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# Path Integrals Without Lattices *

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## Abstract

One can approximate path integrals by expanding the fields in terms of a complete set of functions, by truncating the expansion, and by using Monte-Carlo techniques to evaluate the resulting finite-dimensional integrals. The errors introduced by these approximations are quite small, at least for the harmonic and anharmonic oscillators.

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Euclidean path integrals provide expressions for many physical quantities. The vacuum expected values of time-ordered products of field operators, for example, take the form

$$\langle\Omega|\phi(x_1)\phi(x_2)\cdots\phi(x_k)|\Omega\rangle = \frac{\int D\phi\ \exp(-S[\phi])\phi(x_1)\cdots\phi(x_k)}{\int D\phi\ \exp(-S[\phi])} \quad (1)$$

where $S[\phi]$ is the Euclidean action. One may approximate path integrals by Feynman diagrams when perturbation theory is valid. Otherwise one may use a space-time lattice. Lattice gauge theories have been conspicuously successful in recent years. However, the action functional of a theory must be altered before it can be used in a lattice theory. This is a serious problem for theories of gravity.

The present paper introduces a method for approximating path integrals that does not use a space-time lattice. The method consists of three steps, two of which involve approximations. The first step is to expand each field $\phi$ in terms of a complete set of functions $f_n$,

$$\phi(x) \equiv \phi(x,c) = \sum_{n=1}^{\infty} c_n f_n(x) \quad . \quad (2)$$

By substituting this expansion for each field into the action functional $S[\phi]$ and by integrating over space time, one may express the action as a function $S(c)$ of the expansion coefficients. Thus one converts a path integral into an infinitely multiple integral over the expansion coefficients $c$. For ratios of path integrals, the Jacobian $\det[\partial\phi(x,c)/\partial c_n] = \det[f_n(x)]$ cancels. The second step is to set all but a finite number of the expansion coefficients equal to zero. The

resulting integrals are now of finite dimension:

$$\langle\Omega|\phi(x_1)\cdots\phi(x_k)|\Omega\rangle = \frac{\prod_{n=1}^{N}\int dc_n \exp(-S(c))\phi(x_1,c)\ldots\phi(x_k,c)}{\prod_{n=1}^{N}\int dc_n \exp(-S(c))} \quad (3)$$

The third step is to use Metropolis's method of importance sampling [1] to numerically evaluate the resulting ratio of multi-dimensional integrals.

We have examined the errors introduced by the approximations of steps two and three of our method by applying it to the harmonic and anharmonic oscillators, which may be regarded as field theories in (0+1) dimensions. The hamiltonians are $H_0=(p^2+x^2)/2$ and $H_g=H_0+gx^4$. We computed the Green's function $G(t)=\langle 0|x(t)x(0)|0\rangle$ and its logarithmic derivative, both for $t=0$. As a set of basis functions, we used exponentials in $-|t|$, writing

$$x(t) = \sum_{n=1}^{N} c_n \exp(-n\lambda|t|) \quad (4)$$

where $\lambda$ is an adjustable parameter. We chose these functions because they are easy to work with.

To test the approximation introduced by step two, the truncation of the series (2) to a sum of N functions, we used eqs.(3-4) to write the Green's function G(0) as

$$G(0) = \langle 0|x(0)^2|0\rangle = \sum_{i,j=1}^{N} \langle c_i c_j\rangle, \quad (5)$$

where

$$\langle c_i c_j \rangle = \frac{\prod_{n=1}^{N} \int dc_n \exp(-S(c)) c_i c_j}{\prod_{n=1}^{N} \int dc_n \exp(-S(c))} \quad . \tag{6}$$

We did these Gaussian integrals and expressed G(0) in terms of the inverse of the matrix M

$$M_{i,j} = (\lambda^{-1} + ij\lambda)/(i+j)$$

as

$$G(0) = \frac{1}{2} \sum_{i,j=1}^{N} [M^{-1}]_{i,j} \quad . \tag{7}$$

We used the symbol-manipulating software Macsyma to evaluate this formula for arbitrary $\lambda$ and for various values of N up to 10. The resulting G(0) is of the form

$$G(0) = \frac{1}{2} - \frac{1}{2} \frac{\prod_{n=1}^{N} (-1+n\lambda)^2}{\sum_{n=1}^{N} I_n \lambda^{2n}} \quad . \tag{8}$$

This formula for G(0) yields the exact value, 1/2, at $\lambda = 1/n$ for $1 \leq n \leq N$, due to the double zeros in the numerator of the error term. The coefficients $I_n$ are positive integers and $I_N$ is $(N!)^2$. Even for N=4 the relative error in G(0) introduced by step two is less than three parts in ten thousand for $.2 \leq \lambda \leq 1$. For higher N the errors are even smaller. The errors due to the truncation of the series (2) are tiny for G(0), but may be larger for other physical quantities.

In order to examine the errors introduced by step three, the

Monte-Carlo evaluation of the ratio of the finite-dimensional integrals (3), we used our method to estimate G(0) and its logarithmic derivative for the harmonic and anharmonic oscillators.

The Green's function G(0) for the harmonic oscillator in our units has the value $G(0) = 1/2$ , and its logarithmic derivative at $t = 0$ is the energy gap between the ground state and the first-excited state, $-G(0)'/G(0) = \Delta E = 1$. We used ten functions of the form (4) with $\lambda = .1$ and made 100 runs of 10,000 passes each with a step size of 0.2. We found $G(0) = 0.4979 \pm 0.051$ and $\Delta E = 0.9331 \pm 0.129$. By using just 4 functions, with $\lambda = .25$, and making 100 runs of 10,000 passes each, with a step size of 0.1, we found $G(0) = 0.4932 \pm 0.145$ and $\Delta E = 0.8801 \pm 0.202$. Since in all these runs $\lambda N$ is 1, it follows from the formula (8) for G(0) that no error is introduced by the truncations of the series (2). Thus the errors in G(0) are entirely due to the Monte-Carlo evaluation of the integrals in eq.(6). The errors in $\Delta E$ are due to the Monte-Carlo approximation, to the truncation, and to the fact that the basis functions (4) were even, which is suitable for G(0) but not for $\Delta E$. These errors are typical of Monte-Carlo algorithms. In general, Monte-Carlo evaluations of integrals are better than one might expect for a small number of passes, but become precise only after a huge number of passes. On a Vax 780 each run took 19 sec. for N=4 and 67 sec. for N=10.

For the anharmonic oscillator, the exact values of the energy gap, $\Delta E$, are known for various values of the coupling constant g from the work of Biswas et al. [2]. We calculated the exact

expected values of $x^2$ in the ground state for these values of g, by using as a basis the eigenstates of the harmonic oscillator [3,4,5] and by using the matrix-manipulation software, Matlab [6]. From our path-integral approximation method, using 20 functions of the form (4), we obtained results for both G(0) and $\Delta E$ that were within one standard deviation of these exact results. With just 4 functions, we got good results for G(0). For each value of g, we made 10 runs of 3000 passes each. Our results are presented in the table, which also gives the values of $\lambda$ and of the step size, u.

| Anharmonic Oscillator | | | | | |
|---|---|---|---|---|---|
| g | 0.1 | 0.5 | 1 | 4 | 50 |
| G(0), exact | .4125 | .3058 | .2571 | .1728 | .0773 |
| G(0), N=20 | .409±.06 | .293±.04 | .267±.08 | .192±.07 | .082±.03 |
| G(0), N=4 | .433±.16 | .296±.07 | .269±.08 | .216±.05 | .076±.01 |
| $\Delta E$, exact | 1.2104 | 1.6282 | 1.9341 | 2.8728 | 6.4154 |
| $\Delta E$, N=20 | 1.03±.29 | 1.50±.67 | 1.56±.81 | 2.92±.89 | 4.82±1.6 |
| $\lambda$,u, N=20 | .1,.1 | .2,.06 | .2,.06 | .3,.03 | .3,.03 |
| $\lambda$,u, N=4 | .5,.1 | .5,.1 | .5,.1 | .5,.1 | .5,.1 |

The truncation of the series (2) and the Monte-Carlo evaluation of the integrals in eq.(6) are responsible for all of the error in G(0) and for some of the error in $\Delta E$. The other sources of error for $\Delta E$ are the fact that the basis functions (4) are even and that the logarithmic derivative of G(0) differs from $\Delta E$. On a Vax 780 each run took 25 sec. for N=4 and 17 min. for N=20. We used the random-number generator urand [7] exclusively.

We intend to apply this method to field theories in four dimensions. Such applications will require far more than 20 functions. If the action S(c) depended on all possible products of the expansion coefficients $c_n$, the computer time for one run would be prohibitively long. It is possible to avoid this problem by using basis functions $f_n(x)$ of compact support.

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