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HARMONIC MAPS AND THEIR APPLICATION TO GENERAL RELATIVITY*

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ABSTRACT

A method for obtaining the solutions of the Euler's equations of the theory of harmonic maps is proposed. It is also shown that in two dimensional case different solutions of the Euler's equations of harmonic maps are generally related by the conformal transformations. As an application of the proposed method the solutions of the Ernst equation in general relativity is studied.

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1. Introduction

The theory of harmonic mappings (HM) of Riemannian manifolds has application to a wide variety of problems[1] in Physics such as the Nambu string, solitons, nonlinear σ model, Heisenberg ferromagnet, and Einstein field equations of gravitation. In this paper a method for obtaining the solutions of the Euler's equations of HM is proposed. It is shown that if the hamonic mapping $\phi^A(x) : M \rightarrow M'$ ($A = 1 \ldots m$) are functions of an argument σ alone, where σ is a function of x^μ ($\mu = 1 \ldots n$), then the Euler's equations of HM can be reduced to the geodesic equations in M' space with σ as a line element, and the function $\sigma = \sigma(x)$ satisfies the general covariant D'Alember's equation in M space. It is also shown that in two dimensional case ($m = 2$) different solutions of the Euler's equations of HM are generally related by the conformal transformation $W = F(w)$ where $w = \phi^1 + \phi^2$. As an application of the proposed method, we study the solutions of the Ernst equation in general relativity.

2. Harmonic Maps

Let M and M' be two Riemannian manifolds with x^μ coordinates on M and ϕ^A coordinates on M'. The metrics on M and M' are denoted by

$$d\ell^2 = g_{\mu\nu}(x)dx^\mu \, dx^\nu \qquad (\mu, \nu = 1, 2, \ \ldots, \ n) \tag{1}$$

$$dL^2 = G_{AB}(\phi)d\phi^A \, d\phi^B \qquad (A, B = 1, 2, \ \ldots, \ m) \tag{2}$$

respectively. A mapping

$$\phi : \qquad M \rightarrow M'$$

of M onto M' will be represented in coordinates as

$$\phi^A = \phi^A(x) \tag{3}$$

It will be called a harmonic map if it satisfies the Euler's equations resulting from the variational principle $\delta I = 0$, using the action

$$I = \int d^n x \sqrt{g}\, g^{\mu\nu} \frac{\partial \phi^A}{\partial x^\mu} \cdot \frac{\partial \phi^B}{\partial x^\nu} G_{AB}(\phi) \tag{4}$$

i.e., the conditions for a map to be harmonic are given by the Euler's equations

$$\frac{1}{\sqrt{g}} \cdot \frac{\partial}{\partial x^\mu} \left[\sqrt{g}\, g^{\mu\nu} \frac{\partial \phi^C}{\partial x^\nu} \right] + g^{\mu\nu} \Gamma^C_{AB}(\phi) \frac{\partial \phi^A}{\partial x^\mu} \cdot \frac{\partial \phi^B}{\partial x^\nu} = 0 \qquad (C = 1, 2, ..., m) \tag{5}$$

where Γ^C_{AB} are the Christoffels of the ϕ metric dL^2 on M'.

Since $f'^A_\mu(x') = \partial_\mu \phi^A$ has the vector transformation properties:

$$f'^A_\mu(x') = \frac{\partial x'^\nu}{\partial x^\mu}\ f^A_\nu(x), \qquad f'^A_\mu(x) = \frac{\partial \phi'^A(x)}{\partial \phi^B}\ f^B_\mu(x) \tag{6}$$

it follows that (5) are covariant with respect to coordinate tranformation both on M and M'. That is to say the harmonic mapping is an invariant statement which is not affected by any choice of coordinates on M and M'.

3. The Functional Relations Between the Harmonic Maps

The (5) are nonlinear partial differential equations of variable ϕ^A. We shall study the functional relation between the solutions of the (5). This problem is equivalent to finding the conditions under which the (5) have one and the same form in different coordinates ϕ' and ϕ on M'. It is not difficult to find that these conditions should be

$$G'_{AB}(\phi') = G_{AB}(\phi') \tag{7}$$

This means $G'_{AB}(\phi')$ and $G_{AB}(\phi)$ must be the same functions of ϕ' and ϕ respectively. It follows that $\Gamma'^C_{AB}(\phi')$ and $\Gamma^C_{AB}(\phi)$ are also the same functions of ϕ' and ϕ respectively. The conditions (7) can be expressed as

$$G_{AB}(\phi') = \frac{\partial \phi^C}{\partial \phi'^A} \cdot \frac{\partial \phi^D}{\partial \phi'^B} G_{CD}(\phi) \tag{8}$$

which are the equations for determining the functional relation between the different solutions ϕ' and ϕ of (5).

In two–dimensional case with diagonal metric, denote

$$G_{11} = A(\phi), \qquad G_{22} = B(\phi), \qquad G_{12} = G_{21} = 0$$
$$\phi' = \{\phi^1, \phi^2\}, \qquad \phi = \{\Phi^1, \Phi^2\} \quad . \tag{9}$$

Equation (8) then can be written as

$$A(\Phi)\Big(\frac{\partial\Phi^1}{\partial\phi^1}\Big)^2 + B(\Phi)\Big(\frac{\partial\Phi^2}{\partial\phi^1}\Big)^2 = A(\phi), \tag{10}$$

$$A(\Phi)\Big(\frac{\partial\Phi^1}{\partial\phi^2}\Big)^2 + B(\Phi)\Big(\frac{\partial\Phi^2}{\partial\phi^2}\Big)^2 = B(\phi), \tag{11}$$

and

$$A(\Phi)\frac{\partial\Phi^1}{\partial\phi^1}\cdot\frac{\partial\Phi^1}{\partial\phi^2} + B(\Phi)\frac{\partial\Phi^2}{\partial\phi^1}\cdot\frac{\partial\Phi^2}{\partial\phi^2} = 0 \tag{12}$$

Using (10), (11) and (12) we have the relations

$$\Big(\frac{\partial\Phi^2}{\partial\phi^1}\Big)^2 = \frac{A(\phi)A(\Phi)}{B(\phi)B(\Phi)}\Big(\frac{\partial\Phi^1}{\partial\phi^2}\Big)^2 \tag{13}$$

and

$$\Big(\frac{\partial\Phi^2}{\partial\phi^2}\Big)^2 = \frac{B(\phi)A(\Phi)}{A(\phi)B(\Phi)}\Big(\frac{\partial\Phi^1}{\partial\phi^1}\Big)^2$$

It is easy to see that the expressions satisfying (12) and (13) should be either

$$(a)\begin{cases} \dfrac{\partial\Phi^2}{\partial\phi^1} = -\sqrt{\dfrac{A(\phi)A(\Phi)}{B(\phi)B(\Phi)}}\cdot\dfrac{\partial\Phi^1}{\partial\phi^2} \\ \dfrac{\partial\Phi^2}{\partial\phi^2} = +\sqrt{\dfrac{B(\phi)A(\Phi)}{A(\phi)B(\Phi)}}\cdot\dfrac{\partial\Phi^1}{\partial\phi^1} \end{cases} \tag{14}$$

or

$$(b)\begin{cases} \dfrac{\partial\Phi^2}{\partial\phi^1} = +\sqrt{\dfrac{A(\phi)A(\Phi)}{B(\phi)B(\Phi)}}\cdot\dfrac{\partial\Phi^1}{\partial\phi^2} \\ \dfrac{\partial\Phi^2}{\partial\phi^2} = -\sqrt{\dfrac{B(\phi)A(\Phi)}{A(\phi)B(\Phi)}}\cdot\dfrac{\partial\Phi^1}{\partial\phi^1} \end{cases}$$

In fact equations (b) can be reduced to (a) by writing ϕ^2 for $-\phi^2$.

In the case of equal metric $A = B$, Eq. (14) can be expressed as

$$\frac{\partial \Phi^1}{\partial \phi^1} = \frac{\partial \Phi^2}{\partial \phi^2}, \qquad \frac{\partial \Phi^1}{\partial \phi^2} = -\frac{\partial \Phi^2}{\partial \phi^1} \tag{15}$$

These equations are just the Cauchy–Riemann equations in the theory of complex variables, it follows that $\Phi^1 + i\Phi^2 = F(\phi^1 + i\phi^2)$, where $F(w)$ is a regular function of $w = \phi^1 + i\phi^2$. This means in this case two solutions of the Euler's equations (5) should be related by the conformal transformation

$$W = F(w) \quad , \tag{16}$$

where $W = \Phi^1 + i\Phi^2$. We shall see later that the Ernst equations and its solutions belong to this case. It is well–known that the non–orthogonal metric in two–dimensional space can be transformed into the orthogonal form with equal metric $A(\phi) = B(\phi)$, so in fact the above discussion is the general two–dimensional case.

4. An Important Kind of Solution of the Euler's Equation in HM

In the following we shall study an important kind of solution of the Euler's equation in HM. We investigate the case that ϕ^A $(A = 1, \ ... \ , m)$ are functions of an argument σ alone, where σ is a function of x^μ

$$\sigma = \sigma(x)$$

In this case the Euler's equation can be written as

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left[\sqrt{g} \ g^{\mu\nu} \ \frac{\partial \sigma}{\partial x^\nu}\right] \frac{d\phi^C}{d\sigma} + g^{\mu\nu}\left[\frac{d^2\phi^C}{d\sigma^2} + \Gamma^C_{AB} \frac{d\phi^A}{d\sigma} \frac{d\phi^B}{d\sigma}\right]\frac{\partial \sigma}{\partial x^\mu} \cdot \frac{\partial \sigma}{\partial x^\nu} = 0 \tag{17}$$

From the above equation we can see that if ϕ^A satisfy the geodesic equations on M' space with parameter σ:

$$\frac{d^2\phi^C}{d\sigma^2} + \Gamma^C_{AB} \frac{d\phi^A}{d\sigma} \frac{d\phi^A}{d\sigma} = 0 \tag{18}$$

and $\sigma = \sigma(x)$ satisfies the covariant D'Alember's equation on M space,

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu}\left[\sqrt{g}\; g^{\mu\nu} \frac{\partial \sigma}{\partial x^\nu}\right] = 0 \tag{19}$$

then $\phi^A = \phi^A(\sigma(x))(A = 1, \ldots, m)$ are the solutions of the Euler's equations (5).

In particular when M space is the Minkowski space, the equation (19) for $\sigma(x)$ becomes the usual D'Alember's equation

$$\Box\, \sigma(x) = 0 \tag{20}$$

i.e., $\sigma(x)$ satisfy the wave function and σ is a function of wave argument alone (for example in 1 + 1 dimensional case σ is a function of wave argument $x \pm ct$).

This kind of solution is of importance, because it enables us to find the exact wave solution of some nonlinear partial differential equation, especially to find the exact wave solution of Einstein equation. A study of this problem will be published elsewhere.[4]

In the case ϕ^A is not the function of t (time variable), the function $\sigma(x)$ shall satisfy the Laplace equation

$$\nabla^2 \sigma = 0 \tag{21}$$

which is useful for studying the stationary solution of Einstein equation.

5. The Ernst Equation

In this section we shall study the Ernst equation from the view point of harmonic mapping. Following Ernst[2] we express the axially symmetric line element as

$$dS^2 = f^{-1}[e^{2\mu}(dz^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt - wd\phi)^2 \tag{22}$$

Let M be a flat three–dimensional manifold with metric

$$d\ell^2 = d\rho^2 + dz^2 + \rho^2 d\phi^2 \tag{23}$$

and M' with metric

$$dL^2 = f^{-2}[df^2 + d\psi^2] \tag{24}$$

where

$$G_{AB} = f^{-2}\delta_{AB} \quad , \tag{25}$$

and the function ψ is related to w by

$$\frac{\partial \psi}{\partial \rho} = f^2 \rho^{-1} \frac{\partial w}{\partial z} \quad , \quad \frac{\partial \psi}{\partial z} = -f^2 \rho^{-1} \frac{\partial \omega}{\partial \rho}$$

It has been suggested that the Ernst equations[2]

$$F\nabla^2 f = \nabla f \cdot \nabla f - \nabla\psi \cdot \nabla\psi$$

$$f\nabla^2 \psi = 2\nabla f \cdot \nabla\psi$$

i.e.,

$$f\nabla \mathcal{E} = \nabla\mathcal{E} \cdot \nabla\mathcal{E} \quad , \quad \mathcal{E} = f + i\psi \tag{26}$$

can be obtained from the Euler's equation (5) using the variation principle[3] $\delta I = 0$, where

$$I = \int f^{-2}[(\nabla f)^2 + (\nabla\psi)^2]\rho \; d\rho \; dz \; d\phi \tag{27}$$

That is to say, the Ernst equation can be derived by use of the theory of harmonic map.

In the following we shall study the functional relations between the solutions of the Ernst equation (26). Suppose that (f, ψ) and (u, v) are two pairs of the solutions of the Ernst equation and they are related by

$$\begin{aligned} f &= f(u, v) \\ \psi &= \psi(u, v) \end{aligned} \tag{28}$$

From (10) – (12) and (24), putting $\Phi = (f, \psi)$ and $\phi = (u, v)$, we find that if both ϕ and Φ are solutions of the Ernst equation, they must satisfy the conditions

$$u^2 \left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial \psi}{\partial u}\right)^2\right] = f^2 \tag{29}$$

$$u^2 \left[\left(\frac{\partial f}{\partial v}\right)^2 + \left(\frac{\partial \psi}{\partial v}\right)^2\right] = f^2 \tag{30}$$

and

$$\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v} + \frac{\partial \psi}{\partial u} \cdot \frac{\partial \psi}{\partial v} = 0 \tag{31}$$

It follows from (15) that the expressions satisfying (29), (30), and (31) should be

$$\begin{aligned} \frac{\partial f}{\partial u} &= \frac{\partial \psi}{\partial v} \\ \frac{\partial f}{\partial v} &= -\frac{\partial \psi}{\partial u} \end{aligned} \tag{32}$$

Equations (32) are the Cauchy–Riemann equations and express that $f + i\psi = F(u + iv)$ where $F(\mathcal{E})$ is a regular function of $\mathcal{E} = u + iv$. This means that the transformation (28) should be expressed only as the transformation in terms of the Ernst potential $\mathcal{E}$, that is

$$\mathcal{E}' = F(\mathcal{E}) \tag{33}$$

where $\mathcal{E}' = f + i\psi$. Moreover, from (29) and (32) it can be proved that the transformation function $F(\mathcal{E})$ should satisfy the condition

$$\left|\frac{dF(\mathcal{E})}{d\mathcal{E}}\right| = \frac{f}{u} \tag{34}$$

which means

$$\frac{dF(\mathcal{E})}{d\mathcal{E}} = \frac{f}{u}\, e^{i\alpha}$$

where α is a function of u and v.

It is not difficult to verify that the inverse transformations

$$\mathcal{E}' = \frac{1}{\mathcal{E}} \tag{35}$$

and the Euler's transformation

$$\mathcal{E}' = \frac{d\mathcal{E} + ic}{ib\mathcal{E} + a} \quad , \quad ad + cb = 1 \tag{36}$$

satisfy the condition (34).

Let us now study the functional relation between the static and the stationary solutions of Ernst equation. If we denote the static potential by $\mathcal{E} = u + iv$ with $v = 0$. Then from (26) the Ernst equation for u should be written as

$$u \, \nabla^2 u = \nabla u \cdot \nabla u \tag{37}$$

or

$$\nabla^2 \sigma = 0 \tag{38}$$

Here σ and u are related by

$$u = e^{2\sigma}$$

Let f and ψ be the stationary solution, the transformation (28) should be expressed by functions of u alone

$$f = f(u) \quad , \quad \psi = \psi(u) \tag{39}$$

Since σ satisfies the Laplace equation (38), from (18) and (21) we know that σ coulds be looked upon as the geodesic parameter and the geodesic equation (18) can be written as

$$\begin{aligned} &\frac{d^2 f}{d\sigma^2} - \frac{1}{f}\left[\left(\frac{df}{d\sigma}\right)^2 - \left(\frac{d\psi}{d\sigma}\right)^2\right] = 0 \\ &\frac{d^2\psi}{d\sigma^2} - 2\frac{1}{f}\,\frac{df}{d\sigma}\cdot\frac{d\psi}{d\sigma} = 0 \end{aligned} \tag{40}$$

The solution of the above equation with the boundary condition, namely, of asymptotic flatness at spatial infinity ($\psi = 0, f = 1$), is

$$f^2 + (\psi + b)^2 = a^2 \tag{41}$$

where the real constants a and b are related by

$$1 + b^2 = a^2 \tag{42}$$

This means that the solution of Ernst equation $F = f(u)$ and $\psi = \psi(u)$ should be described by circle on M'.

Employing (39), the condition (29) or (34) can be written as

$$\left(\frac{df}{du}\right)^2 + \left(\frac{d\psi}{du}\right)^2 = \frac{f^2}{u^2} \tag{43}$$

From (41) and (43) it can be proved that the stationary solution of Ernst equation f and ψ should be related to the static solution u by the following relation

$$f = \frac{2au}{a(u^2+1) - b(u^2-1)} \quad , \quad \psi = -\frac{u^2-1}{a(u^2+1) - b(u^2-1)} \tag{44}$$

In the special case $a = 1$ and $b = 0$, the above relation reduced to

$$f = \frac{2u}{u^2+1} \quad , \quad \psi = -\frac{u^2-1}{u^2+1} \tag{45}$$

and

$$f^2 + \psi^2 = 1$$

which represent a circle with unit radius on the M' space.

If we set $a = csc\alpha$ and $b = cot\alpha$, which satisfy (42) obviously, the relation (44) can be expressed as

$$f = \frac{2u}{(u^2+1) - (u^2-1)cos\alpha} \quad , \quad \psi = -\frac{(u^2-1)sin\alpha}{(u^2+1) - (u^2-1)cos\alpha} \tag{46}$$

This pair of solutions represents a trajectory of a circle with radius $a = csc\alpha$ on M' space, which corresponds to the Ernst solution

$$\xi = \frac{1+\mathcal{E}}{1-\mathcal{E}} = -e^{i\alpha}\, coth\, \sigma \quad , u = e^{2\sigma} \tag{47}$$

where $\mathcal{E} = f + i\psi$ and σ satisfy the Laplace equation (38).

From the above discussion we note that the Ernst solution (47) is the solution which satisfies the condition (43) and is related to the static solution by (37).

In the following we shall study the solution of Ernst equation in term of N static solutions. In the case

$$\psi^A = \psi^A(\sigma_1, \sigma_2, ..., \sigma_N)$$

$$\sigma_i = \sigma_i(x) \qquad i = 1, ..., n$$

where it follows from (17) that the solutions of Euler's equation in HM can be obtained from the following two sets of equations:

$$\text{I.} \qquad \frac{1}{\sqrt{g}}\, \partial_\mu\, (\sqrt{g}\ g^{\mu\nu}\ \partial_\nu\, \sigma_i) = 0 \quad , i = 1...N$$

When M is the Minkowski space

$$\Box\, \sigma_i(x) = 0$$

$$\text{II.} \qquad \frac{\partial^2 \phi^C}{\partial\sigma_i \partial\sigma_j} - \Gamma^C_{AB}\ \frac{\partial \phi^A}{\partial \sigma_i} \cdot \frac{\sigma \phi^B}{\partial \sigma_j} = 0 \qquad i, j = 1,\ \ldots\ N$$

From the above two sets of equations, if we put $\phi^1 = f, \phi^2 = \psi$, we can get the solution of Ernst equation in terms of N static solution, that is

$$\begin{aligned} f &= \frac{2a \prod_i\ u_i}{[(\prod_i u_i)^2 + 1] - b[(\prod_i u_i)^2 - 1]} \\ \psi &= \frac{(\prod_i u_u)^2 - 1}{[(\prod_i u_i)^2 + 1] - b[(\prod u_i)^2 - 1]} \end{aligned} \tag{48}$$

where

$$\prod_i u_i = u_1 \, u_2 \, \ldots \, u_N$$

and $u_1, u_2, \ldots u_N$ are static solutions of Ernst equation

$$u_i \, \nabla^2 u_i = \nabla u_i \cdot \nabla u_i \qquad i = 1, 2, \ldots N \tag{49}$$

It had been shown[5] that the Bogomolny equations for the axially symmetric SU(2) gauge fields are equivalent to the Ernst equation. Therefore the method in this paper can be also used to study the problem of monopole theory.

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