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ON THE INTERACTION OF FERMIONS WITH A FINITE-SIZED, NON-ABELIAN MONOPOLE*

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ABSTRACT

The interactions of fermions with a non–abelian monopole is studied without adopting the point–like monopole limit. The fermion is second quantized in a Prasad–Sommerfield background and the absence of dyon solutions is demonstrated. Canonical quantization of the "charge rotator" degree of freedom of the monopole is carried out and gauge invariance elucidated with particular attention to electric charge conservation. Finally, non–conservation of the axial vector charge is demonstrated. A conserved, gauge–dependent axial charge is constructed and the standard anomalous commutator with electric charge found.

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1. Introduction

The interaction of fermions with a non-abelian gauge theory monopole evidently produces some startling effects. In the field theoretic models of the system solved by Callan [1,2] and Rubakov [3], the fermion vacuum exhibits considerable structure and radically alters the monopole-dyon spectrum of the pure gauge theory. Furthermore, the ability of the vacuum to "store" quantum numbers in extended fermion condensates leads, in Grand Unified Models such as minimal SU(5), to baryon number non-conserving reactions at strong interaction rates.

These phenomena have their source in the axial vector anomaly supported by the radial magnetic field of the monopole, and in the fact that the fermionic charge to which the monopole couples is not well defined in the vicinity of the monopole [4,5,6,7]. This latter property in subsumed, in the soluble models, by a charge non-conserving boundary condition on the fermions in the limit that the region of ill-defined charge shrinks to zero, i.e., in the limit of a point-like, abelian monopole. Though there appears to be no reason to challenge the validity of either the limit or the boundary condition, it is interesting to dispense with them and attempt to recover features of the point-like limit.

In this paper we study the interaction of $J = 0$ [1,2,3,8], massless fermions with a non-abelian, finite-sized, monopole in the Prasad-Sommerfield limit [9]. We restrict our attention to an SU(2) gauge theory and one iso-doublet Dirac fermion. Although this system is not soluble, it's possible, at least, to verify a number features of the point-like model.

The plan of the paper is as follows. In the next section the $J = 0$ fermion-monopole system is formulated in the standard way and reduced to an effective two-dimensional field theory. Pertinent aspects of the point-like approximation are briefly reviewed, but rather than adopt this limit, we continue to treat the monopole as an extended object.

In section 3, the fermion eigenmodes in the field of a Prasad-Sommerfield monopole are introduced and normalized. Completeness and orthogonolity are demonstrated and the fermion propagator is derived and compared with that of the point-like limit. With this propagator, we examine a "charged" fermion condensate and discover a short-distance singularity not present in the point-like approximation.

The "charge rotator" degree of freedom of the monopole is turned on in section 4, and we search for a solution to the coupled system with a long range, static field, i.e., a dyon. This problem can be solved exactly and we find that no such solution exists, as did Callan in the point-like limit.

In section 5, the "charge rotator" is quantized, with special attention paid the gauge properties of the system. We adopt a canonical, Hamiltonian approach to quantization which leads, trivally, to charge superselection. In particular, the "charged" condensate discussed in section 3 is not present (or, more accurately, not charged) when gauge invariance is properly taken into account. In addition, we rederive, for the case of fermions in the presence of an extended monopole, the charge quantization results of Witten [10] and Callan [2]. Finally, the anomalous non-conservation of axial vector charge in the monopole field is demonstrated. A conserved, gauge dependent charge is constructed and its anomalous commutator with the electric charge operator found.

A recapitulation and brief concluding remarks are offered in the final section.

2. Formulation

Consider an SU(2) local gauge theory, spontaneously broken to U(1) by an adjoint multiplet of scalars, and coupled to an iso-doublet massless Dirac fermion. The action functional is

$$S[A_\mu, \Phi, \Psi, \bar{\Psi}] = \int d^4x \Big[-\frac{1}{2e^2}\, Tr\, F_{\mu\nu} F^{\mu\nu} - \frac{1}{e^2} Tr D^\mu \Phi D_\mu \Phi - V(\Phi)$$

$$+ \, i\, \bar{\Psi}\, \gamma^\mu (\partial_\mu - iA_\mu)\, \Psi \Big]$$

where $D_\mu\Phi = \partial_\mu\Phi - i[A_\mu, \Phi]$. Our matrix notation is $A_\mu = \sum_{a=1}^{3} A_\mu^a \frac{\tau^a}{2}, \Phi = \sum \Phi^a \frac{\tau^a}{2}, F_{\mu\nu} = \sum F_{\mu\nu}^a \frac{\tau^a}{2} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ with $Tr\, \tau^a \tau^b = 2\delta^{ab}$.

The classical monopole configuration is characterized by values of the gauge and scalar fields which solve thier classical equations of motion with $\Psi = 0$ and make a finite contribution to S. In a non-singular, static gauge,

$$
\begin{aligned}
&A_0^{(0)} = 0, \quad \vec{A}^{(0)} = \hat{r} \times \vec{\tau}\ \frac{1 - K(r)}{2r} \\
&\Phi^{(0)} = \hat{r} \cdot \frac{\vec{\tau}}{2}\, H(r).
\end{aligned}
\tag{1}
$$

The functions $K(r)$ and $H(r)$ satisfy $K(0) = 1$, $H(0) = 0$ and tend exponentially to the limiting values

$$
\begin{aligned}
&K(r) \underset{r\to\infty}{\longrightarrow} 0 \\
&H(r) \underset{r\to\infty}{\longrightarrow} h
\end{aligned}
$$

where h is the expectation value of the Higgs field in the vacuum sector.

The fields, (1), are treated as a fixed background into which quantum fluctuations are introduced. In addition to the fermionic excitations, the gauge field can fluctuate in the space of degenerate configurations reached by performing time independent, spherically symmetric gauge transformations which leave the scalar field expectation value invariant. This "charge rotator" degree of freedom is conveniently parameterized by a collective coordinate $\lambda(r, t)$:

$$
\begin{aligned}
&\vec{A}^{(\lambda)} = U_\lambda\ \vec{A}^{(0)}\ U_\lambda^{-1} + iU_\lambda\ \vec{\nabla}\, U_\lambda^{-1} \quad , \quad A_0^{(\lambda)} = 0 \\
&\Phi^{(\lambda)} = U_\lambda\ \Phi^{(0)}\ U_\lambda^{-1} = \Phi^{(0)}
\end{aligned}
\tag{2}
$$

with $U_\lambda = exp\ i\lambda(r, t)\, \hat{r} \cdot \frac{\vec{\tau}}{2} \quad , \quad \lambda(0, t) = 0.$

Note that the gauge field fluctuations are restricted to the unbroken, residual abelian direction of SU(2). The broken gauge and Higgs field degrees of freedom have large masses and are consequently frozen in the configuration of eq. (1).

The fermion field can be expanded in partial waves of total angular momentum $\vec{J} = \vec{L} + \vec{S} + \vec{I}$, which is conserved in the monopole background. As usual, we will consider only $J = 0$ excitations which evidently account for much of the interesting physics of the system [1,2,3,]:

$$\Psi^{(J=0)}_{A\alpha} = \frac{1}{\sqrt{2}}(\Psi^{(+)}_{A\alpha} + \Psi^{(-)}_{A\alpha}) \quad , \quad \Psi^{(\pm)}_{A\alpha} = \begin{pmatrix} X^{(\pm)} \\ \pm X^{(\pm)} \end{pmatrix}_{A\alpha} \tag{3}$$

with

$$X^{(\pm)}_{A\alpha} = \frac{1}{\sqrt{8\pi}\, r}\Big[[g_\pm(r,t) - i\, p_\pm(r,t)\,\hat{r}\cdot\vec{\sigma}]\sigma^2\Big]_{A\alpha} \quad . \tag{4}$$

In (3) and (4), A and α are, respectively, spin and isospin indices taking on the values 1, 2, and $\gamma^5\Psi^{(\pm)} = \pm\Psi^{(\pm)}$.

Inserting eqs. (1), (2), (3) and (4) into the action functional, we obtain

$$S[\lambda,\chi,\bar{\chi}] = \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \left\{ \frac{2\pi r^2}{e^2}\left[(\lambda')^2 + 2\frac{K^2}{r^2}\dot{\lambda}^2\right] \right.$$

$$\left. + i\,\bar{\chi}^{(+)}\left[\not{\partial} - i\frac{\lambda'}{2}\bar{\gamma}^0 + \frac{K}{r}\bar{\gamma}^5 e^{-i\lambda\bar{\gamma}^5}\right]\chi^{(+)} + i\,\bar{\chi}^{(-)}\left[\not{\partial} + i\frac{\lambda'}{2}\bar{\gamma}^0 + \frac{K}{r}\bar{\gamma}^5 e^{i\lambda\bar{\gamma}^5}\right]\chi^{(-)}\right\} \tag{5}$$

where $\dot{\lambda} = \partial_t\lambda, \lambda' = \partial_r\lambda$ and

$$\chi^{(\pm)}(r,t) = \begin{pmatrix} g\pm(r,t) \\ \pm\, p\pm(r,t) \end{pmatrix} \quad . \tag{6}$$

We have also introduced the two dimensional Dirac matrices $\bar{\gamma}^\mu$, $\mu = 0, 1,$ $\bar{\gamma}^5 = \bar{\gamma}^0\,\bar{\gamma}^1$ which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\, g^{\mu\nu}$; thus, $\not{\partial} = \bar{\gamma}^0\,\partial_t + \bar{\gamma}^1\,\partial_r$.

To round out the reduction to eq. (5) and the variables λ and $\chi^{(\pm)}$, we give here several important observables in terms of these variables. First, various fermion bilinears are easily obtained from eqs. (3), (4) and (6):

$$\bar{\Psi}^{(J=0)}\;\gamma^0\;\tau^a\;\Psi^{(J=0)} = \frac{\hat{r}^a}{4\pi r^2}\Big(\bar{\chi}^{(+)}\,\bar{\gamma}^1\,\chi^{(+)} - \bar{\chi}^{(-)}\bar{\gamma}^1\,\chi^{(-)}\Big)$$

$$\equiv \frac{\hat{r}^a}{4\pi r^2}\Big(\rho^5_+ - \rho^5_-\Big) \equiv \frac{\hat{r}^a}{4\pi r^2}\,\rho^5 \; . \tag{7a}$$

$$\bar{\Psi}^{(J=0)}\;\gamma^i\,\tau^a\,\Psi^{(J=0)} = \frac{1}{4\pi r^2}\Big\{\hat{r}^{\,i}\,\hat{r}^a\Big\{\bar{\chi}^{(+)}\,\bar{\gamma}^0\,\chi^{(+)} - \bar{\chi}^{(-)}\,\bar{\gamma}^0\,\chi^{(-)}\Big\} + (\delta^{ia} - \hat{r}^i\,\hat{r}^a)$$

$$\times\Big(\bar{\chi}^{(+)}\,\chi^{(+)} - \bar{\chi}^{(-)}\,\chi^{(-)}\Big) + i\epsilon^{iaj}\,\hat{r}^j\Big(\bar{\chi}^{(+)}\,\bar{\gamma}^5\,\chi^{(+)} + \bar{\chi}^{(-)}\,\bar{\gamma}^5\,\chi^{(-)}\Big)\Big\} \tag{7b}$$

$$\bar{\Psi}^{(J=0)}\,\gamma^0\gamma^5\,\Psi^{(J=0)} = \frac{1}{4\pi r^2}\Big(\bar{\chi}^{(+)}\,\bar{\gamma}^0\,\chi^{(+)} - \bar{\chi}^{(-)}\,\bar{\gamma}^0\,\chi^{(-)}\Big) \equiv \frac{1}{4\pi r^2}\Big(\rho_+ - \rho_-\Big) \equiv \frac{1}{4\pi r^2}\rho \tag{7c}$$

$$\bar{\Psi}^{(J=0)}\,\gamma^i\gamma^5\,\Psi^{(J=0)} = \frac{1}{4\pi r^2}\hat{r}^i\,\rho^5 \quad . \tag{7d}$$

We have also taken the opportunity to define the radial electric and axial charge densities, ρ^5 and ρ respectively. Note that ρ^5 is also the radial axial vector current density and ρ the radial electric current.

In addition, from eq. (2) and the definition of the gauge invariant electromagnetic field tensor [11], we obtain the radial electric field

$$E_r = \dot{\lambda}' \quad . \tag{8}$$

The gauge properties of the system are quite transparent in terms of λ and $\chi^{(\pm)}$. From eq. (1), the residual gauge symmetry consists of abelian, time-independent transformations of the form

$$\vec{A}^{(\lambda)} \rightarrow U_a(\vec{A}^{(\lambda)} + i\,\vec{\nabla})\,U_a^\dagger$$

$$\Phi^{(\lambda)} \rightarrow U_a\Phi^{(\lambda)}U_a^\dagger = \Phi^{(\lambda)}$$

$$\Psi^{(J=0)}_{A\alpha} \rightarrow (U_a)_{\alpha\beta}\,\Psi^{(J=0)}_{A\beta}$$

where $U_a = exp\ ia(r)\hat{r}\cdot\frac{\vec{\tau}}{2}, a(0) = 0$. In terms of λ and $\chi^{(\pm)}$, these are simply

$$\begin{aligned} \lambda(r,t) &\rightarrow \lambda(r,t) + a(r) \\ \chi^{(\pm)} &\rightarrow e^{\pm ia(r)\frac{1}{2}\bar{\gamma}^5}\chi^{(\pm)} \end{aligned} \tag{9}$$

where the fermionic transformation law is most easily read off from eq. (7a). Returning to eq. (5), it's straightforward to verify the gauge invariance of the reduced action.

Briefly, now, the standard approach to the theory described by eq. (5) is to take the point-like, or, equivalently, infinitely massive, monopole limit. This amounts to setting $K(r)$ equal to zero, its limiting value far from the monopole core [1,2,3]. Since this procedure leads to a non-hermitian hamiltonian, a boundary condition is imposed on the fermion to define the theory. The resulting model is exactly soluble along the lines of the massless Schwinger Model [12]. Subsequently, we will refer to this approach as the point-like limit, approximation or model.

Several authors have pointed out that much of the physics uncovered in this model is dependent on the boundary condition [4,5,7], a somewhat unfamiliar and unsatisfactory circumstance when dealing with an extended system. We propose to investigate the theory without making the point-like approximation. Instead, wherever it's convenient or instructive to have a specific form for $K(r)$, we will adopt the Prasad-Sommerfield approximation, in which [9]

$$K(r) = \frac{m_w r}{sh\, m_w r} \quad . \tag{10}$$

Here m_w sets the scale of the monopole's mass, and m_w^{-1} the radius of the "core", in which non-abelian effects are significant. In general, it should be self-evident that our results depend only minimally, and never crucially, on this approximation. Finally, wherever interesting, we will attempt to compare our results and method to those of the soluble model.

3. Fermions in a Prasad-Sommerfield Background

In this section we examine the "free field" theory of fermions in a Prasad-Sommerfield background. Since this system is tractable, it provides a "free-field" limit for the theory described by eq. (5) and, in addition, is straightforwardly generalized to the case where an additional static, long range electric field is present. The field $\lambda(r,t)$ will be turned on in sections 4 and 5, first as a static background and then as a fully dynamical degree of freedom. Furthermore, it is interesting to compare the free field system with m_w finite, with that which forms the basis of the point-like approximation [13].

The "free" Dirac equation for the $J = 0$ state described by eqs. (3), (4) and (6) is

$$\left\{ i\,\bar{\gamma}^0\,\partial_\tau \;+\; i\,\bar{\gamma}^1\;\partial_r \;+\; i\frac{K(r)}{r}\bar{\gamma}^5 \right\} \chi_0^{(\pm)}(r,t) = 0 \tag{11}$$

with $K(r) = m_w r / sh m_w r$. The two solutions have been obtained by Marciano and Muzinich [14]. In a basis in which $\bar{\gamma}^0 = diag\,(1,-1)$, the physically acceptable positive frequency modes are

$$e^{-iEt}\, u_E(r) = e^{-iEt}\, C(E) \begin{pmatrix} th\,\frac{m_w r}{2}\; cos Er \;+\; \frac{2E}{m_w} sin\; Er \\ \\ cth\,\frac{m_w r}{2}\; sinEr \;-\; \frac{2E}{m_w} cos\; Er \end{pmatrix} \tag{12}$$

where $C(E)$ is a normalization constant yet to be determined. Note that, since both components of $u_E(r)$ vanish as r approaches zero, these solutions are square integrable at the origin.*

The second solution diverges at $r = 0$ and must consequently be rejected as unphysical:

$$e^{-iEt} \begin{pmatrix} th\,\frac{m_w r}{2}\; sinEr \;-\; \frac{2E}{m_w}\; cos\, Er \\ \\ cth\,\frac{m_w r}{2}\; cos\, Er \;+\; \frac{2E}{m_w}\; sin\, Er \end{pmatrix} .$$

This winnowed solution is the finite m_w counterpart of the mode eliminated by the charge non-conserving boundary condition of the point-like model.

Now, $C(E)$ can be fixed by the condition of orthonormality. We regularize the normalization integral at spatial infinity by introducting a box of length L and find

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L dr\, u_E^*(r)\, u_{E'}(r)$$

$$= \lim_{L\to\infty}\; C^*(E)\; C(E') \Bigg[\left(1 + \frac{4EE'}{m_w^2}\right) \frac{sin(E-E')r}{(E-E')L}$$

$$- \frac{2}{m_w L} \left(th\,\frac{m_w r}{2} cos\, Er\, cos\, E'r + cth\,\frac{m_w r}{2} sin\, Er\, sin\, E'r \right) \Bigg]_0^L$$

Equivalently, adiabatic regularization gives

$$\lim_{\alpha\to 0} \int_0^\infty dr e^{-\alpha r} u_E^*(r) u_{E'}(r) = \lim_{\alpha\to 0} C^*(E) C(E') \left(1 + \frac{4EE'}{m_w^2}\right) \frac{\alpha}{\alpha^2 + (E-E')^2} .$$

*That the lower component of u_E vanishes faster than the upper by a power of r is the motivation for the boundary condition adopted in the point-like ($m_w \to \infty$) limit: $(1 - \bar{\gamma}^0)\chi^{(\pm)}\,|_{r=0} = 0$.

Recognizing the standard representations

$$\delta_{E,E'} = \lim_{L\to\infty} \frac{sin(E-E')L}{(E-E')L}$$

$$\delta(E-E') = \lim_{\alpha\to o} \frac{1}{\pi}\frac{\alpha}{\alpha^2+(E-E')^2} \tag{13}$$

we conclude that, modulo an arbitrary phase,

$$C(E) = \left[1 + \left(\frac{2E}{m_w}\right)^2\right]^{-\frac{1}{2}} \tag{14}$$

with a density of states π^{-1}.

We can also find the completeness relation obeyed by our wave functions. Introducing the negative energy solutions to (11),

$$v_E(r) = u_{-E}(r) = \bar{\gamma}^0\, u_E(r) \quad ,$$

we find

$$\frac{1}{\pi}\int_0^\infty dE\left[u_E\,(r)\; u_E{}^\dagger\,(r') \;+\; v_E\,(r)\; v_E{}^\dagger\,(r')\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\left[cos\,E\,(r-r') \;-\; \bar{\gamma}^0\, cos\,E\,(r+r')\right]$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dE}{1+(2E/m_w)^2}\Big\{cos\;E(r-r')\;cth\,\frac{m_w(r-r')}{2}$$

$$\times\left[cthm_w r' - cthm_w r \;+\bar{\gamma}^0\,(cshm_w r - cshm_w r')\right]$$

$$+\,cos\;E(r+r')cth\frac{m_w(r+r')}{2}\left[-cshm_w r - cshm_w r' + \bar{\gamma}^0\,(cthm_w r + cthm_w r')\right]$$

$$-\frac{2E}{m_w}\;sin\;E(r-r')\left[cthm_w r - cthm_w r' - \bar{\gamma}^0\,(cshm_w r - cshm_w r')\right]$$

$$-\frac{2E}{m_w}\;sin\;E(r+r')\left[cshm_w r + cshm_w r' - \bar{\gamma}^0\,(cthm_w r + cthm_w r')\right]\Big\}$$

$$= \delta(r-r') - \bar{\gamma}^0\,\delta(r+r') \quad . \tag{15}$$

Note that this result is independent of m_w. It is interesting, though probably inconsequential, that eq. (15) differs from the completeness relation for free fermions satisfying the charge non-conserving boundary condition of the point-like model:

$$\frac{1}{\pi}\int_{-\infty}^{\infty} dE\, w_E(r)w_E^\dagger(r') = \delta(r-r)' + \bar{\gamma}^0\,\delta(r+r') \tag{16}$$

where

$$w_E(r) = \begin{pmatrix} \cos Er \\ \sin Er \end{pmatrix} \tag{17}$$

Other basic properties of the wave functions $u_E(r)$, notably as scattering states, have been explored elsewhere [14]. Here we will note only that they are eigenstates of neither the radial momentum operator $-i\partial_r$ nor of electric charge, $\bar{\gamma}^5$, but are, instead, energy and position dependent linear combinations of incoming $\bar{\gamma}^5 = -1$ and outgoing $\bar{\gamma}^5 = +1$ states. This is particularly transparent for $m_w r \gg 1$ where

$$u_E(r) \to \begin{pmatrix} \cos(Er-\Omega(E)) \\ \sin(Er-\Omega(E)) \end{pmatrix} \quad , \quad \tan\Omega(E) = \frac{2E}{m_w} \quad . \tag{18}$$

We propose, now, to second quantize the fermion field and derive its two-point function. Our field operator is

$$\chi_0^{(\pm)}(r,t) = \int_0^\infty \frac{dE}{\sqrt{\pi}}\Big[b_{E(\pm)}\, u_E(r)e^{-iEt} \;+\; d^\dagger_{E(\pm)}\, v^*_E(r)e^{iEt}\Big]$$

With $\{b_{E(\pm)}, b^\dagger_{E'(\pm)}\} = \{d_{E(\pm)}, d^\dagger_{E'(\pm)}\} = \delta(E-E')$ and eq. (15), we obtain the standard anti-commutation relation (note that $r, r' \geq 0$)

$$\{\chi_0^{(\pm)}(r,t), \chi_0^{(\pm)\,\dagger}(r',t)\} = \delta(r-r') - \bar{\gamma}^0\,\delta(r+r')$$

Defining the vacuum state, as usual, by

$$b_{E(\pm)}\,\big|0> = d_{E(\pm)}\big|\,0> = 0$$

we obtain the two-point function

$$S^{(0)}(r,t;r't') = \langle 0|\chi_0^{(\pm)}(r,t)\,\bar\chi_0^{(\pm)}(r',t')|0\rangle = \int_0^\infty \frac{dE}{\pi} u_E(r)\,\bar u_E(r') e^{-iE(t-t')}$$

$$= \frac{1}{2\pi i}\frac{\bar\gamma^0(t-t') + \bar\gamma^1(r-r')}{(t-t')^2 - (r-r')^2} - \frac{1}{2\pi i}\frac{(t-t') - \bar\gamma^5(r+r')}{(t-t')^2 - (r+r')^2}$$

$$-\frac{m_w}{4\pi i}\frac{\bar\gamma^0\, ch m_w(r-r')/2 - ch m_w(r+r')/2}{ch m_w(r-r') - ch m_w(r+r')}$$

$$\times\left\{\mathcal{E}_+\left[\frac{m_w}{2}(r-r'),\frac{m_w}{2}(t-t')\right] - \mathcal{E}_+\left[\frac{m_w}{2}(r+r'),\frac{m_w}{2}(t-t')\right]\right.$$

$$\left. - 2\pi i\Theta(r+r'-|t-t'|)\Theta(|t-t'|-|r-r'|) sh m_w|t-t'|/2\right\}$$

$$-\frac{m_w}{4\pi i}\frac{\bar\gamma^1\, sh m_w(r-r')/2 + \bar\gamma^5\, sh m_w(r+r')/2}{ch m_w(r-r') - ch m_w(r+r')}$$

$$\times\left\{\mathcal{E}_-\left[\frac{m_w}{2}(r-r'),\frac{m_w}{2}(t-t')\right] - \mathcal{E}_-\left[\frac{m_w}{2}(r+r'),\frac{m_w}{2}(t-t')\right]\right.$$

$$\left. - 2\pi i\,\Theta(r+r'-|t-t'|)\Theta(|t-t'|-|r-r'|)\epsilon(t-t') ch m_w(t-t')/2\right\} \tag{19}$$

with

$$\mathcal{E}_\pm(x,y) = \begin{cases} \begin{pmatrix}\epsilon(y)\\ 1\end{pmatrix}\left\{e^{|y|}[E_1(|x+y|)+E_1(|x-y|)] \pm e^{-|y|}[Ei(|x+y|)+Ei(|x-y|)]\right\} & for \quad |y|>|x| \\ \begin{pmatrix}\epsilon(x)\\ 1\end{pmatrix}\left\{e^{\epsilon(x)y}[E_1(|x+y|)-Ei(|x-y|) \mp e^{-\epsilon(x)y}[E_1(|x-y|)-Ei(|x+y|)]\right\} & for \quad |x|>|y| \end{cases}$$

E_1 and Ei are the exponential integrals [15]

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}dt \quad , \quad (x>0)$$

$$Ei(z) = \int_{-\infty}^z \frac{e^t}{t}dt \quad .$$

In comparing eq. (19) with the free progator of the point-like limit some care must be exercised. In fact, the manifestly m_w independent terms in $S^{(0)}$ do not match up at all with the $m_w = \infty$ model. It is straightforward, however, to explicity evaluate

$$
\begin{aligned}
S^{(0)}(r,t;r',t') \underset{m_w\to\infty}{\longrightarrow} & \frac{1}{2\pi i}\frac{\bar\gamma^0(t-t')+\bar\gamma^1(r-r')}{(t-t')^2-(r-r')^2} - \frac{1}{2\pi i}\frac{(t-t')-\bar\gamma^5(r+r')}{(t-t')^2-(r+r')^2} \\
& -\frac{m_w}{4\pi i}(e^{-m_w(r+r')/2})(-2e^{m_w(r+r')/2})\frac{2}{m_w}\frac{t-t'}{(t-t')^2-(r+r')^2} \\
& -\frac{m_w}{4\pi i}\bar\gamma^5(e^{-m_w(r+r')/2})(2e^{m_w(r+r')/2})\frac{2}{m_w}\frac{r+r'}{(t-t')^2-(r+r')^2} \\
= & \frac{1}{2\pi i}\frac{\bar\gamma^0(t-t')+\bar\gamma^1(r-r')}{(t-t')^2-(r-r')^2} + \frac{1}{2\pi i}\frac{(t-t')-\bar\gamma^5(r+r')}{(t-t')^2-(r+r')^2}
\end{aligned}
\tag{20}
$$

where asymptotic properties of the exponential integrals have been freely employed. Equation (20) is precisely the propagator associated with the normal modes of eq. (17). This result might be considered a good check of eq. (19), the derivation of which, though basically trivial, involves extensive algebra and manipulation of integrals.

Now consider the two-point function at equal times. It is perhaps not suprising, and is in fact presaged by the discrepency between eqs. (15) and (16), that this limit does not commute with $m_w \to \infty$. Using eq. (13) and properties of the exponential integrals, we find

$$
\begin{aligned}
S^{(0)}(r,t;r',t) = & \frac{1}{2}\delta(r-r')\bar\gamma^0 - \frac{1}{2}\delta(r+r') - \frac{1}{2\pi}\left(\bar\gamma^1\frac{1}{r-r'}+\bar\gamma^5\frac{1}{r+r'}\right) \\
& +\frac{im_w}{2\pi}\,\frac{\bar\gamma^5\, shm_w(r+r')/2+\bar\gamma^1\, shm_w(r-r')/2}{shm_w r\; shm_w r'}\int\limits_{m_w|r-r'|/2}^{m_w(r+r')/2} dt\frac{cht}{t}\,.
\end{aligned}
\tag{21}
$$

Of course, the completeness relation (15) corresponds to $\{S^{(0)}(r,t;r',t),\bar\gamma^0\}$.

The last term in (21) contains a logarithmic short-distance singularity as $r \to r'$, which contributes to the familiar "charge-violating" condensate found in the point-like model [13,16,17,18]:

$$\langle 0 \left| \bar{\chi}_0^{(\pm)} \bar{\gamma}^5 \chi_0^{(\pm)}(rt) \right| 0 \rangle = tr\, S^{(0)}(r,t;r,t)\, \bar{\gamma}^5$$
$$= \frac{-i}{2\pi r} + \frac{i}{\pi} \frac{m_w}{sh m_w r} \int_0^{m_w r} dt\, \frac{cht}{t} \tag{22}$$

where the finite term is the $m_w = \infty$ contribution. The divergence is proportional to the two-dimensional "mass", $K(r)/r$, and doesn't contribute in the point-like limit. In the soluble model, when the monopole's dyon degree of freedom is activated, this condensate (or, more accurately, its gauge invariant counterpart) aquires a supression factor $(e^2 m_w)^{-1}$ as $m_w \to \infty$ [16,17,18]. The suppression is supposed to reflect the high energy cost of exciting the dyon, which maintains charge conservation [17]. It will not, however, kill the singularity in eq. (22), which suggests that the present models are not correctly describing the short-distance behavior of the system. Of course, this is not suprising, since we have frozen several degrees of freedom with masses greater than or of order of m_w. In the point-like model, the fermion modes are effectively cut-off at m_w as well, so that the singularity in (22) does not appear.

To conclude, in this section we have examined several aspects of fermions coupled to a Prasad-Sommerfield monopole. The new results presented here are eqs. (14), (15), (19), and (22). In the next section, we conduct a search for the Prasad-Sommerfield dyon.

4. The Disappearing Prasad-Sommerfield Dyon

We proceed, now, to demonstrate that the theory described by eq. (5) admits no static, long range coulomb field solutions, or dyons. But first, as a reference, the Prasad-Sommerfield dyon will be derived in the absence of fermions.

The pure gauge action is

$$S[\lambda] = \int_{-\infty}^{\infty} dt \int_0^{\infty} dr \frac{2\pi r^2}{e^2} \Big[(\partial_r \dot{\lambda})^2 + \frac{2K^2}{r^2} (\dot{\lambda})^2 \Big]$$
$$= -\int dt\, dr \frac{2\pi}{e^2} r \dot{\lambda} \Big(\partial_r^2 - \frac{2K^2}{r^2} \Big) r \dot{\lambda} + \int dt \Big[\frac{2\pi}{e^2}\, \dot{\lambda}\, r^2\, \partial_r \dot{\lambda} \Big]_{r=\infty} .$$

Varying with respect to λ while fixing $\delta\lambda(t=\pm\infty)=\delta\lambda(r=0)=0$, we obtain the equation of motion

$$\frac{4\pi}{e^2}\, r\Big(\partial_r^2-\frac{2K^2}{r^2}\Big)r\,\ddot{\lambda}\equiv\ -\dot{\Pi}_\lambda=0 \tag{23}$$

with the boundary condition

$$\frac{4\pi}{e^2}r^2\ \partial_r\,\ddot{\lambda}\,|_{r=\infty}\ =0 \tag{24}$$

as well as $\lambda(r=0)=0$. The condition, (24), has a simple physical interpretation, following from eq. (8) for the radial electric field: $4\pi r^2\partial_r\dot{\lambda}\,|_{r=\infty}$ is clearly the total electric charge of the state and (24) expresses its conservation.

Equation (23) can be integrated once, but the integration constant is required, by gauge invariance, to vanish. To see this, note that $\Pi_\lambda=\delta L/\delta\,\dot{\lambda}$ is canonically conjugate to λ. Then, since the gauge symmetry is $\lambda(r,t)\to\lambda(r,t)+a(r)$,

$$G[a]=\int_0^\infty dr\ a(r)\ \Pi_\lambda(r)$$

is the generator of gauge transformations and must vanish for physical, gauge invariant states. Noting that $a(r)$ is arbitrary [up to $a(0)=0$], we have

$$-\Pi_\lambda=\frac{4\pi}{e^2}\ r\Big(\partial_r^2-\frac{2K^2}{r^2}\Big)r\,\dot{\lambda}=0\quad . \tag{25}$$

Of course, recalling that we are working in $A_0=0$ gauge, eq. (25) should simply be Coulomb's law and it is, indeed, straightforward to work out the covariant divergence of the non-abelian electric field:*

$$\partial_i\,F^{(\lambda)oia}+\epsilon^{abc}\,A^{(\lambda)b}\,F^{(\lambda)oic}=\frac{-\hat{r}^a}{r}\,\Big(\partial_r^2-\frac{2K^2}{r^2}\Big)r\,\dot{\lambda}=0\quad .$$

*It is well known that Coulomb's law does not follow from the variational principle in $A_0=0$ gauge.

[It is also worth noting that (25) is the equation of motion obtained by varying S with respect to the gauge invariant variable $\dot{\lambda}$; however, the "global" gauge invariance of this procedure causes us to miss eq. (24).]

Finally, with $K(r)$ given by (10), the two solutions to eq. (25) are

$$r\dot{\lambda} = rg(r) = cthm_w r \quad \text{and} \quad r\dot{\lambda} = rf(r) = m_w r\ cthm_w r - 1\ . \tag{26}$$

The condition $\lambda(r=0)=0$ eliminates the first of these, while $f(r)$ is, in fact, the Prasad-Sommerfield dyon potential. The static radial electric field is

$$E_r = \dot{\lambda}' = \frac{1-K^2}{r^2} \underset{r\to\infty}{\longrightarrow} \frac{1}{r^2} \quad .$$

When λ is quantized in section 5, we will return to and expand upon the gauge structure briefly touched on above.

Replacing the fermions, we have the coupled equations of motion

$$i\Big(\not{\partial} \mp \frac{i\lambda'}{2}\bar{\gamma}^0 + \bar{\gamma}^5\frac{K}{r}e^{\mp i\lambda\bar{\gamma}^5}\Big)\chi^{(\pm)} = 0 \tag{27}$$

$$\Big(\partial_r^2 - \frac{2K^2}{2}\Big)r\ddot{\lambda} = \frac{-e^2}{4\pi r}\frac{\dot{\rho}^5}{2} \tag{28}$$

where ρ^5 is defined in eq. (7a). In addition, eq. (24) is generalized to

$$\frac{4\pi}{e^2} r^2 \partial_r \ddot{\lambda}|_{r=\infty} = \frac{1}{2}\rho\Big|_{r=\infty} \quad . \tag{29}$$

Recalling eq. (7b) and the subsequent discussion, (29) is clearly the statement of charge conservation when fermions are present.

Similarly, (28) is the generalization of the time-derivative of Gauss's law, and must be replaced by its first integral

$$\Big(\partial_r^2 - \frac{2K^2}{r^2}\Big)r\dot{\lambda} = \frac{-e^2}{4\pi r}\frac{1}{2}\rho^5 \quad . \tag{30}$$

Now, consider a solution representing a static coulomb field, i.e., a solution with $\ddot{\lambda} = 0$. The essential point for our proof is that, with this constraint, the Dirac equation is soluble. Writing

$$\chi^{(\pm)} = exp\Big\{\mp\frac{i}{2}\Big[\int_0^r dr'\,\dot{\lambda}(r',t) - \bar{\gamma}^5\lambda(r,t)\Big]\Big\}\chi_\lambda^{(\pm)}(r,t)\ , \tag{31}$$

we find that $\chi_\lambda^{(\pm)}$ solves the "free particle" equation,

$$i\left(\not{\partial} + \bar{\gamma}^5\ \frac{K(r)}{r}\right) \chi_\lambda^{(\pm)} = 0 \tag{32}$$

of the previous section.

To find the charge density induced by the field λ, we make the usual gauge invariant, point split definition [12],

$$\begin{aligned}\langle \rho^5_\pm(r,t)\rangle_\lambda &= i\ \underset{r\to r'}{\text{sym. lim.}}\ Tr\,\bar{\gamma}^1\, S^{(\pm)}(r,t;r',t)\, exp\left[\mp i\,\bar{\gamma}^5 \int_r^{r'} dr''\,\frac{1}{2}\lambda'(r'',t)\right] \\ &= i\ \underset{r\to r'}{\text{sym. lim.}}\ Tr\,\bar{\gamma}^1\, S^{(0)}(r,t;r't)\, exp\left[\pm i \int_r^{r'} dr''\,\frac{1}{2}\dot{\lambda}\,(r'',t)\right]\end{aligned} \tag{33}$$

where $S^{(\pm)}$ is the two-point function for $\chi^{(\pm)}$ and we have used (31) to write (33) in terms of $S^{(0)}$, given in eqs. (19) and (21). [If the second equality in (33) appears to be gauge dependent, note that $\chi_\lambda^{(\pm)}$ is a gauge invariant operator.] Carrying out the limit in (33), using eq. (21), we obtain

$$\langle \rho^5_\pm\,(r,t)\rangle_\lambda \ = \ \mp\,\frac{1}{\pi}\,\frac{\dot{\lambda}\,(r)}{2}$$

or

$$\langle \rho^5\,(r,t)\rangle_\lambda \ = \ -\,\frac{1}{\pi}\,\dot{\lambda}\,(r)\,. \tag{34}$$

In deriving this result, we have treated λ as an arbitrary, externally imposed, classical field. We now ask that it be a self-supporting solution of the system. Self consistency then requires that λ satisfy

$$\left(\partial_r^2 - \frac{2K^2}{r^2}\right) r\,\dot{\lambda} + \frac{e^2}{8\pi r}\langle\rho^5\rangle_\lambda = \left(\partial_r^2 - \frac{2K^2}{r^2} - \frac{e^2}{8\pi^2 r^2}\right) r\,\dot{\lambda} = 0\,. \tag{35}$$

Now, physically acceptable solutions to (35) must vanish at the origin and describe electromagnetic fields which vanish at least as fast as r^{-2} at spatial infinity. Solving (35) in these asymptotic regions, its easy to show that the allowed solution must obey

$$r\,\dot{\lambda}\ \underset{r\to o}{\longrightarrow}\ r^{(2+\alpha/3)}$$

$$r\,\dot{\lambda}\ \underset{r\to\infty}{\longrightarrow}\ r^{-\alpha}$$

with $\alpha = e^2/8\pi^2$. But this behavior implies that the function $r\dot{\lambda}$ has an inflection point at some $\bar{r} \neq 0$:

$$0 = \partial_r^2 \, r\dot{\lambda}\Big|_{r=\bar{r}} = \left(\frac{2K^2}{r^2} + \frac{e^2}{8\pi^2 r^2}\right) r\dot{\lambda}\Big|_{r=\bar{r}} \tag{36}$$

and it's easy to convince oneself that the right hand side of (36) is positive definite for $\bar{r} \neq 0$. We therefore conclude that, in the presence of massless fermions, the Prasad-Sommerfield dyon disappears.

What has become of the dyon? The solution corresponding to the classical dyon field behaves like $\dot{\lambda} \sim r^\alpha$ as $r \to \infty$ and, as $\alpha \to o$, coincides with $f(r)$ in eq. (26). However, the transition is non-analytic: for any finite α, the dyon state is pushed off to infinite mass $[E_r \underset{r\to\infty}{\to} r^{\alpha-1} + O(r^{-2})]$. As has been previously noted, this is a consequence of the ability of the fermions to totally screen the dyon's electric field in the infinite volume limit [1,19].*

*A derivation of the "disappearing dyon" has been given by Sonoda which is similiar to this one in that the point-like approximation is not adopted [20]. However, the author assumes the anomalous non-conservation of the axial vector current and adopts it as a classical equation of motion.

5. Canonical Quantization

To quantize the theory of eq. (5), it is convenient to introduce the decomposition

$$\lambda(r,t) \;=\; \alpha(t)\,\frac{f(r)}{m_w} \;+\; \hat{\lambda}\,(r,t) \tag{37}$$

where $f(r)$ is given in eq. (26), and

$$\begin{aligned} \hat{\lambda}\,(r,t)\,|_{r=\infty} &= 0 \\ \lambda(r,t)\,|_{r=\infty} &= \alpha(t) \end{aligned}$$

Using eq. (25), we find that (37) diagonalizes the gauge action; we have

$$\begin{aligned} S[\hat{\lambda}\,,\alpha,\chi^{(\pm)}] = \frac{2\pi}{e^2}\int dt\,\frac{\dot{\alpha}^2}{m_w} + \int dt \int_0^\infty dr \Big\{ &\frac{-2\pi}{e^2}\,\dot{\hat{\lambda}}\, r\Big(\partial_r^2 - \frac{2K^2}{r^2}\Big) r\,\dot{\hat{\lambda}} \\ &+ i\,\bar{\chi}^{(+)}\Big(\not{\partial} - i\frac{\lambda'}{2}\bar{\gamma}^0 + \bar{\gamma}^5\frac{K}{r}\,e^{-i\lambda\bar{\gamma}^5}\Big)\chi^{(+)} \\ &+ i\,\bar{\chi}^{(-)}\Big(\not{\partial} + i\frac{\lambda'}{2}\bar{\gamma}^0 + \bar{\gamma}^5\frac{K}{r}\,e^{i\lambda\bar{\gamma}^5}\Big)\chi^{(-)}\Big\} \end{aligned} \tag{38}$$

from which follow the equations of motion

$$i\Big(\not{\partial} \mp \frac{1}{2}\lambda'\,\bar{\gamma}^0 + \frac{K}{r}\bar{\gamma}^5\,e^{\mp i\lambda\bar{\gamma}^5}\Big)\chi^{(\pm)} = 0 \tag{39}$$

$$\Big(\partial_r^2 - \frac{2K^2}{r^2}\Big) r\,\ddot{\hat{\lambda}} = \frac{e^2}{4\pi r}\Big[\frac{1}{2}\rho' - \frac{K}{r}\Big(\bar{\chi}^{(+)}\,e^{-i\lambda\bar{\gamma}^5}\,\chi^{(+)} - \bar{\chi}^{(-)}\,e^{i\lambda\bar{\gamma}^5}\chi^{(-)}\Big)\Big] = \frac{-e^2}{4\pi r}\frac{1}{2}\dot{\rho}^5 \tag{40}$$

$$\ddot{\alpha} = \frac{e^2 m_w}{4\pi}\Big[\int_0^\infty dr\,\frac{f(r)}{m_w}\frac{1}{2}\dot{\rho}^5 + \frac{1}{2}\rho\,|_{r=\infty}\Big] = \frac{e^2 m_w}{4\pi}\int_0^\infty dr\Big[\frac{f(r)}{m_w}\frac{\dot{\rho}^5}{2} + \Big(\frac{f(r)}{m_w}\frac{\rho}{2}\Big)'\Big]\,. \tag{41}$$

To obtain the last equality in eq. (40), we have used the partial conservation law, which can be derived from the Dirac equation, (39),

$$\frac{1}{2}\,\partial_\mu\epsilon^{\mu\nu}j_\nu \equiv \frac{1}{2}\Big(\dot{\rho}^5 + \rho'\Big) = \frac{K}{r}\Big(\bar{\chi}^{(+)}\,e^{-i\lambda\bar{\gamma}^5}\,\chi^{(+)} - \bar{\chi}^{(-)}\,e^{\,i\lambda\bar{\gamma}^5}\,\chi^{(-)}\Big) \tag{42}$$

where $j^\mu = \bar{\chi}^{(+)}\bar{\gamma}^\mu \chi^{(+)} - \bar{\chi}^{(-)}\bar{\gamma}^\mu \chi^{(-)}$. Equation (42) expresses the near conservation of what would normally be identified as, and in the point-like limit is, the abelian fermionic electromagnetic current, $\epsilon^{\mu\nu} j_\nu$ [cf. eqs. (7a) and (7b)]. The right hand side is significantly different from zero only in the vicinity of the monopole ($r \lesssim m_w^{-1}$) where the physics is genuinely non-abelian and the fermionic charge is not well-defined.

This is not to say that the theory doesn't possess a "global" gauge symmetry, only that we must be careful to account for its underlying non-abelian nature. At this point, we will digress a bit to consider the true conservation laws following from the gauge symmetry, eq. (9). We must distinguish two kinds of transformation. These are shifts of λ by time-independent functions which, in one case, vanish at spatial infinity and, in the other, tend to a non-zero constant. Evidently, these can be represented by

$$\hat{\lambda}(r,t) \to \hat{\lambda}(r,t) + \delta\hat{\lambda}(r) \tag{43}$$

$$\alpha(t) \to \alpha(t) + \delta\alpha \quad . \tag{44}$$

It will become clear that these transformations correspond, respectively, to the local and global gauge symmetries of the theory.

Application of Noether's construction for (43) leads to the "charge density"

$$\kappa(r) = \frac{-4\pi r}{e^2}\left(\partial_r^2 - \frac{2K^2}{r^2}\right) r\dot{\hat{\lambda}} - \frac{1}{2}\rho^5 \tag{45}$$

while, for (44), we obtain the current

$$\begin{aligned} J^0 &= -\frac{f(r)}{m_w}\frac{1}{2}\,\rho^5 + \frac{4\pi}{e^2 m_w}\,\dot{\alpha}\,\partial_r\left(\frac{f}{m_w} r^2 \partial_r f\right) \\ J^1 &= -\frac{f(r)}{m_w}\frac{\rho}{2} \quad . \end{aligned} \tag{46}$$

Conservation of $\kappa(r)$ and of the current J_μ is clearly ensured by the equations of motion (40) and (41). Thus these equations are merely statements of the gauge symmetry and are consequently constraints, rather than true dynamical equations. It's plain that $Q_{tot} = \int_0^\infty dr J^0$ should be interpreted as the conserved electric charge of the theory and that

$$J_F^\mu = -\frac{f(r)}{m_w}\epsilon^{\mu\nu} j_\nu$$

is the fermionic electromagnetic current. Note that this identification is in agreement with the considerations of the previous section [cf. eq. (29)]:

$$\int dr\, \dot{J}^0 = \frac{4\pi}{e^2}\frac{\ddot{\alpha}}{m_w} - \int dr \frac{f(r)}{m_w}\frac{\dot{\rho}^5}{2} = \frac{4\pi}{e^2}\Big(\frac{\ddot{\alpha}}{m_w} + r^2\partial_r\ddot{\hat{\lambda}}\,\big|_{r=\infty}\Big) = \frac{4\pi}{e^2} r^2 \partial_r \ddot{\lambda}\,\big|_{r=\infty}$$

$$= -J^1\big|_0^\infty = \frac{1}{2}\rho\Big|_{r=\infty}$$

where (40) and (25) have been used. Furthermore, $4\pi\dot{\alpha}/e^2 m_w$ is clearly the charge lodged on the monopole core. [In this connection, it should be recalled that $f(r)m_w^{-1}$ is appreciably different from zero only for $r \gtrsim m_w^{-1}$ and tends to unity exponentially in that region.]

Finally, we note that α plays the role of an angle variable in the following sense: shifts of α by integral multiples of 2π are equivalent, as far as the action, eq. (38), is concerned, to shifts of $\hat{\lambda}$ by $2\pi n[1-f(r)m_w^{-1}]$. The consequence of this is that physical states which are locally gauge invariant must also evince periodicity in α.

Before discussing $\kappa(r)$ and eq. (45), we proceed to construct the Hamiltonian and quantization rules. From (38), we find the canonical momenta

$$\frac{\delta L}{\delta\dot{\hat{\lambda}}} \equiv \Pi_{\hat{\lambda}} = \frac{-4\pi}{e^2}\, r\Big(\partial_r^2 - \frac{2K^2}{r^2}\Big) r\,\dot{\hat{\lambda}} \tag{47}$$

$$\frac{\delta L}{\delta\dot{\alpha}} \equiv \frac{q_\lambda}{e} = \frac{4\pi}{e^2}\frac{\dot{\alpha}}{m_w} \quad . \tag{48}$$

To invert $\dot{\lambda}$ in terms of the momenta, we need the Green function, $D(r,r')$, satisfying

$$-r\Big(\partial_r^2 - \frac{2K^2}{r^2}\Big) r\; D(r,r') = \delta(r-r')$$

$$D(o,r') = D(r,o) = D(r,r')\Big|_{\substack{r\ or\ r'\\ =\infty}} = 0\ .$$

$D(r,r')$ is readily constructed from the functions $f(r)$ and $g(r)$:

$$D(r,r') = \frac{1}{m_w}\begin{cases} g(r)f(r') & \text{for } r > r' \\ g(r')f(r) & \text{for } r' > r \end{cases} \quad .$$

Thus,

$$\dot{\lambda}(r,t) = \frac{e}{4\pi} q_\lambda(t) f(r) + \frac{e^2}{4\pi}\int_0^\infty dr' D(r,r')\Pi_{\hat{\lambda}}(r',t),$$

and the Hamiltonian is

$$\begin{aligned} H &= \dot{\alpha}\frac{q_\lambda}{e} + \dot{\hat{\lambda}}\Pi_{\hat{\lambda}} + i\int dr\Big(\chi^{(+)\dagger}\dot{\chi}^{(+)} + \chi^{(-)\dagger}\dot{\chi}^{(-)}\Big) - L \\ &= \int_0^\infty dr\Big\{-i\chi^{(+)\dagger}e^{i\lambda\bar{\gamma}^5/2}\Big(\bar{\gamma}^5\partial_r + \bar{\gamma}^1\frac{K}{r}\Big)e^{-i\lambda\bar{\gamma}^5/2}\chi^{(+)} \\ &\quad - i\chi^{(-)\dagger}e^{-i\lambda\bar{\gamma}^5/2}\Big(\bar{\gamma}^5\partial_r + \bar{\gamma}^1\frac{K}{r}\Big)e^{i\lambda\bar{\gamma}^5/2}\chi^{(-)}\Big\} \\ &\quad + q_\lambda^2(t)\frac{m_w}{8\pi} + \frac{e^2}{8\pi}\int_0^\infty drdr'\,\Pi_{\hat{\lambda}}(r,t)D(r,r')\Pi_{\hat{\lambda}}(r,t)\,. \end{aligned} \tag{49}$$

The penultimate term, here, clearly accounts for the coulomb self-energy of the charge lodged on the monopole core. The last contribution is also an instantaneous coulomb interaction, but of the extended charge distribution beyond the core. The interaction energy between between these charges is hidden in the exponential factors of the fermionic part of H.

To quantize, impose the canonical equal-time commutation relations

$$[\Pi_{\hat{\lambda}}(r,t),\ \hat{\lambda}(r',t)] = -i\delta(r-r')$$

$$\Big[\frac{1}{e}q_\lambda(t),\ \alpha(t)\Big] = -i$$

$$\{\chi^{(\pm)\dagger}(r,t)\ ,\ \chi^{(\pm)}(r',t)\} = \delta(r-r')$$

with all other equal-time brackets vanishing. Hamilton's equations simply reproduce the Euler-Lagrange system; we can write them in the form

$$\frac{d}{dt}\Big(e^{\mp i\lambda\bar{\gamma}^5/2}\chi^{(\pm)}\Big) = -\Big(\bar{\gamma}^5\partial_r + \bar{\gamma}^1\frac{K}{r} \pm i\frac{\dot{\lambda}}{2}\bar{\gamma}^5\Big)e^{\mp i\lambda\bar{\gamma}^5/2}\chi^{(\pm)} \tag{50}$$

$$\dot{\Pi}_{\hat{\lambda}} = \frac{1}{2}\dot{\rho}^5 \tag{51}$$

$$\dot{q}_\lambda = -e \int_0^\infty dr \partial_\mu J_F^\mu \quad . \tag{52}$$

Equation (51) can immediately be integrated to give the conservation law, (45),

$$\Pi_{\hat{\lambda}} - \frac{1}{2}\rho^5 = \kappa \ , \ \dot{\kappa} = 0 \quad .$$

Now we can identify $\kappa(r)$ as the generator of gauge transformations which vanish at infinity. By forming

$$G[a] = \int_0^\infty dr\, a(r)\, \kappa(r)$$

with $a(0) = a(\infty) = 0$, and, using the commutation relations, we have

$$exp\left(i\,G[a]\right)\lambda(r,t) exp\left(-i\,G[a]\right) = \lambda(r,t) + a(r)$$

$$exp\left(i\,G[a]\right)\chi^{(\pm)}(r,t)\ exp\left(-i\,G[a]\right) = exp\left(\pm i a(r)\frac{\bar{\gamma}^5}{2}\right)\chi^{(\pm)}(r,t)\ .$$

As usual, though indicated by the gauge invariance, we cannot set $\kappa(r) = 0$ as an operator identity without upsetting the commutation relations. Instead, we must impose

$$G[a]\ |\psi\rangle = 0 \tag{53}$$

as a subsidiary condition on physical states.

The content of eq. (52) is, again, evidently charge conservation. In addition, the operator

$$\frac{1}{e}Q \equiv \frac{1}{e}q_\lambda + \int_0^\infty dr \frac{f(r)}{m_w}\frac{\rho^5}{2}$$

generates "global" gauge transformations, as in eq. (44):

$$e^{i\delta\alpha Q/e}\lambda(r,t)e^{-i\delta\alpha Q/e} = \lambda(r,t) + \delta\alpha\frac{f(r)}{m_w}$$

$$e^{i\delta\alpha Q/e}\chi^{(\pm)}(r,t)e^{-i\lambda\alpha Q/e} = exp\left(\pm i\delta\alpha\frac{f(r)}{m_w}\frac{\bar{\gamma}^5}{2}\right)\chi^{(\pm)}(r,t)\ .$$

Not suprisingly, the fermions carry a position dependent charge which tends quickly to $\pm 1/2$ beyond the monopole's core.

Now, to help realize the constraint, (53), we perform a canonical transformation to a gauge invariant fermion operator:

$$\begin{aligned}
&\chi^{(\pm)} \to e^{\mp i\lambda\bar{\gamma}^5/2}\chi^{(\pm)} \quad &, \quad &\chi^{(\pm)\dagger} \to \chi^{(\pm)\dagger}e^{\pm i\lambda\bar{\gamma}^5/2} \\
&\hat{\lambda} \to \hat{\lambda} &, \quad &\Pi_{\hat{\lambda}} - \tfrac{1}{2}\rho^5 \equiv \kappa \\
&\alpha \to \alpha &, \quad &q_\lambda \to q_\lambda - Q_F \equiv Q \quad .
\end{aligned}$$

It is easily verified that the commutation relations are invariant under this transformation. The new fermion operator is not only locally gauge invariant ($[\kappa, \chi^{(\pm)}] = 0$) but carries zero electric charge ($[Q, \chi^{(\pm)}] = 0$). In terms of the new variables, the Hamiltonian is cyclic in α and $\hat{\lambda}$:

$$H = \int_0^\infty dr \Big\{ -i\chi^{(+)\dagger}\Big(\bar{\gamma}^5\,\partial_r + \bar{\gamma}^1\frac{K}{r}\Big)\chi^{(+)} - i\chi^{(-)\dagger}\Big(\bar{\gamma}^5\,\partial_r + \bar{\gamma}^1\frac{K}{r}\Big)\chi^{(-)}\Big\} + \frac{m_w}{8\pi}Q^2$$

$$+ \frac{em_w}{4\pi}Q\int_0^\infty dr\,\frac{f(r)}{m_w}\frac{\rho^5}{2}(r,t) + \frac{e^2}{8\pi}\int_0^\infty dr dr'\,\frac{\rho^5}{2}(r,t)\,\tilde{D}(r,r')\,\frac{\rho^5}{2}(r',t)$$

where $\tilde{D}(r,r') = D(r,r') + m_w^{-1}f(r)f(r')$ and, in view of (53), several terms of the form $G[a]$ have been dropped. The equations of motion have been reduced to

$$\dot{\chi}^{(\pm)} = -\Big\{\bar{\gamma}^5\,\partial_r + \bar{\gamma}^1\frac{K}{r} \pm \frac{ie}{8\pi}\bar{\gamma}^5\Big[Qf(r) + e\int_0^\infty dr'\,\tilde{D}(r,r')\frac{\rho^5}{2}(r',t)\Big]\Big\}\chi^{(\pm)}$$

$$\dot{Q} = 0 \quad .$$

It's clear now that the physical states, those on which (53) is satisfied, are generated by the locally gauge invariant operators $\chi^{(\pm)}$, Q and $e^{\pm i\alpha}$. This last operator is included pursuant to the comment, above, on periodicity in α. That is, since $\alpha \to \alpha + 2\pi$ is equivalent to a local gauge transformation, only this function of α is gauge-invariant.*

*Put another way: we do not insist on global gauge invariance, but instead "allow" only those global transformations which go to the identity at spatial infinity and thus have no effect on locally gauge invariant states. We will somewhat relax this constraint below.

Using the formalism developed thus far, we propose to investigate two familiar aspects of the point-like model. The first of these is the spectrum of Q, the existance of charge superselection and the consequent absence of a charged fermion condensate around the monopole [17,18]. The second is the chiral anomaly, specifically the anomalous commutation relation of electric and chiral charge [19,21,22].

Since we have constructed a conserved operator, Q, and identified it as the total electric charge of the system, charge superselection is assured. However, we would like to demonstrate this property explicitly by finding the specturm of Q and, in the process, rederive the charge quantization results of Witten [10] and Callan [2].

Charge quantization follows trivially from the inclusion of $e^{\pm i\alpha}$ in the gauge invariant algebra. Since

$$[e^{\pm i\alpha}, Q/e] = \mp\, e^{\pm i\alpha} \ ,$$

repeated application of $e^{\pm i\alpha}$ to any eigenstate of Q generates a tower of eigenstates obeying $\Delta Q = \pm e$. The initial value of Q is arbitrary, so we obtain the spectrum

$$Q = e\left(n + \frac{\theta}{2\pi}\right) \quad , \quad n = 0,\ \pm 1,\ \pm 2,\ \ldots \tag{54}$$

where $0 \leq \theta < 2\pi$, but is otherwise undetermined.

We can construct the eigenstates of Q by introducing states $|\bar{\alpha}, \chi>$, where the symbol χ in the ket denotes the fermionic content of the state, and

$$\alpha\, |\bar{\alpha}, \chi> = \bar{\alpha}\, |\bar{\alpha}, \chi>$$
$$< \bar{\alpha}, \chi\, |\bar{\alpha}', \chi'> = \delta(\bar{\alpha} - \bar{\alpha}')\delta_{\chi,\chi'} \quad .$$

States with different $\bar{\alpha}$ are clearly degenerate in energy; consider the linear combination

$$|\,\bar{q}, m, \chi> \equiv \frac{1}{\sqrt{2\pi}} \int_{2m\pi}^{2(m+1)\pi} d\,\bar{\alpha}\, e^{-i\bar{q}\bar{\alpha}}\, |\bar{\alpha}, \chi> \quad . \tag{55}$$

The states $|\bar{q}, m, \chi>$ and $|\bar{q}, m \pm 1, \chi>$ are connected by the "topologically nontrivial", global gauge transformations

$$e^{\mp i 2\pi Q/e} \tag{56}$$

which simply shift α by $\pm\, 2\pi$. Thus, there is a one-paremeter family of states stable under (56),

$$|\bar{q}\,,\theta> = \sum_{m} e^{im\theta}\, |\bar{q}\,,\, m> \quad . \tag{57}$$

Evidently, we have

$$Q\,|\bar{q}\,,\theta> = e\Big(\bar{q} + \frac{\theta}{2\pi}\Big)|\bar{q}\,,\theta> \quad , \quad \bar{q} = 0,\, \pm\, 1,\, \pm 2,\, \ldots\,, \tag{58}$$

which ensures that

$$e^{\mp i2\pi Q/e}\, |\bar{q}\,,\theta> = e^{\mp i\theta}\, |\bar{q}\,,\theta> \quad .$$

The result (54) or (58) is just that of Witten [10]. The vacuum angle, θ, is the usual one, here appearing as the phase aquired by the states under topologically non-trivial gauge transformations, rather than as a parameter in the Lagrangian. Of course, since our theory contains massless fermions, it should be possible to rotate θ away, though at this point, it is unclear how this is to be done.

Charge superselection is now explicit in the fact that any charge carrying operator must contain factors of $e^{\pm i\alpha}$ which act as ladder operators on the physical states $|\bar{q}\,,\theta>$. Thus, the expectation value of any charged operator vanishes. In particular, consider the charged fermion condensate discussed in section 3. As explained below, the relevant operator is

$$\bar{\chi}^{(\pm)}_{ch}\,\bar{\gamma}^5\,\chi^{(\pm)}_{ch} \;= cos\alpha\,\bar{\chi}^{(\pm)}\,\bar{\gamma}^5\,\chi^{(\pm)} \tag{59}$$

where the subscript "*ch*" denotes a fermion carrying the global $U(1)$ charge, as is defined in eq. (60). Working to <u>zeroeth</u> order in e, we have

$$< \bar{q}\,,\theta,\Omega\left|\bar{\chi}^{(\pm)}_{ch}\,\bar{\gamma}^5\,\chi^{(\pm)}_{ch}\,\right|\bar{q}\,,\theta,\,\Omega> = <\bar{q}\,,\theta\left|\frac{1}{2}(e^{i\alpha} + e^{-i\alpha})\right|\bar{q}\,,\theta>$$

$$\times <\Omega\left|\bar{\chi}^{(\pm)}\,\bar{\gamma}^5\,\chi^{(\pm)}\right|\Omega>$$

where Ω denotes the fermion vacuum. The last factor is given by eq. (22), but the first evidently vanishes. In view of the definition, (55), this result can be thought of as a consequence of averaging the operator over gauge equivalent field configurations.

The vanishing of the expectation value, eq. (22), when the dyon degree of freedom is turned on has sometimes been attributed to "radiative corrections" [5]. Here we see that this choice of words is somewhat misleading. The disappearance of the condensate is independent of the coupling constant and is a consequence only of correctly accounting for gauge invariance.

Now, the spectrum (58) is incomplete since it cannot account for the possible presence of an odd number of charge $\pm 1/2$ fermions. The reason for this is clear: strictly speaking such a state is not gauge invariant. A transformation which rotates the fields through 2π at infinity will not leave the charged fermion invariant, owing to the fact that it's an $SU(2)$ doublet.

The field operator for a fermion carrying a half unit of the global $U(1)$ charge is

$$\chi_{ch}^{(\pm)} = e^{\pm i\alpha \frac{1}{2}\bar{\gamma}^5} \chi^{(\pm)} \; , \tag{60}$$

which satisfies

$$[Q, \chi_{ch}^{(\pm)}] = \pm \frac{e}{2} \bar{\gamma}^5 \chi_{ch}^{(\pm)} \quad .$$

This operator is obviously locally gauge invariant, but under the topologically nontrivial transformation, (56), it changes sign.* Evidently, we must relax, slightly, the gauge invariance constraint, and add $e^{\pm i\alpha/2}$ and $-e^{\pm i\alpha/2}$ to the gauge invariant algebra in order to account for the fermions. Pursuant to the charge quantization argument above, this yields the required spectrum

$$Q = e\left(\frac{n}{2} + \frac{\theta}{2\pi}\right) \quad , \quad n = 0, \; \pm 1, \; \pm 2, \; \ldots \quad . \tag{61}$$

*Note, however, that $e^{\pm 4\pi i Q/e}$ leaves $\chi_{ch}^{(\pm)}$ invariant, in agreement with the underlying SU(2) structure.

In the way of a recapitulation, we can note that all this is familiar from the Schwinger model [12]. As discussed by Rothe and Swieca [23], one constructs a Fock space representation for the algebra of that model by considering only local gauge transformations which vanish at spatial infinity. By relaxing this constraint to allow transformations which tend to the identify at infinity, a family of inequivalent representations is discovered, each built on a different vacuum. These representations correspond to our different charge sectors satisfying $\Delta Q = \pm e$. The underlying fermionic structure of the Schwinger model is only manifested when the constraint is further relaxed to allow transformations which tend to -1 at infinity. This step is completely analogous to the inclusion of the operators $\pm e^{\pm i\alpha/2}$ in the algebra which generates the spectrum, (61), consistent with the presence of arbitrary numbers of charge $\pm 1/2$ fermions.

Thus far, we have concentrated on the gauge symmetry of the monopole-fermion system. Also important for understanding the dynamics of this interaction is the global axial vector symmetry generated, in the $J = 0$ sector, by the current (7c) and (7d). At the level of the classical equations of motion, this current is conserved. But quantum effects produce an anomalous source term in the continuity equation [21];

$$\partial_\mu J^{\mu 5}_{(J=0)} = \frac{1}{4\pi^2} \, Tr \, F_{\mu\nu} \, \tilde{F}^{\mu\nu} = \frac{1}{2\pi^2} \, Tr \, \vec{E} \cdot \vec{B} \tag{62}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$ is the dual of the gauge field strength tensor. This equation implies that, in the presence of long range, parallel electric and magnetic fields, the axial charge

$$Q^5 = \int d^3 \vec{r} \, J^{o5}_{(J=0)} \tag{63}$$

is not conserved. In the radial $\vec{B}$ field of a monopole which fluctuates within its internal charge space and, thus, into configurations with a non-vanishing radial electric field, the anomaly is obviously important.

Equation (62) is a very familiar feature of the physics of the vacuum sector of gauge theories. Since we are not working in the vacuum sector, rather than simply render (62) in terms of our canonical variables, we'll draw on previous results to demonstrate the non-conservation of Q^5 in the presence of the monopole and, in effect, derive the anomaly equation. Our result will suggest an obvious definition of a conserved, but gauge dependent, axial charge, and lead us naturally to the standard anomalous commutator of the axial and electric charges. This commutator is immediate in, and is responsible for much of the non-trivial dynamics of, the point-like model [19]; our derivation, of course, will make no use of that approximation.

Now, from eqs. (63) and (7), we have

$$< \dot{Q}^5 >= \int_0^\infty dr < \dot{\rho} >= -\int_0^\infty dr\ \partial_r < \rho^5 >= - < \rho^5 > |_{r=\infty} \ .$$

For the expectation value of ρ^5, we simply use the result of section 4, eq. (34). In so doing, we are assuming that, for present purposes, we can make an adiabatic approximation for the field $\dot{\lambda}$ and neglect its time dependence. We obtain

$$< \dot{Q}^5 >= \frac{1}{\pi}\dot{\lambda}\Big|_{r=\infty} = \frac{\dot{\alpha}}{\pi} \quad . \tag{64}$$

Strictly speaking, this result in exact only for $\ddot{\alpha} = 0$, which, in view of the conclusions of section 4, is not possible. However, eq. (64) should correspond, in general, to the usual one-loop, "triangle-graph" calculation, since it is also exact to lowest order in e [i.e., $\ddot{\alpha} = O(e^2)$].* In fact, this result is identical to that obtained by assuming the anomaly equation, (62).

Equation (64) leads us to define a new axial charge,

$$\tilde{Q}^5 \equiv Q^5 - \frac{\alpha}{\pi} \quad . \tag{65}$$

Though conserved, $\tilde{Q}^5$ is clearly gauge dependent; under the transformation (56) it obeys $\Delta\tilde{Q}^5 = \pm 2$, a result familiar from anomaly physics in the vacuum sector [25]. Consequently, $\tilde{Q}^5$ has the anomalous commutator with the electric charge,

$$[Q, \tilde{Q}^5] = [Q, -\alpha/\pi] = \frac{ie}{\pi} \quad . \tag{66}$$

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*The one-loop non-abelian calculation actually involves higher order contributions [24], but the quantity we are computing depends only on the fields at infinity where the physics is essentially abelian. This explains a "non-abelian" discrepancy between the local versions of (64), $\dot{\rho} = \frac{1}{\pi}\dot{\lambda}'$, and eq. (62).

This equation has also been obtained by Balachandran and Schecter, but only after taking the point-like limit and by adopting an effective Hamiltonian which included an anomaly term [19].

Note that conservation of $\tilde{Q}^5$ depends on the masslessness of the fermions and that, as a symmetry generator, it allows us to rotate away the vacuum angle, θ. We have, from eqs. (55) and (57)

$$e^{-i\pi\theta\tilde{Q}^5/2\pi}|\bar{q},\theta> = |\bar{q}-\theta/2\pi,\theta> = |\bar{q},\theta=0> \tag{67}$$

where the second equality follows from the fact that only the combination $\bar{q}+\theta/2\pi$ is measurable.

As emphasized in reference [19], the real significance of eq. (66) is that the operator $e^{-i\pi\Theta\tilde{Q}^5}$ generates a dyonic state, with charge Θ, from the pure monopole ground state, and which is degenerate with the ground state by virtue of $\dot{\tilde{Q}}^5 = i[\tilde{Q}^5, H] = 0$.

Unfortunately, though the formalism adopted here is well suited to the limited analysis above, the dynamical consequences of the anomaly, particularly for the structure of the fermion vacuum, which are readily obtained in the point-like limit, are, at present, beyond our ability to explicitly exhibit in this model. This is because these effects are genuinely radiative and, thus, intractable in our, basically, tree-level investigation of the extended monopole problem. It's possible that an effective Hamiltonian approach, akin to that adopted in reference [19] for the point-like model, would allow a more complete description, but for the present, we'll not pursue this possibility.

Nevertheless, it's clear that the monopole-fermion ground state is an eigenstate of neither $\tilde{Q}^5$ nor Q^5. Thus, it's not suprising that an operator like $\bar{\chi}^{(+)}\chi^{(-)}$ should develop a vacuum expectation value, as it does in the point-like model [1,3].

6. Conclusion

In this report, we have attempted to study the interaction of massless fermions with a non-abelian magnetic monopole, without resorting to the point-like monopole limit. Exact, second quantized solutions were given, in sections 3 and 4, to the problem of fermions in the background field of a Prasad-Sommerfield monopole and dyon. In the latter case, we have verified the absence of dyon solutions in the presence of massless fermions.

In section 5, the "charge rotator" was canonically quantized. This system is not soluble for a finite-sized monopole. However, it was possible to rederive and elucidate several familiar results of the point-like model including gauge invariance and the ultimate conservation of electric charge despite its apparent non-conservation by the fermion sector. Our exact treatment of the "non-abelian" core lent itself particularly well to the accounting in this effect. In addition, the discrete spectrum of the electric charge operator was derived for the finite-sized monopole coupled to fermions.

Finally, the non-conservation of fermionic axial vector charge was demonstrated as a one-loop, quantum effect, and a conserved, but gauge-non-invariant version constructed. The standard anomalous commutation relation of the axial vector and electric charges was immediate.

Many of these results have been reported previously elsewhere without, however, the consistent context we have tried to present here. In particular, other authors have ended up in the point-like approximation or have adopted the anomalous divergence of the axial vector current as an equation of motion [7,17,19,20].

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