\alpha\beta}}{\partial \phi_{\mu A}} = -\,\delta^\mu_{(\alpha}\,\phi_{\beta)A} \\
&\frac{\partial \alpha^{\alpha\beta}}{\partial \phi_{\mu A}} = -2\Big[\alpha^{\alpha(\mu}\,\phi^{\beta)}_A + {}^{(g)}\varepsilon^{\alpha\lambda\mu}\,\phi^\beta_B\,\omega_{\lambda BA}\Big] \\
&\frac{\partial \alpha^{\alpha\beta}}{\partial \phi_{\mu A\nu}} = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,\phi^\beta_A \quad , \quad \frac{\partial \alpha^{\alpha\beta}}{\partial \omega_{\mu AB}} = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,\phi^\beta_A\,\phi_{\nu B} \\
&\frac{\partial\theta^{\alpha\beta}}{\partial\phi_{\mu A}} = 2\,{}^{(g)}\varepsilon^{\alpha\nu\lambda}\,{}^{(g)}\varepsilon^{\beta\mu\sigma}\,\phi_{\sigma B}\,F_{\nu\lambda AB} = 2\,\theta^{\alpha\beta}\,\phi^\mu_A \\
&\frac{\partial\,\theta^{\alpha\beta}}{\partial\,\omega_{\mu AB}} = 2\,{}^{(g)}\varepsilon^{\alpha\lambda\mu}\,{}^{(g)}\varepsilon^{\beta\sigma\nu}\,\phi_{\nu E}\,(\phi_{\sigma B}\,\omega_{\lambda EA} - \phi_{\sigma A}\,\omega_{\lambda EB}) \\
&\frac{\partial\,\theta^{\alpha\beta}}{\partial\,\omega_{\mu AB\nu}} = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,{}^{(g)}\varepsilon^{\beta\tau\sigma}\,\phi_{\tau A}\,\phi_{\sigma B}
\end{aligned}
\tag{5.9}
$$

Substituting (5.9) into (5.5) and (5.6) and using the chain rule of differentials, we obtain

$$
\begin{aligned}
&\sigma^{\mu a} = J^a_\nu\,\frac{\partial W}{\partial E^{(e)}_{\mu\nu}} \quad , \quad \sigma^{\mu\nu}_A = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,\phi^\beta_A\,\frac{\partial W}{\partial\,\alpha^{\alpha\beta}} \\
&\sigma^{\mu\nu}_{AB} = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,{}^{(g)}\varepsilon^{\beta\tau\sigma}\,\phi_{\tau A}\,\phi_{\sigma B}\,\frac{\partial W}{\partial\theta^{\alpha\beta}} \\
&f^\mu_A = -\,\phi_{\nu A}\,\frac{\partial W}{\partial E^{(e)}_{\mu\nu}} - 2\Big[\alpha^{\alpha(\mu}\,\phi^{\beta)}_A + {}^{(g)}\varepsilon^{\alpha\lambda\mu}\phi^\beta_B\,\omega_{\lambda BA}\Big]\frac{\partial W}{\partial\alpha^{\alpha\beta}} + 2\,\theta^{\alpha\beta}\,\phi^\mu_A\,\frac{\partial W}{\partial\theta^{\alpha\beta}} \\
&f^\mu_{AB} = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,\phi^\beta_A\,\phi_{\nu B}\,\frac{\partial W}{\partial\alpha^{\alpha\beta}} \\
&\qquad + 2\,{}^{(g)}\varepsilon^{\alpha\lambda\mu}\,{}^{(g)}\varepsilon^{\beta\sigma\nu}\,\phi_{\nu E}\,(\phi_{\sigma B}\omega_{\lambda EA} - \phi_{\sigma A}\omega_{\lambda EB})\,\frac{\partial W}{\partial\theta^{\alpha\beta}}
\end{aligned}
\tag{5.10}
$$

From (5.10), we may see that the stresses $\sigma^{\mu\nu}_A$ are antisymmetric in the indices μ, ν and $\sigma^{\mu\nu}_{AB}$ are also antisymmetric in both μ, ν and A, B, that is,

$$
\sigma^{(\mu\nu)}_A = 0 \quad , \quad \sigma^{(\mu\nu)}_{AB} = \sigma^{\mu\nu}_{(AB)} = 0 \tag{5.11}
$$

Substitution of (5.10) into the basic governing equations (5.4) leads to the following expressions

$$\begin{aligned} \mathcal{D}_\mu^{(0)}\sigma^{\mu a} \; - \; F^a &= \rho_r\,(\ddot{y}^a + \nabla_b\,\dot{y}^a\cdot\dot{y}^b) \\ \mathcal{D}_\nu^{(0)}\sigma_A^{\mu\nu} \; - \; F_A^\mu &= 0 \\ \mathcal{D}_\nu^{(0)}\sigma_{AB}^{\mu\nu} \; - \; F_{AB}^\mu &= 0 \end{aligned} \tag{5.12}$$

where the operator $\mathcal{D}_\mu^{(0)}$ acting on stress fields $\sigma^{\mu a}$, $\sigma_A^{\mu\nu}$ and $\sigma_{AB}^{\mu\nu}$ represents the total covariant differentials with respect to all three different indices μ, a and A therefore

$$\begin{aligned} \mathcal{D}_\mu^{(0)}\sigma^{\mu a} &= \sigma^{\mu a}_{,\mu} + \overset{\circ}{\Gamma}{}^{\lambda}_{\lambda\mu}\,\sigma^{\mu a} + \Gamma^a_{bc}\,\sigma^{\mu b}\,y^c_\mu \\ \mathcal{D}_\mu^{(0)}\sigma_A^{\mu\nu} &= \nabla_\nu^{(0)}\sigma_A^{\mu\nu} - \omega_{\nu AB}\,\sigma_B^{\mu\nu} \\ \mathcal{D}_\nu^{(0)}\sigma_{AB}^{\mu\nu} &= \nabla_\nu^{(0)}\sigma_{AB}^{\mu\nu} - \omega_{\nu AE}\,\sigma_{EB}^{\mu\nu} - \omega_{\nu BE}\,\sigma_{AE}^{\mu\nu} \end{aligned} \tag{5.13}$$

and

$$F_A^\mu = -\phi_{\nu A}\,\frac{\partial W}{\partial E^{(e)}_{\mu\nu}} - 2\,\alpha^{\alpha(\mu}\,\phi_A^{\beta)}\,\frac{\partial W}{\partial\alpha^{\alpha\beta}} + 2\,\theta^{\alpha\beta}\,\phi_A^\mu\,\frac{\partial W}{\partial\theta^{\alpha\beta}}$$

$$F_{AB}^\mu = -2\,{}^{(g)}\varepsilon^{\alpha\mu\nu}\,\phi_A^\beta\,\phi_{\nu B}\,\frac{\partial W}{\partial\alpha^{\alpha\beta}} = -F_{BA}^\mu$$

Because it has been proved that $E^e_{\mu\nu}$, $\alpha^{\mu\nu}$ and $\theta^{\mu\nu}$ are invariant for the gauge transformation and are transformed as a conventional tensor with the coordinate transformation, the system of Eq. (5.11) is seen to be covariant for the combined transformation. We should also notice that if the Lagrangian function is taken from the n-state instead of the r-state, the variational calculus done above still hold true, therefore, instead of (5.11), the Euler equations can be written in the covariant form by using the total covariant derivative defined based on the christoffel of $g_{\mu\nu}$ for the practical application, the form given in (5.11) appears more useful.

In the following, we shall deal with, as an example, the small deformation theory, in which the internal energy W is taken to be only a quadratic form as

$$
\begin{aligned}
W =& \frac{1}{2} A^{\mu\nu\lambda\sigma} E^e_{\mu\nu} E^e_{\lambda\sigma} + B^{\mu\nu}_{\lambda\sigma} E^e_{\mu\nu} \alpha^{\lambda\sigma} \\
&+ C^{\mu\nu}_{\lambda\sigma} E^e_{\mu\nu} \theta^{\lambda\sigma} + \frac{1}{2} D_{\mu\nu\lambda\sigma} \alpha^{\mu\nu} \alpha^{\lambda\sigma} \\
&+ G_{\mu\nu\lambda\sigma} \alpha^{\mu\nu} \theta^{\lambda\sigma} + \frac{1}{2} K_{\mu\nu\lambda\sigma} \theta^{\mu\nu} \theta^{\lambda\sigma}
\end{aligned}
\tag{5.14}
$$

where $A^{\mu\nu\lambda\sigma}, \cdots, K_{\mu\nu\lambda\sigma}$ are the material constants, thus we have

$$
\begin{aligned}
\frac{\partial W}{\partial E^e_{\mu\nu}} &= A^{\mu\nu\lambda\sigma} E^e_{\lambda\sigma} + B^{\mu\nu}_{\lambda\sigma} \alpha^{\lambda\sigma} + C^{\mu\nu}_{\lambda\sigma} \theta^{\lambda\sigma} \\
\frac{\partial W}{\partial \alpha^{\mu\nu}} &= B^{\lambda\sigma}_{\mu\nu} E^e_{\lambda\sigma} + D_{\mu\nu\lambda\sigma} \alpha^{\lambda\sigma} + G_{\mu\nu\lambda\sigma} \theta^{\lambda\sigma} \\
\frac{\partial W}{\partial \theta^{\mu\nu}} &= C^{\lambda\sigma}_{\mu\nu} E^e_{\lambda\sigma} + G_{\lambda\sigma\mu\nu} \alpha^{\lambda\sigma} + K_{\mu\nu\lambda\sigma} \theta^{\lambda\sigma}
\end{aligned}
\tag{5.15}
$$

Substituting (5.14) into (5.10), we obtain the stress-strain relations in a useful form. Furthermore, if the material body is assumed to be macroscopically istropic, the internal energy depends only on the first and second principal invariants of the tensors E^e, α and θ, and takes the form

$$
\begin{aligned}
W =& A I_E^2 + B II_E + C I_\alpha^2 + D II_\alpha + E I_\theta^2 + F II_\theta + G I_E I_\alpha \\
&+ H I_E I_\theta + F I_\alpha I_\theta
\end{aligned}
\tag{5.16}
$$

where $A, B, \ldots, F$ are the material constants, and $I_E, \ldots, II_\theta$ are the principal invariants of E, α and θ and are given by

$$
\begin{aligned}
I_E &= t_r E^e \quad , \quad II_E = \frac{1}{2}\left[(t_r E^e)^2 - t_r(E^{e^2})\right] \\
I_\alpha &= t_r \alpha \quad , \quad II_\alpha = \frac{1}{2}\left[(t_r \alpha)^2 - t_r \alpha^2\right] \\
I_\theta &= t_r \theta \quad , \quad II_\theta = \frac{1}{2}\left[(t_r \theta)^2 - t_r \theta^2\right]
\end{aligned}
\tag{5.17}
$$

After the material constants in (5.14) or (5.15) are determined, we substitute (5.13) or (5.15) into (5.10), then (5.11) the basic governing equation for y^a, $\phi_{\mu A}$ and $\omega_{\mu AB}$ could be solved assuming certain boundary conditions.

Finally, we should notice that the derivation is based on the assumption that the internal energy has no dependence on the time derivatives of $\phi_{\mu A}$ and $\omega_{\mu AB}$. If these time derivatives do exist, the dislocation and disclination currents must be introduced. It is not difficult to generalize the method to this case and will be discussed in some detail in a separate paper.

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