SLAC-PUB-3268
ANL-HEP-PR-83-56
September, 1983

LATTICE ACTION FORMS STABLE UNDER RENORMALIZATION*

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ABSTRACT

We review the role of the generalized gaussian solution to the Migdal-Kadanoff renormalization group for SU(N) lattice gauge theories, and point out that it can be continued down to very low values of the inverse coupling β. We thus explain the long distance stable line of actions observed in numerical investigations of SU(2), and propose a simple SU(3) mixed action which should exhibit improved scaling behavior.

*Work performed under the auspices of the United States Department of Energy.
†On leave from the Department of Physics and Astronomy, Tel-Aviv University. Work supported by the D.O.E. under contract number DE-AC03-76SF00515.

The approach of lattice gauge theories to the continuum limit is generally hampered by the smallness of the lattices available in conventional simulations. As a result, it is not immaterial in practice what action is used in a Monte Carlo simulation. Is it possible to improve the approach to the continuum by a judicious choice of the action?

Consider the effective action which results out of renormalizing a theory defined on a lattice, by integrating out degrees of freedom (e.g. block spins), so as to obtain a lattice with fewer sites. The resulting effective action should, for a lattice of a given size, provide an improved approach to the continuum, as its irrelevant operators are suppressed. In general, the effective action resulting out of the renormalization of single plaquette actions is non-local, and cannot itself be described in terms of single plaquettes.[1,2] However, in the Migdal-Kadanoff (MK) approximation[3,4] to the real space renormalization operation, the effective action lies in the space of single plaquettes, just like the original bare actions.

Since the MK effective actions are definable in terms of single plaquettes, they are reasonably easy to incorporate into a conventional program and to manipulate without the complications typical of the corresponding more exact multiplaquette expressions. In the past, Bitar, Gottlieb, and Zachos[5] observed that the MK effective action for SU(2) gauge theory is described essentially by

$$
\begin{aligned}
S &= \beta\left[\chi_{1/2} - 0.18\chi_1\right] \\
&= \beta\left[\mathrm{Tr}U - 0.18\left((\mathrm{Tr}U)^2 - 1\right)\right] .
\end{aligned}
\tag{1}
$$

This action is universal in that it is the long distance attractor of a large domain of possible bare actions entering the renormalization process--see Fig.1. Since it reflects properties of actions defined on lattices with spacings smaller than the spacing of the lattice on which it itself is defined, this action was conjectured[5] to approach the continuum limit faster than the Wilson action commonly used.

Following this clue, Otto and Randeria[6] computed the physical ratio of the mass of the lightest glueball (0^+) to the square root of the string tension in the SU(2) pure glue theory, for several values of the coupling β. They noted that this ratio varies with β significantly less when the action of Eq.(1) is used, as compared to the case when the Wilson action is used. They therefore concluded that this Long Distance Effective Action (LDEA), Eq.(1), improves the approach to the continuum limit, since it is less dependent on lattice artifacts like variation with the coupling.

A natural extension of the above investigation would be to find the corresponding LDEA for SU(3). Could one perhaps avoid carrying out the cumbersome analog of the renormalization calculation of Ref.5? In fact, this turns out to be possible, provided we find a generic characterization of the LDEA's within the framework of the MK approximation to the renormalization kernel.

Actually there exists empirical information on the generic form of the LDEA's of the MK kernel.[3,7,9,8,5,10] In some analogy to the central limit theorem of statistics[11], they are gaussians generalized to the appropriate group manifold, quite close to the heat kernel action[8,12-14]. Here, we will try to make this characterization somewhat more quantitative. By analogy with Eq.(1) for SU(2), we will further conjecture that the following SU(3) action exhibits improved scaling properties:

$$S(U) = \beta \, \mathrm{Re}\left[\chi_3(U) - 0.26\chi_8(U) - 0.10\chi_6(U)\right] \tag{2}$$

where $\chi_3(U) = \mathrm{Tr}U$ constitutes the Wilson action, and $\chi_6 = (\chi_3)^2 - (\chi_3)^*$, $\chi_8 = |\chi_3|^2 - 1$.

Let us start by a review of the MK renormalization recursions. We will follow the conventions of Refs.9,5,10. The actions for the gauge theories considered are class functions, i.e. they cannot distinguish among different group elements which belong to the same equivalence class. As a consequence, these actions can be expanded in terms of the characters of the group, and so can their Gibbs factors (their exponentials which enter into the functional integral):

$$F(U) \equiv e^{-S(U)} = \sum_r F_r d_r \chi_r(U)$$

$$F_r = \frac{1}{d_r} \int dU \; e^{-S(U)} \chi_r^*(U) \; . \tag{3}$$

Here $\chi_r(U)$ denotes the trace of U in the irreducible representation labeled by r; $d_r \equiv \chi_r(1)$ is the dimensionality of that representation; and dU is the normalized group invariant Haar measure.

If every other link is integrated out in all directions, the ensuing Gibbs factor will describe the exponential of the renormalized action. In general, this doubling of the basic length scale yields single plaquette effective actions like the original ones only in the special case of two spacetime dimensions. Nonetheless, Migdal[3] proposed to extend the two dimensional result to arbitrary numbers of dimensions d and scaling factors λ. His one-shot approximation relies on judicious processing of the link variables which reduces the problem to a two dimensional one.

The (Migdal) renormalized Gibbs factor reads:

$$F'(U) \equiv e^{-S'(U)} = \left(\sum_r F_r^{\lambda^2} d_r \chi_r(U)\right)^{\lambda^{d-2}} . \tag{4}$$

This recursion has the correct $d = 2$, $\lambda = 2$ limit, and, of course, the necessary $\lambda = 1$, and $S(U) = \text{const.}$ limits.

A closely related, perhaps more intuitive approximation has been provided by Kadanoff[4]. In addition, there have been attempts[15] to improve both approximations systematically, but at the heavy price of formal complication. For instance, the desirable feature of remaining in the original space of functions of single plaquettes is lost. We will thus not be discussing these improvements here.

The following joint recursion[5]:

$$e^{-S'(U)} = \left[\sum_r \left(\frac{1}{d_r} \int dV e^{-S(V)\lambda^b} \chi_r^*(V)\right)^{\lambda^2} d_r \chi_r(U)\right]^{\lambda^{d-2-b}} \tag{5}$$

describes both the Migdal ($b = 0$) and the Kadanoff ($b = d - 2$) prescriptions through the different settings of the formal parameter b.

Note that in a succession of transformations Eq.(5) for a given b, there is no dependence on b in all intermediate exponentiations. Furthermore, for an upscaling by a small factor $\lambda = 1 + \varepsilon$, the recursion Eq.(5) reads:

$$F' = F + \varepsilon\left[(d-2)F\ \ell nF + 2\sum_r F_r \ell nF_r\ d_r\chi_r\right] + O(\varepsilon^2)\ . \tag{6}$$

The dependence on b starts only at the second order in ε, which is to say that the infinitesimal renormalization kernel is identical for the Migdal and the Kadanoff transformations.[16] In what follows, we will thus focus only on the Migdal prescription, Eq.(4), without loss of generality as far as the infinitesimal transformation is concerned.

We will now proceed to search for fixed lines of actions of the recursion Eq.(4) (or Eq.(6)), that is actions which preserve their form under renormalization and only vary with respect to one parameter identifiable with the coupling. Clearly, in two dimensions, a large class of actions with $\ell n\ F_r = f(\beta)g(r)$ will do, provided the uniform rescaling of $f(\beta)$ dictated by the recursion Eq.(4) can be reinterpreted as a definition of the renormalized coupling: $\lambda^2 f(\beta) = f(\beta')$. A particularly simple family with this structure is the heat kernel action[12,13,8] defined through:

$$F_r = e^{-C_r/\beta} \tag{7}$$

where C_r is proportional to the quadratic Casimir invariant of the relevant group. For U(1): $C_r = r^2/4$, and for SU(2): $C_r = 2r(r + 1)$. In consequence, in two dimensions these actions maintain their form, while exhibiting asymptotic freedom (and attract nearby renormalization trajectories). Does this feature extend to higher numbers of dimensions, when the recursions Eq.(4)-(6) are no longer exact?

In higher numbers of dimensions, the situation is less clear, since raising F(U) to a power maintains its form only if it happens that $\ell n\ F(U) =$

$\bar{f}(\beta)\bar{g}(U)$. This is not true for any known families of the type specified above. However, it is approximately true for the heat kernel actions in the weak coupling regime, as we will now discuss. For U(1), the heat kernel action is equal to the periodic gaussian (Villain) action:

$$\sum_{r=-\infty}^{\infty} e^{-r^2/4\beta} \cos r\theta = \sqrt{4\pi\beta} \sum_{\ell=-\infty}^{\infty} e^{-\beta(\theta - 2\pi\ell)^2} . \tag{8}$$

For SU(2) Menotti and Onofri have generalized this to[8]:

$$\sum_{r} e^{\frac{-2r(r+1)}{\beta}} d_r \chi_r(\theta) = n(\beta) \sum_{\ell=-\infty}^{\infty} \frac{\theta/2 + 2\pi\ell}{\sin \theta/2} e^{-\beta(\theta/2 - 2\pi\ell)^2/2} . \tag{9}$$

The logarithm of $n(\beta)$ is an irrelevant additive constant in the action, which may be obtained by normalizing Eq.(9) at $\theta = 0$: like all constant shifts in the action, it will not be crucial in the discussion that follows, and will thus be ignored. In general, for SU(N), the appropriate periodic gaussian representation of the Gibbs factor is proportional to[8]:

$$\sum_{\{\ell\}=-\infty}^{\infty} \prod_{i<j} \frac{\phi_i - \phi_j + 2\pi(\ell_i - \ell_j)}{2\sin \frac{1}{2}[\phi_i \phi_j + 2\pi(\ell_i - \ell_j)]} e^{-\beta \sum_j (\phi_j + 2\pi \ell_j)^2/4} . \tag{10}$$

Here the N invariant angles are dependent for SU(N): $\sum_{i=1}^{N} \phi_i = 0$; they reduce to the N-1 independent class variables corresponding to the rank of the group. (The angle in the N = 2 case of this formula is normalized by 2, Eq.(9), to accord with standard angular momentum conventions.)

For weak coupling (large β), the U(1) Villain action is dominated by a periodic gaussian. For instance, in the Brillouin zone $[-\pi,\pi]$, the Gibbs factor of Eq.(8) goes like:

$$e^{-\beta\theta^2}[1 + e^{-\beta(2\pi)^2} 2\cosh(4\pi\beta\theta) + \ldots] \; . \qquad (11)$$

Consequently the action is essentially $\beta\theta^2_{[-\pi,\pi]}$, i.e. Manton's action[17], up to terms suppressed exponentially in β—they smooth out this action's cusps on the boundaries ±π of the Brillouin zone.

Since, in this approximation, the action has the requisite form, it follows by inspection of the renormalization recursion Eq.(4) that the renormalized coupling is $\beta' = \beta\lambda^{d-4}$. As a result, U(1) has a fixed point behavior for $d = 4$. Thus, for any large β, the theory is essentially free, as observed in studies of the iterated recursion.[3,7,10] In these studies there is moreover an extremely slight renormalization towards smaller β's. This flow becomes more apparent for smaller β's, as the suppression of the terms ignored in the above approximation weakens. For $d < 4$ and $d > 4$, inspection of the same weak coupling approximation reveals asymptotic freedom and anti-asymptotic freedom respectively. (In the strong coupling regime asymptotic freedom prevails for all d's, which dictates a phase transition for $d > 4$.)

The situation for SU(2) is somewhat more complicated, because of the additional presence of the crucial measure $\frac{\theta/2 + 2\ell\pi}{\sin\theta/2}$, which accounts for asymptotic freedom in four dimensions, as we will now discuss. Let us first take the logarithm of this measure so as to incorporate it in the action, and then focus on the first Brillouin zone $[-2\pi,2\pi]$. In analogy to the U(1) case, for large β, the zone is dominated by its central region $\theta \sim 0$—note that the singularities at $\theta = 0$ cancel between each $\pm\ell$ pair of terms. The important part of the action in this region is then $-\frac{\beta}{2}(\frac{\theta}{2})^2 + \ln(\frac{\theta/2}{\sin\theta/2})$. It turns out that the logarithmic term can be approximated reasonably well by a parabola in this region:

$$\ell n\left(\frac{\theta/2}{\sin\,\theta/2}\right) = \frac{1}{6}\left(\frac{\theta}{2}\right)^2 + \frac{1}{180}\left(\frac{\theta}{2}\right)^4 + \dots\ . \qquad (12)$$

Hence the dominant component in the action is a periodic gaussian $-\frac{1}{8}\left(\beta - \frac{1}{3}\right)\theta^2_{[-2\pi,2\pi]}$ with the requisite form for stability under renormalization[3,9,8,5].

A remarkable feature of Eq.(12) is the smallness of the contribution of the $O(\theta^4)$ terms: it amounts to a less than 10% correction to the Gibbs factor for all θ less than 3.8. This indicates that the gaussian approximation will hold for quite small β's well below 1, as discussed later.

The renormalized β' in the Migdal approximation is read off from Eq.(4):

$$\left(\beta' - \frac{1}{3}\right) = \lambda^{d-2}\left(\frac{\beta}{\lambda^2} - \frac{1}{3}\right)\ . \qquad (13)$$

We should, however, reinterpret $\beta - 1/3$ as the effective coupling $\bar{\beta}$. The renormalized $\bar{\beta}'$ is then:

$$\bar{\beta}' = \lambda^{d-4}\left(\bar{\beta} + \frac{1-\lambda^2}{3}\right)\ . \qquad (14)$$

This coupling ($\bar{\beta}$) is essentially identifiable with β_F of the LDEA of Ref.5 and Fig.1, since the projection of this gaussian action on the lowest SU(2) characters is, apart from an irrelevant additive constant:

$$-\frac{\bar{\beta}}{8}\,\theta^2_{[-2\pi,2\pi]} \approx \frac{7.11}{8}\,\bar{\beta}\left[\chi_{1/2} - 0.21\chi_1 + 0.08\chi_{3/2} + \dots\right]\ . \qquad (15)$$

Inspection of Eq.(14) directly reveals the presence of an unstable fixed point (and the concomitant phase transition) for d > 4:

$$\bar{\beta}_c = \frac{(\lambda^2 - 1)\lambda^{d-4}}{3(\lambda^{d-4} - 1)} . \tag{16}$$

For example, for d = 5 and $\varepsilon = 0.1$ we obtain $\bar{\beta}_c = 2/3 + \varepsilon$, which corresponds to $\beta_{Fc} = 0.68$, in accord with Refs.3,9.

It is also clear by inspection of Eq.(14) that asymptotic freedom prevails for d < 4. In four dimensions, $\bar{\beta}$ decreases with a speed independent of its value:[18]

$$\Delta\bar{\beta} = \bar{\beta}' - \bar{\beta} = -\frac{2}{3}\varepsilon + O(\varepsilon^2) . \tag{17}$$

This agrees well with the LDEA results of numerical MK iterations (Table II of Ref.5, and Ref.9) down to $\beta = \beta_F \sim 0.4$. In addition, down to the same coupling, the fixed line of Fig.1 is straight, with local slope 0.21 (Table II of Ref.5).

The above remarks suggest that the analytical treatment discussed here holds for quite large couplings ($\beta_F > 0.4$), even though it relies on the weak coupling approximation. For smaller β's, the quartic term in Eq.(12) becomes significant and upsets the fixed proportion among the characters, so that the LDEA begins to curve, aligning itself with the Wilson axis. For sufficiently small β's, it is evident that the renormalization recursion Eq.(4) diminishes the Wilson component less than all higher ones:

$$e^{\beta_F \chi_{1/2} + \beta_A \chi_1 + \cdots} = 1 + \beta_F \chi_{1/2} + \beta_A \chi_1 + \cdots$$

$$\rightarrow 1 + \frac{\lambda^{d-2} \beta_F^{\lambda^2}}{2^{\lambda^2 - 1}} \chi_{1/2} + \frac{\lambda^{d-2} \beta_A^{\lambda^2}}{3^{\lambda^2 - 1}} \chi_1 + \cdots \tag{18}$$

$$= e^{\lambda^{d-2} (\beta_F^{\lambda^2} \chi_{1/2}/2^{\lambda^2-1} + \beta_A^{\lambda^2} \chi_1/3^{\lambda^2-1} + \ldots)}$$

Since the LDEA we are studying is not a straight line near the origin of the β's (Fig.1), its straight portion does not quite extrapolate to the origin. However, since the corresponding intercepts are small, we choose to fit it with a straight line of slightly smaller slope Eq.(1), in the interest of computational simplicity.

Bitar[20] has suggested a refinement by providing a strong coupling approximation to the universal trajectory with parabolic behavior. He is guided by saturation of the Osterwalder-Schrader positivity bound[21,22], which is evident in weak coupling. In weak coupling, the two approximations to the LDEA, namely the heat kernel action Eq.(9) and the Manton action Eq.(15) satisfy and violate OS positivity, $F_r > 0$, respectively.[22] As they approach each other for large β, they bracket the boundary which separates the OS region from the domain of negative norms. Bitar traces this boundary to strong coupling and parameterizes on it the curved part the LDEA (the connection to the OS positivity boundary is however only empirical). In any case, the resulting parameterization is more elaborate than the simplest mixed actions we are proposing for Monte Carlo simulations.

Let us now summarize our discussion of the SU(2) LDEA's. For weak coupling and even well beyond the crossover region, they are reasonably well approximated by the heat kernel and the Manton actions, both of which exhibit smoother crossovers and improved scaling in the Monte Carlo simulations of Lang _et al._[23] Moreover, the two leading terms in the character expansion of the appropriate gaussian provide a simple mixed action close to Eq.(1), the fit to the universal LDEA which was specified by direct iteration of the MK kernel. This action was empirically observed to exhibit improved continuum behavior[6]. Let us now extend this reasoning to SU(N), and, as a consequence, obtain the LDEA for SU(3).

The extension of the above weak coupling approximation to the general SU(N) case[8] is straightforward. In the central Brillouin zone, all the angles of the LDEA in Eq.(10) are forced to lie near zero, and thus the logarithmic terms are well approximated by parabolas:

$$\sum_{i<j} \ln \frac{(\phi_i - \phi_j)}{2 \sin \frac{1}{2}(\phi_i - \phi_j)} \approx \frac{1}{6} \sum_{i<j} \left(\frac{\phi_i - \phi_j}{2}\right)^2 = \frac{N}{24} \sum_i \phi_i^2 \ . \tag{19}$$

We are thus led to the LDEA gaussian for SU(3):

$$-\left(\frac{\beta}{4} - \frac{1}{8}\right) \sum_{i=1}^{3} \phi^2_{i[-\pi,\pi]} = -\left(\frac{\beta}{2} - \frac{1}{4}\right)(\theta^2 + \phi^2 + \theta\phi)_{[-\pi,\pi]} \tag{20}$$

where $\phi_1 \equiv \theta$, $\phi_2 \equiv \phi$, and $\phi_3 = -(\theta + \phi)$. The analog of Eq.(17) is now $\Delta\bar{\beta} = -\varepsilon + O(\varepsilon^2)$. The character expansion of this (real) action is:

$$\theta^2 + \phi^2 + \theta\phi = C_1 + C_3\left(\frac{\chi_3 + \chi_{\bar{3}}}{2} - 3\right) + C_6\left(\frac{\chi_6 + \chi_{\bar{6}}}{2} - 6\right) + C_8(\chi_8 - 8) + \ldots \tag{21}$$

The real coefficients C_i are obtained by performing the two dimensional integrals:

$$\begin{aligned} C_1 &= \int dU(\theta^2 + \phi^2 + \theta\phi) = 4.88 \\ C_3 &= \int dU(\theta^2 + \phi^2 + \theta\phi)(\chi_3 + \chi_{\bar{3}}) = -2.73 \\ C_6 &= \int dU(\theta^2 + \phi^2 + \theta\phi)(\chi_6 + \chi_{\bar{6}}) = 0.28 \\ C_8 &= \int dU(\theta^2 + \phi^2 + \theta\phi)\chi_8 = 0.70 \end{aligned} \tag{22}$$

The Haar measure in this parameterization may be found in Ref.9. Projecting onto the lowest three characters and taking the intercepts of the extension of this line to be zero, we obtain the mixed action of Eq.(2). Taking into account the measure, this action agrees with Eq.(20) over most of the variable range reasonably well, and represents the approximate generic renormalization trajectory for SU(3).

Since approach to this universal trajectory upon scale expansion involves suppression of irrelevant components in the action (lattice artifacts), we conjecture it to provide better access to the continuum limit.[24] As in the case if its SU(2) analog, Eq.(1), we therefore wish to attract attention to this mixed action as a convenient, improved alternative to the Wilson action. Of course, since there is no agreement on the reliability of the MK framework, the superior performance of Eq.(2), or some action close to it, is an open "experimental" question. It thus appears to us that a Monte Carlo study of it should be quite worthwhile.

We are obliged to K. Bitar, M. Karliner and P. Reinartz for helpful conversations and numerical assistance. This work was partly carried out at Fermilab and the Max Planck Institut für Physik und Astrophysik, whose theory groups we also wish to thank.

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FIGURE CAPTION

Figure 1. Reproduced from Ref.5. The SU(2) MK renormalization trajectories of bare actions with a Wilson (β_F) and an adjoint (β_A) component. Within a large domain around the Wilson axis, all trajectories are attracted to and coalesce with a line of effective long distance actions. They then flow along this universal line of actions to the infrared fixed point at the origin.