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CHIRAL SYMMETRY BREAKING IN A COMPOSITE MODEL WITH SCALARS BASED ON LATTICE GAUGE THEORY*

PISIN CHEN†

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

and

CHARLES B. CHIU

Center for Particle Theory, Department of Physics
University of Texas, Austin, Texas 78712

ABSTRACT

In a composite model, based on SO(3) gauge group, in which there are both fermion and scalar fundamental fields, we determine whether there is spontaneous breaking of chiral symmetry and look for the mass gap between the ground state and the one composite-fermion state. The chiral symmetry is realized in the strong-coupling lattice Hamiltonian with the fundamental fermions being massless and fundamental scalars being massive. This calculation is based on the mean-field approximation to the state wave functions. Similar to the calculations of Quinn, Drell and Gupta in models without scalars, we also find that the chiral symmetry is spontaneously broken, and the composite fermions are massive. The extension of our calculation to SO(N) cases is shown to be straight forward.

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† Present address: Department of Physics, University of California, Los Angeles, CA 90024.

1. Introduction

In the context of strong coupling lattice gauge theories, Quinn, Drell and Gupta[1] (QDG) have shown that the chiral symmetry is spontaneously broken and composite leptons and quarks, which are made out of fundamental fermions are always massive. They argue that in going to the continuum limit there can only be two alternatives, either the chiral symmetry is realized in the Nambu-Goldstone mode with massive leptons and quarks, or there is a phase transition at some finite coupling. In the first case, the masslessness of leptons and quarks cannot be achieved; whereas in the second case the desired property of confinement and asymptotic freedom is lost.

Our goal in this paper is to extend this no-go senario to the type of composite models in which the fundamental matter fields involves both scalars and fermions. Due to the limitation of our technical tools, we can only deal with those models having SO(N) as their gauge groups. A generalization to SU(N) groups awaits for future efforts.

We shall first briefly review the QDG senario, then go to a SO(3) model and demonstrate that the no-go senario can be extended to the case involving fermion and scalar fundamental fields. We then argue that this can be straight forwardly extended to SO(N) cases. Although we are not able to cover the no-go senario to all models of this type (in particular, the SU(N) cases like the Fritzsch-Mandelbaum model[2] or the Abbott-Farhi model[3]), the implication is already interesting. Some discussions are given at the end.

2. The Quinn-Drell-Gupta No-Go Senario

As mentioned previously, Quinn, Drell and Gupta consider the composite models in strong-coupling lattice gauge theories. They use the long-range form of the lattice gradient operator (the SLAC gradient), which for an infinite-volume lattice is

$$\partial_\mu \psi(\vec{j}) \equiv \sum_{\vec{j}'} \left[\prod_{\hat{\eta} \neq \hat{\mu}} \delta_{j_{\hat{\eta}}\, j'_{\hat{\eta}}} \right] \frac{(-1)^{(j_{\hat{\mu}} - j'_{\hat{\mu}})}}{j_{\hat{\mu}} - j'_{\hat{\mu}}}\, \psi(\vec{j}') \quad , \tag{2.1}$$

in order to explicitly maintain chiral symmetry without the fermion "doubling" problem. Consider the problem of a simple hypercolor gauge group, say SU(N), with fermions (preons) assigned to some set of representations R with dimension d_R and number of flavors f_R. The Hamiltonian in $A_0 = 0$ gauge is given by

$$\begin{aligned} H = \frac{1}{a} \Bigg\{ \sum_{\text{links}} g^2 E^2 + \frac{1}{g^2} \sum_{\text{plaquettes}} \sum_R \left[\operatorname*{Tr}_{\Box} \left(U^R U^R U^{R\dagger} U^{R\dagger} \right) + h.c. \right] \\ + \sum_R \sum_{a=1}^{f_R} \sum_{\vec{j}, \hat{\mu}, \ell} \left[\frac{(-1)^\ell}{\ell}\, \bar{\psi}^a_R(\vec{j})\, \alpha_\mu \sum_{\vec{j}' = \vec{j}}^{\vec{j} + (\ell - 1)\hat{\mu}} U^R(\vec{j}', \hat{\mu})\, \psi^a_R(\vec{j} + \ell\, \hat{\mu}) \right] \Bigg\} , \end{aligned} \tag{2.2}$$

where the lattice spacing a is the only dimensionful quantity and α_μ is the Dirac matrix $\gamma_0 \gamma_\mu$. For strong-coupling effective Hamiltonian we separate H into

$$H_0 = \sum g^2 E^2 , \quad \text{and} \quad V = H - H_0 \tag{2.3}$$

and perform degenerate perturbation theory in the sector of flux-free states. This requires that the fermion states at every site is a gauge group singlet. By retaining the terms only up to order $1/g^2$, which corresponds to acting V twice, exciting a flux then annihilating it, gives[4]

$$\begin{aligned} H_{eff} = \frac{1}{g^2} \sum_R \sum_{\vec{j}, \ell, \hat{\mu}} \psi^{\dagger a}_{R\alpha}(\vec{j})\, \alpha_\mu \psi^{\beta a}_R(\vec{j} + \ell\, \hat{\mu})\, \psi^{\dagger b}_{R\beta}(\vec{j} + \ell\, \hat{\mu})\, \alpha_\mu\, \psi^{\alpha b}_R(\vec{j}) \left(\frac{(-1)^\ell}{\ell} \right)^2 \frac{N_R}{g^2 C_R \ell} \\ + O\left(\frac{1}{g^4} \right) \quad , \end{aligned} \tag{2.4}$$

where

$${}^{\alpha}U^R_\mu ({}^{\sigma}U^R_\nu)^\dagger = N_R\, \delta^\alpha_\sigma \delta^\nu_\mu + \text{nonsinglet pieces}$$

and $g^2 C_R \ell$ is the energy denominator from H_0 corresponding to a string of length ℓ in representation $R - \bar{R}$.

Notice that the Hamiltonian has a nearest-neighbor symmetry

$$S_{nn} = \prod_R \left[SU(4f_R) \otimes U(1)_R \right] \quad . \tag{2.5}$$

The $U(1)$ charge is

$$\begin{aligned} Q_R(\vec{j}) &= b_R^\dagger(\vec{j})\, b_R(\vec{j}) - d_R^\dagger(\vec{j})\, d_R(\vec{j}) \\ &= \tilde{\psi}_R^\dagger(\vec{j})\, \tilde{\psi}_R(\vec{j}) - 2 f_R\, d_R \quad , \\ Q_R &= \sum_{\vec{j}} Q_R(\vec{j}) \quad , \end{aligned} \tag{2.6}$$

where

$$\tilde{\psi}_R(\vec{j}) \equiv \alpha_x^{j_x}\, \alpha_y^{j_y}\, \alpha_z^{j_z}\, \psi_R(\vec{j}) \quad ,$$

$$\tilde{\psi}_R^{\alpha a} = \begin{pmatrix} b_R^{\alpha a} \\ d_R^{\dagger \alpha a} \end{pmatrix} \quad ,$$

and b and $d^\dagger$ are two-component spinors. The generators of the $SU(4f_R)$ are

$$\begin{aligned} Q_R^k(\vec{j}) &= \tilde{\psi}_R^\dagger(\vec{j})\, M^k\, \tilde{\psi}_R(\vec{j}) \quad , \\ Q_R^k &= \sum_{\vec{j}} Q_R^k(\vec{j}) \quad , \end{aligned} \tag{2.7}$$

where M^k are the $(4f_R \times 4f_R)$ Hermitian unitary matrix representations of $SU(4f_R)$. By performing a Fierz transformation and using Eqs. (2.6) and (2.7), we can rewrite Eq. (2.4) in a compact form

$$\begin{aligned} H_{eff} = \frac{1}{g^2} \sum_R \sum_{\vec{j},\ell,\hat{\mu}} \frac{N_R}{4 f_R C_R} \\ \times \left[Q_R(\vec{j})\, Q_R(\vec{j} + \ell \hat{\mu}) + \chi^2 \sum_k (\eta^{\mu k})^{\ell+1}\, Q_R^k(\vec{j})\, Q_R^k(\vec{j} + \ell \hat{\mu}) \right] \frac{1}{\ell^3} \, , \end{aligned} \tag{2.8}$$

where

$$\alpha_\mu M^k \alpha_\mu = \eta^{\mu k} M^k \qquad (\eta^{\mu k} = \pm 1) \tag{2.9}$$

and χ is a normalization factor. Since the interactions fall off rapidly as $(\ell)^{-3}$, the odd-neighbor terms ($\ell = 1$) provide the dominant part of H_{eff}. The smaller terms ($\ell = 2$) provide the symmetry breaking perturbations. The odd-terms are of the form

$$\chi^2 \sum_k Q_R^k(\vec{j})\, Q_R^k\Big[\vec{j} + (2n+1)\,\hat{\mu}\Big] + Q_R(\vec{j})\, Q_R\Big[\vec{j} + (2n+1)\,\hat{\mu}\Big]$$

and hence are antiferromagnetic in character, tending to antialign $SU(4f_R) \times U(1)$ spins on sites separated by odd numbers of lattice spacings. The even-ℓ terms tend to reinforce or compete with this antialignment depending on whether the sign $\eta^{\mu k}$ is negative or positive. By using the mean-field ansatz with two overlapping sublattices (see Ref. 4, and more specifically the simplified version in Ref. 5), QDG showed that the states that develop an expectation value for $M^k = \gamma_0$, that is for the operator

$$\bar{\psi}(\vec{j})\, \psi(\vec{j}) = (-1)^{j_x + j_y + j_z}\, \tilde{\psi}^\dagger(\vec{j})\, \gamma_0\, \tilde{\psi}(\vec{j}) \qquad , \tag{2.10}$$

are among the degenerate set of possible ground states that minimize the energy density

$$\mathcal{E} = \frac{\langle\phi|H_{eff}|\phi\rangle}{\text{volume}} \qquad . \tag{2.11}$$

Any infinitesimal mass term added to the Hamiltonian will select this chiral symmetry breaking state about which the mass acts as a perturbation.

QDG also showed that by acting on any site of the lattice with a composite fermion operator on the chosen ground state, there will be a mass gap created and of order $1/g^2$. Hence the composite leptons and quarks are massive. This conclusion does not change for any choice of fermion representation content and gauge group for four-component fermions. And the no-go senario of QDG as stated at the beginning of this paper follows.

3. The Ground State Structure in a SO(3) Lattice Gauge Theory

Now we extend the above no-go senario to composite models involving fermions and scalars as fundamental constituents. We begin with the model where the hypercolor SO(3) is the gauge group, both fermions and scalars are assigned to the fundamental triplet representations.

The Hamiltonian in the continuum limit is given by

$$H_c = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \bar{\psi}^{\alpha} D\psi_{\alpha} + (D_{\mu}\phi^{\alpha})^{\dagger}(D^{\mu}\phi_{\alpha}) + \mu^2 \phi^{\dagger\alpha}\phi_{\alpha} + \frac{\lambda}{4!} (\phi^{\dagger\alpha}\phi_{\alpha})^2 \quad (3.1)$$

with $\lambda > 0$. Notice that the Yukawa term $\phi\bar{\psi}\psi$ is not allowed due to the assignment of the representations. The quartic coupling between scalars and fermions is purposely avoided by assuming that the scalars and fermions interact only through the gauge fields.

Although the Hamiltonian H_c thus constructed preserves chiral symmetry, one may suspect that when $\phi^{\alpha\dagger}\phi_{\alpha}$ obtains a nonzero vacuum expectation value the fermions may still acquire masses through the box diagram and its higher order iterations (see Fig. 1), as a result the chiral symmetry of the system will be broken explicitly. We argue that this worry is unnecessary, because the contributions of these diagrams have an explicit dependence on the fermion mass m (see Appendix A). Since we assume $m = 0$ in the first place, these contributions should vanish.

Let us now consider a lattice Hamiltonian, which preserves chiral symmetry and it will reduce to Eq. (3.1) as the lattice size $a \to 0$. Recall that in the standard construction of the Hamiltonian in a lattice gauge theory, one assigns the matter fields on lattice sites and the gauge fields on the links connecting the sites. Within the SLAC long range gradient formalism,[4,6] the present Hamiltonian in the $A_0 = 0$ gauge can be written as:

$$
\begin{aligned}
H = \frac{1}{a}\Bigg\{ & \sum_{\text{links}} \frac{1}{2} g^2 (E^\alpha)^2 - \frac{1}{g^2} \sum_{\text{plaquettes}} \left[\mathrm{Tr}\,(UUU^\dagger U^\dagger) + h.c. \right] \\
& + \sum_{\vec{j},\ell,\hat{\mu}} \left[\frac{(-1)^\ell}{\ell}\, \psi^{\dagger\alpha}(\vec{j})\, \alpha_\mu \left(\prod_{\vec{j}\,'=\vec{j}}^{\vec{j}+(\ell-1)\hat{\mu}} U_{\alpha\beta}(\vec{j}\,', \hat{\mu}) \right) \psi^\beta(\vec{j} + \ell\,\hat{\mu}) + h.c. \right] \\
& + \sum_{\vec{j}} \Bigg[\sum_{\ell,\hat{\mu}} \frac{2(-1)^\ell}{\ell}\, \phi^{\dagger\alpha}(\vec{j}) \left(\prod_{\vec{j}\,'=\vec{j}}^{\vec{j}+(\ell-1)\hat{\mu}} U_{\alpha\beta}(\vec{j}\,', \hat{\mu}) \right) \phi^\beta(\vec{j} + \ell\,\hat{\mu}) \\
& \qquad + D(0)\, \phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j}) + h.c. \Bigg] \\
& + \sum_{\vec{j}} \left[\pi^{\dagger\alpha}(\vec{j})\, \pi_\alpha(\vec{j}) + \mu_0^2\, \phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j}) + \frac{\lambda_0}{4!} \left[\phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j}) \right]^2 + h.c. \right] \Bigg\} .
\end{aligned} \tag{3.2}
$$

Notice that the term $D(0)\,\phi^{\dagger\alpha}(\vec{j})\,\phi_\alpha(\vec{j})$ comes from the spatial derivative of the scalars at the same site. Numerically $D(0) = \pi^2$ in three spatial dimensions. The scalar momentum $\pi_\alpha(\vec{j})$ is defined as

$$
i\pi_\alpha(\vec{j}) = \sqrt{\kappa(\vec{j})/2}\, \left[-a^\dagger_{+\alpha}(\vec{j}),\ a_{-\alpha}(\vec{j}) \right] \tag{3.3}
$$

so that

$$
\left[\pi_\alpha(\vec{j}),\ \phi_\beta(\vec{j}\,') \right] = \delta_{\alpha\beta}\, \delta(\vec{j} - \vec{j}\,') \quad . \tag{3.4}
$$

All the fields and parameters in Eq. (3.2) have been properly redefined in such a way that the lattice spacing a is the only dimensionful quantity.

In the strong coupling limit, we neglect the plaquette terms and separate the rest of the Hamiltonian into two parts:

$$
H_0 = \frac{1}{a} \sum \frac{1}{2}\, g^2\, E^\alpha\, E_\alpha \quad \text{and} \quad V = H_\ell - H_0 \quad . \tag{3.5}
$$

Since $E^{\alpha}(\vec{j},\hat{\mu})$ measures hypercolor flux-excitations, the ground state of H_0 corresponds to that sector of the Hilbert space, which does not involve flux-excitations. So we are looking for perturbative effects due to V. Rewrite V as

$$V = V_m + V_s + V_f \quad , \tag{3.6}$$

where

$$\begin{aligned} V_m =& \frac{1}{a} \sum_{\vec{j}} \Big[\pi^{\dagger}_{\alpha}(\vec{j})\, \pi^{\alpha}(\vec{j}) + D(0)\, \phi^{\dagger}_{\alpha}(\vec{j})\, \phi^{\alpha}(\vec{j}) + \mu_0^2\, \phi^{\dagger}_{\alpha}(\vec{j})\, \phi^{\alpha}(\vec{j}) \\ & + \frac{\lambda_0}{4!} \left[\phi^{\dagger}_{\alpha}(\vec{j})\, \phi^{\alpha}(\vec{j}) \right]^2 + h.c. \Big] \ , \\ V_s =& \frac{1}{a} \sum_{\vec{j},\ell,\hat{\mu}} \left[\frac{2(-1)^{\ell}}{\ell}\, \phi^{\dagger\alpha}(\vec{j}) \left(\prod_{\vec{j}'=\vec{j}}^{\vec{j}+(\ell-1)\hat{\mu}} U_{\alpha\beta}(\vec{j}',\hat{\mu}) \right) \phi^{\beta}(\vec{j}+\ell\,\hat{\mu}) + h.c. \right] \ , \\ V_f =& \frac{1}{a} \sum_{\vec{j},\ell,\hat{\mu}} \left[\frac{(-1)^{\ell}}{\ell}\, \psi^{\dagger\alpha}(\vec{j})\, \alpha_{\mu} \left(\prod_{\vec{j}'=\vec{j}}^{\vec{j}+(\ell-1)\hat{\mu}} U_{\alpha\beta}(\vec{j}',\hat{\mu}) \right) \psi^{\beta}(\vec{j}+\ell\,\hat{\mu}) + h.c. \right] \ . \end{aligned}$$

Since we are working in the strong coupling limit, we retain the perturbative Hamiltonian only up to the second order. In other words,

$$H_{eff} \approx H^{(1)} + H^{(2)} \quad , \tag{3.7}$$

with

$$H^{(1)} = V_m$$

and

$$H^{(2)} = \sum \frac{1}{E_0 - E_\ell} \left\{ V_f^2 + V_s^2 + 2V_f V_s \right\}$$

$$= \sum_{\vec{j},\ell,\hat{\mu}} -\frac{2N}{ag^2 \ell C_F} \left\{ \frac{(-1)^{2\ell}}{\ell^2} \psi^{\dagger\alpha}(\vec{j})\, \alpha_\mu \psi^\beta(\vec{j} + \ell\,\hat{\mu})\, \psi^\dagger_\beta(\vec{j} + \ell\,\hat{\mu})\, \alpha_\mu\, \psi_\alpha(\vec{j}) + h.c. \right.$$

$$+ \frac{4(-1)^{2\ell}}{\ell^4} \phi^{\dagger\alpha}(\vec{j})\, \phi^\beta(\vec{j} + \ell\,\hat{\mu})\, \phi^\dagger_\beta(\vec{j} + \ell\,\hat{\mu})\, \phi_\alpha(\vec{j}) + h.c.$$

$$\left. + \frac{4(-1)^{2\ell}}{\ell^3} \psi^{\dagger\alpha}(\vec{j})\, \alpha_\mu \psi^\beta(\vec{j} + \ell\,\hat{\mu})\, \phi_\beta(\vec{j} + \ell\,\hat{\mu})\, \phi^\dagger_\alpha(\vec{j}) + h.c. \right\}$$

$$\equiv H_f + H_s + H_{fs} \quad .$$

After a Fierz transformation, H_{eff} can be expressed as

$$H_{eff} = \frac{1}{a} \sum_{\vec{j}} \left[\pi^{\dagger\alpha}(\vec{j})\, \pi_\alpha(\vec{j}) + [D(0) + \mu_0^2]\, \phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j}) \right.$$

$$\left. + \frac{\lambda_0}{4!} [\phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j})]^2 + h.c. \right]$$

$$+ \frac{2N}{ag^2 C_F} \sum_{\vec{j},\ell,\hat{\mu}} \left\{ \frac{1}{4\ell^3} \left[Q_f(\vec{j})\, Q_f(\vec{j} + \ell\,\hat{\mu}) \right. \right. \tag{3.8}$$

$$\left. + \chi^2 \sum_{k=1}^{15} (\eta^{\mu k})^{\ell+1}\, Q_k(\vec{j})\, Q_k(\vec{j} + \ell\,\hat{\mu}) + h.c. \right]$$

$$- \frac{4}{\ell^5} \left[[\phi^{\dagger\alpha}(\vec{j})\, \phi_\alpha(\vec{j})]\, [\phi^{\dagger\alpha}(\vec{j} + \ell\,\hat{\mu})\, \phi_\alpha(\vec{j} + \ell\,\hat{\mu})] + h.c. \right]$$

$$\left. - \frac{4}{\ell^4} \left[[F^\dagger(\vec{j})\, \alpha_\mu^{\ell+1}\, F(\vec{j} + \ell\,\hat{\mu}) + h.c. \right] \right\} \quad .$$

where

$$F^\dagger(\vec{j}) \equiv \phi^\dagger_\alpha(\vec{j})\, \psi^{\dagger\alpha}(\vec{j}) \equiv \vec{\phi}^\dagger(\vec{j}) \cdot \vec{\psi}^\dagger(\vec{j}) \quad , \tag{3.9}$$

is the creation operator for the composite fermions.

Let us compare our H_{eff} in Eq. (3.8) with the H_{eff} in Eq. (2.8). The U(1) $\times$ SU(4) charges in H_f is identical to those in Eq. (2.8). But, in addition to the H_s terms, there are two main differences between the two effective Hamiltonians: First, there are zeroth order (in $1/g^2$) contributions $H^{(1)}$ to H_{eff} due to the scalar fields in our case. This plays a dominant role in the construction of the ground state bosonic wave function. Second, H_{fs} corresponds to the kinetic energy term which moves the composite fermion from site $\vec{j}+\ell\hat{\mu}$ to site $\vec{j}$. While the kinetic term in QDG's case is of order $1/g^4$, our kinetic term is of order $1/g^2$ and can in principle compete with the potential terms in H_f. Next we proceed to construct the ground state of the system.

A. A General Trial State for the Vacuum

We anticipate that the trial state for the ground state or the vacuum should take the general form

$$
\begin{aligned}
|\Omega\rangle &= \sum_{\ell,\{\alpha,\beta\}} C^{\ell}_{\alpha\beta}\Big[\sum_m \sqrt{2\ell+1}\,\langle 0,0|\ell,m;\ell,-m\rangle \Phi^{(\alpha)\dagger}_{\ell m}\;\Psi^{(\beta)\dagger}_{\ell -m}\Big]\Big|0\Big\rangle \\
&= \sum_{\ell,\{\alpha,\beta\}} C^{\ell}_{\alpha\beta}\Big[\Phi^{(\alpha)\dagger}_{\ell}\cdot\Psi^{(\beta)\dagger}_{\ell}\Big]\Big|0\Big\rangle \ ,
\end{aligned}
\tag{3.10}
$$

where the SO(3) scalar product is defined and where Φ's are functions of boson field operators with ℓ, m being the SO(3) hypercolor labels and (α) some additional labels, and Ψ's are the corresponding hypercolor conjugated functions of fermion field operators. Summing over m above amounts to taking the scalar product in the hypercolor space to ensure hypercolor singlet states. The C's are numerical weight factors for different assignments of $\ell,\{\alpha,\beta\}$.

(1) The Boson Wave Functions

The first order Hamiltonian for the scalar part has the form [cf. Eq. (3.6)]:

$$
\begin{aligned}
H^{(1)} = \frac{1}{a}\sum_{\vec{j}}\Big[&\pi^{\dagger}_{\alpha}(\vec{j})\,\pi^{\alpha}(\vec{j}) + D(0)\,\phi^{\dagger}_{\alpha}(\vec{j})\,\phi^{\alpha}(\vec{j}) + \mu_0^2\,\phi^{\dagger}_{\alpha}(\vec{j})\,\phi^{\alpha}(\vec{j}) \\
&+\frac{\lambda_0}{4!}\Big[\phi^{\dagger}_{\alpha}(\vec{j})\,\phi^{\alpha}(\vec{j})\Big]^2 + h.c.\Big] \ .
\end{aligned}
$$

Our present problem may be mapped to the nonrelativistic quantum mechanics problem with

$$H^{(1)} \rightarrow \frac{p^2}{2m} + \frac{1}{2}\ m\omega^2 r^2 + \frac{\lambda_0\, r^4}{4!} \quad . \tag{3.11}$$

In particular all we need is to make the identification:

$$2m = a\ ,\ \frac{\omega^2}{4} = \frac{D_0 + \mu_0^2}{a^2}\ ; r^2 = \phi_\alpha^\dagger \phi^\alpha = \vec{\phi}\cdot\vec{\phi}\ ,\ \text{and}\ p^2 = \pi_\alpha^\dagger \pi^\alpha = \vec{\pi}\cdot\vec{\pi}\ . \tag{3.12}$$

Now we consider the case where λ_0 is sufficently small, so that we may make use of the perturbation approach and describe the complete set of the orthonormal wave functions for the tensor products of $\vec{\phi}$, in terms of eigenfunctions of the Hamiltonian:

$$H^{(1)} \simeq \frac{p^2}{2m} + \frac{1}{2}\ m\omega^2 r^2 \quad . \tag{3.13}$$

Later we shall make precise on the range of λ_0 to be assumed.

Next we proceed to discuss the solutions for $H^{(1)}$ of Eq. (3.13). The radial part of the Schroedinger equation for $H^{(1)}$ is given by:

$$-\frac{1}{2\mu}\cdot\frac{d^2u}{dr^2} + \left[\frac{\ell(\ell+1)}{2\mu r^2} + \frac{1}{2}\ \mu\omega^2 r^2\right] u = Eu \quad . \tag{3.14}$$

where the full wave function is

$$\Phi \sim \frac{u(r)}{r}\ Y_{\ell m}(\theta,\phi) \quad , \tag{3.15}$$

which satisfies $H^{(1)}\Phi = E\Phi$. Following the standard approach (see for example Ref. 7), let

$$u = r^{\ell+1}\ \exp\left(-\frac{\mu\omega^2}{2}\ r^2\right) f(r) \tag{3.16}$$

and denote $z = m\omega r^2$. Substituting Eq. (3.16) into Eq. (3.14), we get

$$z\,\frac{d^2 f}{dz^2} + \frac{df}{dz}\left(\ell + \frac{3}{2} - z\right) + \left[\frac{E}{2w} - \left(\frac{\ell}{2} + \frac{3}{4}\right)\right] f = 0 \quad . \tag{3.17}$$

Recall that the associated Lageurre polynomial satisfies the differential equation

$$z\,\frac{d^2w}{dz^2}+\frac{dw}{dz}\,(p+1-z)+qw=0 \quad . \tag{3.18}$$

Comparing Eqs. (3.17) and (3.18), one obtains

$$f(z)=L_q^{\ell+\frac{1}{2}}(z) \quad \text{and} \quad E=\left(2q+\ell+\frac{3}{2}\right)w \quad . \tag{3.19}$$

Denote $z=\sqrt{D_0+\mu_0^2}\,\phi^2$. For our present complete orthonormal set of bosonic functions, we write

$$\Phi_{\ell m}^{(\alpha)}\equiv\Phi_{\ell m}^{(q)}\sim z^{\ell/2}\exp\left(-\frac{z}{2}\right)L_q^{\ell+\frac{1}{2}}(z)\,Y_{\ell m}(\theta,\phi) \quad , \tag{3.20}$$

where the overall constant can be specified through normalization of the states.

(2) The Fermion Wave Functions

To describe fermion states we must take into account the two degrees of freedom for spin 1/2 and make the particle-antiparticle distinctions. Denote the field operators for the creation of fermions by $b^\dagger_{+,m}$, $b^\dagger_{-,m}$ and for the antifermions $d^\dagger_{+,-m}$ and $d^\dagger_{-,-m}$. Here the subscript m takes the value $m=-1,0,1$, since all these particles are in $\ell=1$ triplet states. Note also a particle in the m state is color-conjugate to an antiparticle in the $-m$ state.

The resultant color content of the products of fermionic field operators can now be defined through the usual procedure of the Clebsch-Gordan additions. More explicitly one would write:

$$\Psi_{\ell m}^{\beta}\equiv\Psi_{\ell m}^{(n_+,n_-,\bar{n}_+,\bar{n}_-)}=\sum_{\substack{\ell_1 m_1\\ \ell_2 m_2}}\psi_{\ell_1 m_1}^{(n_+,n_-)}\,\psi_{\ell_2 m_2}^{(\bar{n}_+,\bar{n}_-)}\,\langle\ell m|\ell_1 m_1;\ell_2 m_2\rangle \quad , \tag{3.21}$$

with

$$\psi_{\ell_1 m_1}^{(n_+,n_-)}=\sum_{\ell' m',\ell'' m''}\psi_{\ell' m'}^{n_+}\,\psi_{\ell'' m''}^{n_-}\,\langle\ell_1 m_1|\ell' m',\ell'' m''\rangle \quad ,$$

and a similar definition for the antiparticles. The general trial state of Eq. (3.10) is now defined through Eqs. (3.20) and (3.21).

B. The Inclusion of the $\lambda_0 r^4$ Term

Now we return to the full expression of $H^{(1)}$ [cf. Eq. (3.11)] by adding in the $\lambda_0 r^4$ term as a perturbation to the previous treatment.

By invoking WKB approximation for the harmonic oscillators (see Appendix B), we have, for $\ell = 0$, the phase condition when without the $\lambda_0 r^4$ term

$$\sqrt{2m} \int dr \sqrt{E_0 - \frac{1}{2} m\omega^2 r^2} = \frac{3\pi}{4} , \quad \text{gives} \quad E_0 = \frac{3\omega}{2} \quad ; \qquad (B.4)$$

and for $\ell > 0$, the phase condition

$$\sqrt{2m} \int dr \sqrt{E_0 - \frac{1}{2} m\omega^2 r^2 - \frac{\ell(\ell+1)}{2mr^2}} = \frac{\pi}{2} , \quad \text{gives} \quad E_\ell \approx E_0 + \ell w \quad . \ (B.10)$$

Now we proceed to consider the effect of including the $\lambda_0 r^4$ term. Denote the eigenvalues we are looking for by:

$$E_0' = E_0 + \Delta_0 \qquad \text{and} \qquad E_\ell' = E_\ell + \Delta_\ell \quad . \qquad (3.22)$$

And they satisfy the corresponding conditions:

$$\sqrt{2m} \int dr \sqrt{E_0' - \frac{1}{2} m\omega^2 r^2 - \lambda_0 r^4} = \frac{3\pi}{4} \quad , \qquad (3.23)$$

and

$$\sqrt{2m} \int dr \sqrt{E_0' - \frac{1}{2} m\omega^2 r^2 - \frac{\ell(\ell+1)}{2mr^2} - \lambda_0 r^4} = \frac{\pi}{2} \quad . \qquad (3.24)$$

Application of the mean value theorem on Eqs. (3.23) and (3.24) reveals following crucial properties for the energy shifts. For $\lambda_0 > 0$, Δ_i's are all positive quantities and they are monotonic functions of λ_0. Since $E_\ell' - E_0' = \ell\omega + \Delta_\ell - \Delta_0$, so long as Δ_0 is maintained with some finite value below w, E_ℓ' for $\ell > 0$ would always be at least a finite gap away from E_0'. In other words, we have now arrived at a sufficient condition which ensures a finite gap between

E_0^l and E_ℓ^l for $\ell > 0$. To reiterate, this is achieved when λ_0 is kept in the range such that

$$\Delta_0 < (1-\eta)w \tag{3.25}$$

with η being some arbitrary finite fraction.

Now we proceed to make a quantitative estimate on the range of λ_0. First we calculate the approximate ground state energy based on the variational technique. We choose the form of the ground state wave function for the harmonic oscillator to be that for the trial wave function. We get

$$\langle 0|H^{(1)}|0\rangle = \frac{\int r^2\,dr\ \exp\left(-\frac{\beta}{4}\,r^2\right) H^{(1)} \exp\left(-\frac{\beta}{4}\,r^2\right)}{\int r^2\,dr\ \exp\left(-\frac{\beta}{2}\,r^2\right)} \quad , \tag{3.26}$$

with

$$H^{(1)} = -\frac{1}{r^2}\frac{d}{dr}\,r^2\,\frac{d}{dr} + \frac{\omega^2}{4}\,r^2 + \frac{L^2}{r^2} \quad , \tag{3.27}$$

and

$$H^{(1)} \exp\left(-\frac{\beta}{4}\,r^2\right) = \left(\frac{3\beta}{2} + \frac{\omega^2-\beta^2}{4} + r^2 + \frac{\lambda_0}{4!}\,r^4\right) \exp\left(-\frac{\beta}{2}\,r^2\right) \quad . \tag{3.28}$$

Use the mathematical identity:

$$\int_0^\infty r^n\,dr\ \exp\left(-\frac{\beta r^2}{2}\right) = \frac{1}{2}\left(\frac{2}{\beta}\right)^{(n+1)/2} \Gamma\left(\frac{n+1}{2}\right) \quad . \tag{3.29}$$

Equation (3.26) becomes:

$$\langle H^{(1)}\rangle = \frac{3}{4}\left(\beta + \frac{\omega^2}{\beta} + \frac{5\lambda}{6\beta^2}\right) = f(\beta) \quad . \tag{3.30}$$

Now let us allow β to be a parameter. The minimum value of $\langle H^{(1)}\rangle$ is obtained by minimizing $f(\beta)$. As a rough estimate on λ_0, we solve for $\langle H^{(1)}\rangle$ iteratively. To the zeroth order in λ_0, the stationary point occurs at $\beta \sim \omega$, so

$$\langle H^{(1)}\rangle \sim \frac{3\omega}{2} + \frac{5\lambda_0}{2w^2} \quad . \tag{3.31}$$

The condition of Eq. (3.25) now gives,

$$\lambda_0 < \frac{2}{5}(1-\eta)\omega^3 \quad . \tag{3.32}$$

It is important to point out that while from Eq. (3.26) onward, we have used the perturbative method to estimate the quantitative range of λ_0, our argument in arriving at the finite-gap conclusion is independent of the perturbation theory.

4. Chiral Symmetry Breaking and Composite Fermion Mass

Recall that the full effective Hamiltonian H_{eff} includes $H^{(1)}$ and $H^{(2)}$, where $H^{(2)}$ is of order $1/g^2$. The fact that the eigenvalues of $H^{(1)}$ are separated by finite gaps suggests that the $\ell = 0$ component in the ground state $|\Omega\rangle$ [defined in Eq. (3.10)] dominates over other components.

A. <u>The Mixing Angle in the Ground State</u>

We shall simplify our ground state structure by truncating away all the $\Phi^\dagger_{\ell m}, \Psi^\dagger_{\ell,-m}$ components for $\ell \geq 2$. Meanwhile we introduce a mixing angle θ to parametrize the relative weight between the $\Phi^\dagger_0 \cdot \Psi^\dagger_0$ and $\Phi^\dagger_1 \cdot \Psi^\dagger_1$ components:

$$\begin{aligned} |\Omega\rangle &= [\cos\theta\, \Phi^\dagger_0 \cdot \Psi^\dagger_0 + \sin\theta\, \Phi^\dagger_1 \cdot \Psi^\dagger_1]|0\rangle \\ &\equiv |\Omega_0\rangle + |\Omega_1\rangle \quad . \end{aligned} \tag{4.1}$$

Note that the normalization for these states are: $\langle\Omega|\Omega\rangle = \langle\Omega_0|\Omega_0\rangle = \langle\Omega_1|\Omega_1\rangle = 1$. Let us define

$$\begin{aligned} &\langle\Omega_0|H_f|\Omega_0\rangle = \cos^2\theta\, E_{f0} \ , \quad \langle\Omega_1|H_f|\Omega_1\rangle = \sin^2\theta\, E_{f1} \ ; \\ &\langle\Omega_0|H_s|\Omega_0\rangle = \cos^2\theta\, E_{s0} \ , \quad \langle\Omega_1|H_s|\Omega_1\rangle = \sin^2\theta\, E_{s1} \ ; \end{aligned} \tag{4.2}$$

and

$$\langle\Omega_0|H_{fs}|\Omega_1\rangle = \langle\Omega_1|H_{fs}|\Omega_0\rangle = \cos\theta\, \sin\theta\, E_{fs} \quad .$$

Notice that all E_{fi}'s, E_{si}'s and E_{fs} are at least of order $1/g^2$ inherited from their corresponding Hamiltonians. Recall that

$$\langle\Omega_0|H^{(1)}|\Omega_0\rangle = \cos^2\theta\, E_0 \ , \quad \langle\Omega_1|H^{(1)}|\Omega_1\rangle = \sin^2\theta\, E_1 \ ; \tag{4.3}$$

where E_0 and E_1 are given according to Eq. (3.22), and the energy gap between E_0 and E_1 is finite (of order zero in $1/g^2$).

Putting everything together we have

$$\begin{aligned}\mathcal{E} &\equiv \langle\Omega|H_{eff}|\Omega\rangle \\ &= \langle\Omega|(H^{(1)} + H^{(2)})|\Omega\rangle \\ &= \cos^2\theta(E_0 + E_{f0} + E_{s0}) + \sin^2\theta(E_1 + E_{f1} + E_{s1}) + 2\cos\theta\sin\theta\, E_{fs} \quad .\end{aligned} \tag{4.4}$$

Minimize this vacuum expectation value by varying θ,

$$\frac{\partial\mathcal{E}}{\partial\theta} = 0 \quad ,$$

we get

$$\tan 2\theta = -\frac{E_{fs}}{(E_1 - E_0) + (E_{f1} - E_{f0}) + (E_{s1} - E_{s0})} \sim O\left(\frac{1}{g^2}\right) \quad , \tag{4.5}$$

or

$$\theta \sim O\left(\frac{1}{g^2}\right) \quad . \tag{4.6}$$

If we were to include the $\ell \geq 2$ contributions in Eq. (4.1) it can be shown that our conclusion of Eq. (4.6) still prevails.

B. Chiral Symmetry Breaking

Now that $\theta \sim O(1/g^2)$, we know that $\cos^2\theta \sim O(1 - 1/g^4)$ and $\sin^2\theta \sim O(1/g^4)$. Thus

$$\langle\Omega_0|H_f|\Omega_0\rangle = \cos^2\theta\, E_{f0} \sim O\left(\frac{1}{g^2}\right)$$

and

$$\langle\Omega_1|H_f|\Omega_1\rangle = \sin^2\theta\, E_{f1} \sim O\left(\frac{1}{g^6}\right) \quad . \tag{4.7}$$

Since the part of the Hamiltonian H_{eff} which deals with chiral symmetry breaking is H_f [see Eq. (3.8)], we see from Eq. (4.7) that the part of the vacuum which dominates this breaking situation comes from $|\Omega_0\rangle \equiv \cos\theta\, \Phi_0^\dagger \cdot \Psi_0^\dagger|0\rangle$ where $\Psi_0^\dagger$ corresponds to creating all possible fermionic hypercolor-singlets in SO(3). On the other hand, $|\Omega_1\rangle$, which consists of fermionic hypercolor-nonsinglets, will contribute only up to order $1/g^6$.

In the context of our present discussion, we remind the reader an important point: In QDG's argument for chiral symmetry breaking, they consider the case where there are only colorless fermionic states present, that is the state $|\Omega_0\rangle = |\Psi_0\rangle$. (Note for $\ell = 0$ chiral symmetry breaking is independent of the scalar contribution.) Their chiral symmetry breaking conclusion is arrived at by showing that a certain nonchiral symmetric ground state gives the lowest energy expectation value E_{f0}. At the same time chiral symmetric trial states give expectation values at least of the order $1/g^2$ higher. For our case, from Eq. (4.7) one sees that the $|\Omega_1\rangle$ state contribution is suppressed by an additional factor of $1/g^4$ and can be ignored. So our conclusion of the spontaneous breaking of chiral symmetry now follows in the same way as that of QDG.

C. The Composite Fermion Mass

In the case of QDG, the "potential energy" for creating a composite fermion from the mean-field state is positive and of order $1/g^2$:

$$|\langle Q_R^k\rangle| \;>\; |\langle F Q_R^k F^\dagger\rangle| \;\sim\; \mathcal{O}\left(\frac{1}{g^2}\right) \quad . \tag{4.8}$$

On the other hand, the "kinetic energy" terms that move composite fermions (made of three or more fermionic preons) from one site to another are able to reduce the mass gap but only to leading order $1/g^4$. Hence one cannot alter the mass gap result by making a zero momentum superposition of local composite fermion states. We now turn to our case.

The evaluation of the composite fermion mass for our case is complicated by two additional effects. First the presence of the composite fermion kinetic term H_{fs}, which as we shall see gives rise to a $\mathcal{O}(1/g^2)$ effect. Second, even more importantly, the presence of the scalar term which gives rise to $\mathcal{O}(1)$ effect.

(i) General considerations involving composite fermion operators.

Define a composite fermion state as a composite creation operator acting on the mean-field ground state:

$$F^\dagger(\vec{j})|\Omega\rangle \equiv |F(\vec{j})\rangle \cdot \prod_{\substack{\text{even}\\ i\neq j}} |\Omega_e(\vec{i})\rangle \cdot \prod_{\substack{\text{odd}\\ k\neq j}} |\Omega_o(\vec{k})\rangle \tag{4.9}$$

where $|\Omega_e\rangle$ and $|\Omega_o\rangle$ are the even and odd site mean-field ground states, respectively, and $F^\dagger \equiv \vec{\phi}^\dagger \cdot \vec{\psi}^\dagger$.

In momentum space, introduce

$$|\mathcal{F}(\vec{k})\rangle \equiv \sum_{\vec{j}} \exp\,(i\,\vec{k}\cdot\vec{j})\; F^\dagger(\vec{j})|\Omega\rangle\;, \quad 0 \leq |\,\vec{k}\,| \leq \frac{\pi}{a} \quad . \tag{4.10}$$

The expectation value of the kinetic term defined by the momentum state is given by

$$\begin{aligned} &\langle \mathcal{F}(\vec{k})|F^\dagger(\vec{j})\,F(\vec{j}+\ell\,\hat{\mu})|\mathcal{F}(\vec{k})\rangle + h.c.\\ &= \sum_{\vec{m},\vec{m}'} \left\{\exp\left[i\,\vec{k}\cdot(\vec{m}-\vec{m}')\right]\delta(\vec{m}-\vec{j})\,\delta(\vec{j}+\ell\,\hat{\mu}-\vec{m}') + h.c.\right\} \qquad (4.11)\\ &= 2\cos\,(\vec{k}\cdot\ell\,\hat{\mu}) \quad . \end{aligned}$$

Clearly, the maximum value corresponds to $\vec{k}=0$, i.e.

$$|\mathcal{F}(\vec{k}=0)\rangle = \sum_{\vec{j}} F^\dagger(\vec{j})|\Omega\rangle \quad . \tag{4.12}$$

This turns out will lower the expectation value of H_{eff} the most. So the $|\mathcal{F}(\vec{k}=0)\rangle$ state is appropriate for the evaluation of the mass gap.

(ii) Evaluation of the mass gap.

The Hamiltonian of the kinetic terms is H_{fs} and can be expressed in terms of the composite fermion operators

$$H_{fs} = -\frac{2N}{ag^2C_F}\sum_{\hat{\mu}}\left[4F_e^\dagger\,F_o + \frac{1}{4}\,F_e^\dagger\,\alpha_\mu\,F_e\right] \quad , \tag{4.13}$$

where the first term on the right-hand side, $F_e^\dagger F_o$, corresponds to movings between nearest-neighbors, and the second term corresponds to hoppings between next-to-the-nearest-neighbors. These are all along the $\hat{\mu}$ direction.

In appendix D we find that

$$
\begin{aligned}
\left(\phi_1^\dagger \cdot \psi_1^\dagger\right)\left(\Phi_0^\dagger \cdot \Psi_0^\dagger\right) &= \Phi_1^{\prime\dagger} \cdot \Psi_1^{\prime\dagger} \\
\left(\phi_1^\dagger \cdot \psi_1^\dagger\right)\left(\Phi_0^\dagger \cdot \Psi_0^\dagger\right) &= \Phi_0^{\prime\prime\dagger} \cdot \Psi_0^{\prime\prime\dagger} + \Phi_1^{\prime\prime\dagger} \cdot \Psi_1^{\prime\prime\dagger} + \Phi_2^{\prime\prime\dagger} \cdot \Psi_2^{\prime\prime\dagger}
\end{aligned}
\tag{4.14}
$$

Again, keeping only $\ell < 2$ terms then the state $|\mathcal{F}(\vec{k} = 0)\rangle$ is approximately equal to

$$
\begin{aligned}
|\mathcal{F}(\vec{k} = 0)\rangle &\simeq (\phi_1^\dagger \cdot \psi_1^\dagger)\left(\cos\ \theta\Phi_0^\dagger \cdot \Psi_0^\dagger + \sin\theta\Phi_1^\dagger \cdot \Psi_1^\dagger\right)|0\rangle \\
&\simeq \sin\ \theta|\Omega_0''\rangle + \cos\theta|\Omega_1'\rangle \quad ,
\end{aligned}
\tag{4.15}
$$

where $|\Omega_0''\rangle \equiv \Phi_0^{\prime\prime\dagger} \cdot \Psi_0^{\prime\prime\dagger}|0\rangle$ and $|\Omega_1'\rangle \equiv \Phi_1^{\prime\dagger} \cdot \Psi_1^{\prime\dagger}|0\rangle$. The superscript states are not properly normalized (see appendix D). Denote some normalization factor $N_F = \langle\mathcal{F}(\vec{k} = 0)|\mathcal{F}(\vec{k} = 0)\rangle^{-1}$,

$$
N_F\langle\mathcal{F}(\vec{k} = 0)|H_f|\mathcal{F}(\vec{k} = 0)\rangle \simeq \sin^2\ \theta E_{f0}'' + \cos^2\ \theta E_{f1}' \quad ,
$$

with E_{f0}'' defined to be equal to $N_F\langle\Omega_0''|H_f|\Omega_0''\rangle$, and is of order $\mathcal{O}(1/g^2)$, E_{f1}' equal to $N_F\langle\Omega_1'|H_f|\Omega_1'\rangle$, also of order $\mathcal{O}(1/g^2)$. So

$$
N_F\langle\mathcal{F}(\vec{k} = 0)|H_f|\mathcal{F}(\vec{k} = 0)\rangle \sim \mathcal{O}\left(\frac{1}{g^2}\right) \quad . \tag{4.16}
$$

whereas for H_{fs},

$$
N_F\langle\mathcal{F}(\vec{k} = 0)|H_{fs}|\mathcal{F}(\vec{k} = 0)\rangle \sim \sin\theta\ \cos\theta\, E_{fs} \sim \mathcal{O}\left(\frac{1}{g^4}\right) \quad . \tag{4.17}
$$

To arrive at Eq. (4.17), note that for the expectation value of the second term on the right hand side of Eq. (4.13) upon averaging over the spin orientations of the composite fermions, this term vanishes. So in our case the kinetic energy is also of order $1/g^4$ and cannot help to close the mass gap for the composite fermion state.

Now we come to the second effect mentioned earlier. The fact that there exists a $H^{(1)}$ part in the total Hamiltonian and that $H^{(1)}$ is of zeroth order in $1/g^2$ expansion, tells us that the mass gap for the composite fermion state is of zeroth order in $1/g^2$. To be more specific,

$$N_F\langle \mathcal{F}(\vec{k}=0)|H_{eff}|\mathcal{F}(\vec{k}=0)\rangle \sim \sin^2\theta\, E_0'' + \cos^2\theta\, E_1'$$

and using the notation of Eq. (3.22), we have

$$\langle \Omega|H_{eff}|\Omega\rangle \sim \cos^2\theta\,(E_0+\Delta_0) + \sin^2\theta\,(E_1+\Delta_1) \quad .$$

Thus to leading order in $1/g^2$

$$\Delta\mathcal{E} = N_F\langle \mathcal{F}|H_{eff}|\mathcal{F}\rangle - \langle \Omega|H_{eff}|\Omega\rangle \sim \cos^2\theta(E_1'' - E_0 - \Delta_0) \sim O(1) \quad . \tag{4.18}$$

In the last step we have used $E_1'' \simeq E_1 + \Delta_1$. Note in general to zeroth order in $1/g^2$, for fixed ℓ the corresponding energy eigenvalue for $|\Omega_\ell\rangle$ is the same as that for $|\Omega_\ell''\rangle$. We thus conclude that, in the case of SO(3) composite model involving both fermionic and scalar preons, the chiral symmetry is also spontaneously broken and the composite fermions (leptons and quarks) are massive in the strong-coupling lattice calculation.

D. Generalization To Other Cases

(i) Extension to the case with Higg's phase.

For $D_0 + \mu_0^2 < 0$, the minimum of the expectation value is

$$H_\phi = \pi^2 + (D_0+\mu_0^2)\phi^2 + \frac{\lambda_0}{4!}\,\phi^2 \quad , \tag{4.19}$$

occurs at

$$\phi^2 = v^2 = \frac{|a|}{2\lambda_0}\ , \quad \text{with} \quad |a| = -(D_0+\mu_0^2) \quad . \tag{4.20}$$

With the usual redefinition of the dynamical fields: $\tilde{\phi} = \phi - v$, we have

$$H_{\phi} = \tilde{\pi}^2 + 2|a|\ \tilde{\phi}^2 + 4\lambda_0 v \tilde{\phi}^3 + \lambda_0 \tilde{\phi}^4 \quad . \tag{4.21}$$

Now we can apply the procedure as before to evaluate the energy of the trial ground state as a function of the parameter β. After a similar manipulation, we arrive at

$$E_0 = \frac{3}{2}\, w \left[1 + \lambda_0 \left(\frac{4}{3}\sqrt{\frac{2}{3}}\ \frac{1}{w^{5/2}} + \frac{5}{12 w^3}\right)\right] \quad . \tag{4.22}$$

The absence of $\ell > 0$ components in the ground state can be ensured when

$$\lambda_0 \ll \frac{2}{3}\left(\frac{4}{3}\sqrt{\frac{2}{3}}\ \frac{1}{w^{5/2}} + \frac{5}{12 w^3}\right)^{-1} \quad . \tag{4.23}$$

(ii) Extension from SO(3) to SO(N).

So far for definiteness, we have considered the SO(3) case, which is locally isomorphic to SU(2). Our considerations can, in a straight forward manner be extended to general case of SO(N). A particular case which might be of physical interest is to assign the fermions and the scalars and their corresponding antiparticles to the irreducible representations of SO(6), which contains SU(3) as a subgroup. Here we will be looking at a theory which has a symmetry slightly larger than that of QCD and it has additional scalars.

Bander and Itzykson[8] have shown that for SO(N), the corresponding Laplacian operator is given by:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\,\frac{\partial}{\partial r} - \frac{L^2}{r^2} \tag{4.24}$$

where L^2 is the generalization of the total angular momentum and it operates on angular variables with

$$L^2\ Y^{(N)}_{\ell,\{m_i\}}(\hat{n}) = \ell(\ell + N - 2)\ Y^{(N)}_{\ell,\{m_i\}}(\hat{n}) \tag{4.25}$$

where $\hat{n}$ is an arbitrary unit vector in the N-dimensional space.

Define the reduced radial function by:

$$u = r^a R \quad , \quad a = -\frac{N-1}{2} \quad . \tag{4.26}$$

The present SO(N) theory can now be associated with the corresponding radial equation:

$$-\frac{1}{2\mu}\frac{d^2u}{dr^2}+\left[\frac{4\ell(\ell+N-2)-(N-1)(N-3)}{8\mu r^2}+\frac{1}{2}\mu\omega^2 r^2+\frac{\lambda_0 r^4}{4!}\right]u = Eu \quad . \tag{4.27}$$

WKB approximation can again be applied to solve for the energy eigenvalues. Once again one concludes that as long as the parameter λ_0 is kept below a certain value, the nonhypercolor-singlet ($\ell > 0$) components in Φ_ℓ and Ψ_ℓ are appropriately suppressed in the ground state. In turn the no-go senario advocated in Ref. [1], is also applicable here.

5. Further Remarks

So far we have demonstrated that the QDG no-go scenario extends to the composite models based on SO(N) groups. Note that our result follows rather trivially once one recognizes the smallness of the mixing angle θ. On the other hand we find that without the explicit calculations of the eigenvalues E_1 and E_0, it would be difficult to argue a priori about the smallness of θ and thus the breaking of chiral symmetry.

Unfortunately we are not able to enlarge the scope to those models based on SU(N) local gauge groups, be it either having both scalars and fermions assigned to the fundamental representations (like the Fritzsch-Mandelbaum model), or having right-handed and left-handed fermions transform differently under the gauge group (like the Abbott-Farhi model), or other constructions. But the message we get from the previous sections is quite clear. For composite models based on QCD-like non-Abelian gauge theories, the chiral symmetry is very hard, if not impossible, to be retained. There is, however, an exception suggested by Banks and Kaplonovsky[9], who use a variant of a Kogut-Susskind gradient which is only possible for groups with real representations, such as O(n), and with a single flavor of fermion. But these theories would give massive composite fermions when treated using any of the standard lattice gradients.[1]

An interesting alternative is given by Buchmüller, Peccei and Yanagida,[10] in which the lightness of leptons and quarks is not supposed to be protected by the chiral symmetry of a "normal" Yang-Mill gauge theory. Rather, the light composite leptons and quarks are quasi Goldstone fermions arising from the spontaneous breaking of a global symmetry in a supersymmetric theory. In this framework they treat the weak interactions as residual interactions of the quasi Goldstone fermions. The simplest preon model realizing this global symmetry breakdown leads to the supersymmetric extension of the standard model of electroweak interactions. So this model can be viewed as a supersymmetric generalization of the model of Abbott and Farhi.[3] But there are shortcomings in this supersymmetric model. Namely, there is only one family of leptons and quarks realizing in the coset space $U(6)/U(4) \times SU(2)$ that they suggest. Thus the main theoretical motivation for lepton-quark substructure, namely, the family (or generation) replication and the fermion mass spectrum, is far from being addressed. Despite the fact that no realistic composite model has been advanced thus far, the notion that quarks and leptons are composite particles remains to be intriguing.

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APPENDIX A
The Absence of $\phi^\dagger\phi\,\bar{\psi}\,\psi$ Term in Effective Hamiltonian

We argue that the Hamiltonian H_c will not develop an effective $\phi^\dagger\phi\,\bar{\psi}\,\psi$ term. Since we assume that ϕ and ψ can interact only through the gauge fields, the lowest order contribution is the box diagram in Fig. 1(a). The scattering amplitude in the U(1) case is

$$A_2 \propto \int d^4q \, \frac{(2k-q)_\mu \, [\gamma^\mu(\not{p} + \not{q} + m)\gamma^\nu] \, (2k-q)_\nu}{[(k-q)^2 - M^2] \, [(p+q)^2 - m^2] \, q^4} \quad , \tag{A.1}$$

where k and p are the momenta of the scalar and the fermion, respectively. The quantity q is the hypergluon momentum, M and m are the scalar and fermion masses, respectively. The resulting terms in the effective Hamiltonian are given by

$$\sum_\lambda \mathrm{Tr}\,(\Gamma_\lambda A_2)\, \phi^\dagger\phi\,\bar{\psi}\Gamma^\lambda\psi \quad . \tag{A.2}$$

The candidate mass term is $\phi^\dagger\phi\,\bar{\psi}\,\psi$, with its coefficient given by the trace of A_2,

$$\begin{aligned} \mathrm{Tr}\; A_2 &\propto \int d^4q \, \frac{(2k-q)_\mu \; \mathrm{Tr}\,(\gamma^\mu\gamma^\nu m)\,(2k-q)_\nu}{[(k-q)^2 - M^2]\,[(p+q)^2 - m^2]\,q^4} \\ &= 4m \int d^4q \frac{(2k-q)^2}{[(k-q)^2 - M^2]\,[(p+q)^2 - m^2]\,q^4} \quad . \end{aligned} \tag{A.3}$$

We see that $\mathrm{Tr}\,A_2$ depends explicitly on the fermion mass m. Since we assume $m = 0$ to begin with, this box diagram does not contribute to an effective $\phi^\dagger\phi\,\bar{\psi}\,\psi$ term.

Next we look at the higher order iterations, which is diagramatically shown in Fig. 1(b). The corresponding coefficient for its contribution to the mass term is proportional to $\mathrm{Tr}\, A_N$,

$$\mathrm{Tr}\, A_N \propto \int d^4q_1 \ldots d^4q_{N-1}(\ldots)$$
$$\times \mathrm{Tr}\left[\gamma^{\mu_1}(\not{p}'_1 + m)\gamma^{\mu_2}(\not{p}'_2 + m) \ldots \gamma^{\mu_{N-1}}(\not{p}'_{N-1} + m)\gamma^{\mu_N}\right] \quad . \tag{A.4}$$

Notice that the mass term should be associated with even charge-conjugation configurations, so N must be even. This in turn tells us that there are odd number of $(p'_i + m)$ terms inside the trace. As a result the only term which is independent of m [i.e., $\mathrm{Tr}\,(\gamma^{\mu_1}\ \not{p}'_1\ \gamma^{\mu_2}\ \not{p}'_2 \ldots \gamma^{\mu_{N-1}}\ \not{p}'_{N-1}\ \gamma^{\mu_N})$] should vanish. All other expansions depend at least on one power of m, and should be zero when $m = 0$. So we conclude that the term $\phi^\dagger\phi\,\bar{\psi}\,\psi$ should never occur in the effective Hamiltonian.

APPENDIX B
WKB Approximation for the Harmonic Oscillator Problem

In this appendix we illustrate that the WKB approximation does indeed reproduce, at least approximately, the spectrum of Eq. (3.19). We shall illustrate the effect of the perturbative terms in this context.

1. WKB Approximation for the Ground State Energy

Consider Eq. (3.14). With $\ell = 0$, the WKB approximation gives[7]:

for $0 < r < a$,

$$u \sim \frac{2}{\sqrt{k}} \cos\left(\phi_{ra} - \frac{\pi}{4}\right) \quad \text{where} \quad k = \sqrt{E - V(r)} \quad ; \tag{B.1}$$

and for $r > a$,

$$u \sim \frac{1}{\sqrt{\kappa}} \exp\left(-\int_a^r \kappa \, dr\right) \quad \text{where} \quad \kappa = \sqrt{V(r) - E} \quad . \tag{B.2}$$

In these equations, a is the classical turning point. For the present case:

$$a = \sqrt{2E/m\omega^2} \ , \quad V(r) = \frac{1}{2} m\omega^2 r^2 + \frac{\ell(\ell+1)}{2mr^2} \ , \quad \phi_{12} = \int_{r_1}^{r_2} k \, dr \quad .$$

To ensure the vanishing of u at $r = 0$, we need the condition

$$\phi_{0a} - \frac{\pi}{4} = (2q+1)\,\frac{\pi}{2} \quad . \tag{B.3}$$

The ground state corresponds to $q = 0$ or the condition

$$\int_0^a dr \ \sqrt{2m\left(E - \frac{1}{2} m\omega^2 r^2\right)} = \frac{3}{4}\,\pi \quad . \tag{B.4}$$

Evaluating the left hand side of Eq. (B.4) gives

$$\text{LHS} = a\sqrt{2mE}\ \frac{\pi}{4} \quad .$$

This leads to $E = \frac{3}{2}\, w$ in agreement with the exact solution given in Eq. (3.19).

2. <u>To verify the WKB Method for $\ell \geq 1$</u>

For $\ell \geq 1$, the WKB method gives:

$$u \sim \frac{1}{\sqrt{\kappa}} \exp\left(-\int_r^b \kappa\, dr\right) \quad \text{for} \quad 0 < r < b \quad , \tag{B.5a}$$

and

$$u \sim \frac{2}{\sqrt{k}} \cos\left(\phi_{br} - \frac{\pi}{4}\right) \quad \text{for} \quad b < r < a \quad . \tag{B.5b}$$

As usual, we rewrite the right-hand side of Eq. (B.5b) as:

$$\begin{aligned} \frac{2}{\sqrt{k}} \cos\left(\phi_{br} - \frac{\pi}{4}\right) = \frac{2}{\sqrt{k}} \Bigg[& \cos\left(\phi_{ba} - \frac{\pi}{2}\right) \cos\left(\phi_{xa} - \frac{\pi}{4}\right) \\ & + \sin\left(\phi_{ba} - \frac{\pi}{2}\right) \sin\left(\phi_{xa} - \frac{\pi}{4}\right)\Bigg] \quad . \end{aligned} \tag{B.6}$$

Now we recall that the connection formulae in the WKB method give

$$\frac{2}{\sqrt{k}} \cos\left(\phi_{ra} - \frac{\pi}{4}\right) \;\rightarrow\; \frac{1}{\sqrt{\kappa}} \exp\left(-\int_a^r \kappa\, dr\right) \quad , \tag{B.7}$$

$$\frac{1}{\sqrt{k}} \sin\left(\phi_{ra} - \frac{\pi}{4}\right) \;\rightarrow\; -\frac{1}{\sqrt{\kappa}} \exp\left(+\int_a^r \kappa\, dr\right) \quad . \tag{B.8}$$

Equation (B.8) shows that the appearance of $\sin[\phi_{xa} - (\pi/2)]$ factor in Eq. (B.6) is not acceptable. This could be avoided with

$$\phi_{ba} - \frac{\pi}{2} = q\pi \quad ,$$

or

$$\phi_{ba} = \left(q + \frac{1}{2}\right)\pi \,, \quad q = 0, 1, 2, \ldots \quad . \tag{B.9}$$

For arbitrary ℓ and q, Eq. (B.9) takes the form:

$$\int_b^a dr \sqrt{2\mu\left(E - \frac{1}{2}\mu\omega^2 r^2 - \frac{\ell(\ell+1)}{2\mu r^2}\right)} = \left(q + \frac{1}{2}\right)\pi \ . \qquad (B.10)$$

Rewrite LHS in the form

$$\begin{aligned} \text{LHS} &= \frac{\mu\omega}{2}\int_{b^2}^{a^2} \frac{dr^2}{r^2}\sqrt{(r^2-b^2)(a^2-r^2)} \\ &= \frac{\mu\omega}{2}\cdot\frac{\pi}{2}(a-b)^2 = \left(q+\frac{1}{2}\right)\pi \ . \end{aligned} \qquad (B.11)$$

The second equality can be proven by means of, among other things, hypergeometric function identities.[11] Some details are given in the Appendix C.

Comparing Eqs. (B.10) and (B.11), we get

$$ab = \frac{\sqrt{\ell(\ell+1)}}{\mu\omega} \qquad \text{and} \qquad a^2 + b^2 = \frac{2E}{\mu\omega^2} \ . \qquad (B.12)$$

From Eqs. (B.11) and (B.12)

$$(a-b)^2 = \frac{2}{\mu\omega}\left(\frac{E}{\omega} - \sqrt{\ell(\ell+1)}\right) = \frac{4}{\mu\omega}\left(q+\frac{1}{2}\right) \ , \qquad (B.13)$$

or

$$E = 2q + 1 + \sqrt{\ell(\ell+1)} \approx 2q + \ell + \frac{3}{2} \ . \qquad (B.14)$$

Some observant reader might have already notice that the WKB wave function given in Eq. (B.5a) does not have the correct near $r = 0$ behavior for $\ell \geq 1$. Despite of this discrepancy, we observe that so far as the eigenvalues of (3.19) are concerned, the WKB method does a reasonable job in reproducing the spectrum. Notice that for $\ell = 1$,

$$\sqrt{\ell(\ell+1)} = 1.41$$

our approximation in (B.14) amounts to replacing this value by 1.5, which is a reasonable approximation. Furthermore the approximation in replacing $\sqrt{\ell(\ell+1)}$ by $\ell + \frac{1}{2}$ becomes more and more accurate as ℓ increases.

APPENDIX C
Derivation of Equation (B.11)

Mathematical identities (all quotations are referred to those given in Ref. 11) that we need are:

I.1

$$\int_a^b dx\ \frac{(x-a)^{\nu-1}(b-x)^{\mu-1}}{x} = \frac{(b-a)^{\mu+\nu-1}}{b}\ B(\mu,\nu) \tag{3.228.3}$$

$$\times {}_2F_1\Big(1,\mu\,;\,\mu+\nu\,;\,\frac{b-a}{b}\Big), \quad \text{for} \quad b>a>0 \ .$$

I.2

$${}_2F_1(\alpha,\beta\,;\,2\beta\,;\,z) = \Big(1-\frac{z}{2}\Big)^{-\alpha} F\Big[\frac{\alpha}{2},\frac{\alpha+1}{2}\,;\,\beta+\frac{1}{2}\,;\,\Big(\frac{z}{2-z}\Big)^2\Big] \ , \tag{9.134.1}$$

I.3

$${}_2F_1\Big[\frac{1}{2},1\,;\,2\,;\,4z(1-z)\Big] = \frac{1}{1-z} \ , \ |z| \le \frac{1}{2} \ , \ |z(1-z)| \le \frac{1}{4} \ . \tag{9.121.25}$$

We begin with the LHS of Eq. (3.11). From (I.1):

$$\text{LHS} = \frac{\mu\omega}{2}\cdot\frac{(a^2-b^2)^2}{a^2}\cdot\frac{\pi}{8}\cdot\ {}_2F_1\Big(1,\frac{3}{2}\,;\,3\,;\,\frac{a^2-b^2}{a^2}\Big) \ . \tag{C.1}$$

From (I.2),

$${}_2F_1\Big(1,\frac{3}{2}\,;\,3\,;\,\frac{a^2-b^2}{a^2}\Big) = \frac{2a^2}{a^2+b^2}\ {}_2F_1\Big[\frac{1}{2},1\,;\,2\,;\,\Big(\frac{a^2-b^2}{a^2+b^2}\Big)^2\Big] \ . \tag{C.2}$$

Identify

$$4z(1-z) = \Big(\frac{a^2-b^2}{a^2+b^2}\Big)^2 \ . \tag{C.3}$$

This gives

$$\frac{1}{1-z} = \frac{2(a^2+b^2)}{(a+b)^2} \ . \tag{C.4}$$

From Eqs. (C.2), (C.3) and (I-3), Eq. (C.1) can be rewritten as

$$\text{LHS} = \frac{\mu\omega\pi}{4}\ (a-b)^2 \ , \tag{C.5}$$

and the expression in Eq. (B.11) follows.

APPENDIX D

Derivation of Eq. (4.14)

To arrive at Eq. (4.14), we need to reexpress each relevant bilinear form of scalar products of SO(3) vectors in terms of linear forms of scalar products of new SO(3) vectors. Consider the general expression

$$\begin{aligned} P &= (a \cdot b)(c \cdot d) \\ &= \sum_{m,n} [(-1)^{1-m}\, a_{1m}\, b_{1-m}]\,[(-1)^{1-n}\, c_{1n}\, d_{1-n}] \quad , \end{aligned} \tag{D.1}$$

where a, b, c and d are vectors in the triplet representation. Introduce the new SO(3) vector $A_{JM}, B_{J'M'}$ defined

$$\begin{aligned} a_{1m}\, c_{1n} &= \sum_{JM} A_{JM} \langle J, M | 1, m; 1, n \rangle \quad , \\ b_{1-m}\, d_{1-n} &= \sum_{J'M'} B_{J'M'} \langle J', M' | 1, -m; 1, -n \rangle \end{aligned} \tag{D.2}$$

with J ranging from 0 to 2. Now the Clebsch–Gorden identities imply that

$$\begin{aligned} &\langle J', M' | 1, -m; 1, -n \rangle = (-1)^{J'} \langle J', -M' | 1, m; 1, n \rangle \\ &\sum_{m,n} \langle J, M | 1, m; 1, n \rangle \langle J', -M' | 1, m; 1, n \rangle = \delta_{JJ'} \delta_{M-M'} \end{aligned} \tag{D.3}$$

with $M = m + n$. Substituting Eq. (D.2) and (D.3) into Eq. (D.1) we get

$$\begin{aligned} P &= (a \cdot b)(c \cdot d) \\ &= \sum_{JM} A_{JM}\, B_{J-M} (-1)^{J-M} \\ &= A_0 \cdot B_0 + A_1 \cdot B_1 + A_2 \cdot B_2 \\ &= 1 \quad . \end{aligned} \tag{D.4}$$

Now we introduce Φ'_ℓ and Φ''_ℓ through following expressions:

$$\phi_1^\dagger \Phi_0^\dagger = \Phi_1'^\dagger$$

$$\begin{aligned} a_{1m}\, c_{1n} &= \phi_{1m}^\dagger \Phi_{1n}^\dagger \\ &= \Phi_0''^\dagger \langle 0,0|l,m;l,n\rangle + \sum_M \Phi_{1M}''^\dagger \langle 1,M|1,m;1,n\rangle \\ &\quad + \sum_{M'} \Phi_{2M'}''^\dagger \langle 2,M'|1,m;1,n\rangle , \end{aligned} \tag{D.5}$$

and similar expressions for Ψ'_ℓ and Ψ''_ℓ. Substituting Eq. (D.5) and the corresponding expressions into Eq. (D.4) one arrives at Eq. (4.14).

Consider the case when $a \cdot a = b \cdot b = c \cdot c = d \cdot d = 1$. Note that for this case by setting A_J equal to B_J in Eq. (D.4), A_J and B_J are not properly normalized. However, note that the multiplicative factor needed to restore the normalization is some finite factor of $\mathcal{O}(1)$. For example,

$$\begin{aligned} A_0 &= \sum_{m,n} a_{1m}\, c_{1n} \langle 0,0|1,m;1,n\rangle \\ &= \frac{1}{\sqrt{3}} a_1 \cdot c_1 \end{aligned} \tag{D.6}$$

When $a_1 = c_1, A_0 = 1/\sqrt{3}$. In general, $a_1 \cdot c_1$ deviates from unity by some finite amount. One can easily see that this complies with the above statement of normalization.

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Figure Caption

Fig. 1. (a) The box diagram that could give rise to an effective $\phi^\dagger \phi \bar{\psi} \psi$ term; and (b) its higher order iterations.

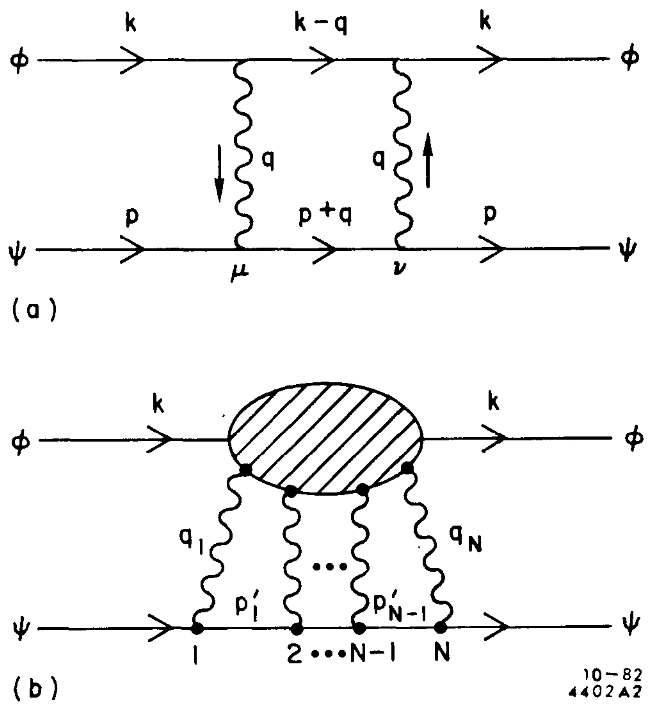

Fig. 1