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THE RUNNING COUPLING CONSTANT IN QCD*

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ABSTRACT

The running coupling constant $\bar{g}(Q^2)$ in QCD is investigated using the general principle of renormalization and renormalization group theory. From the formal expression of color dielectric function, an exact functional relation between $\bar{g}^2$ and momentum transfer Q^2 has been obtained. For large Q^2 the theory is consistent with the perturbative QCD. For small Q^2 the running coupling constant behaves as $\bar{g}^2(Q^2) = (M^2/Q^2)^k$ where k > 0. It is shown that in the infrared region the scale transformation $\bar{g}'^2 = \lambda\,\bar{g}^2$ is essential to the renormalization group theory in QCD for small Q^2.

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1. Introduction

Quantum Chromodynamics is a renormalizable non–Abelian gauge field theory of color quarks and gluons. In this theory the renormalization group running coupling constant or the effective charge plays an important role. One of the fundamental problems in QCD is to find the momentum dependence of the effective charge $\bar{g}^2(Q^2)$. QCD became a most attractive theory, only when it was understood to be asymptotically free.[1,2] For large momentum transfer or short distance the effective charge becomes small, this allows perturbative calculations to be performed for large Q^2. The study of the effective charge for small Q^2 or large distance (i.e. the problem of confinement) has known only a limited success. The investigation in this case encounters great difficulties due to a failure of perturbative theory. Several methods were developed based on Schwinger–Dyson equation,[3–7] dispersion relation[8] and self–consistance condition imposed by Ward–Slavnov–Taylor identities.[9]

Since QCD is a non–Abelian gauge theory, the gluons themselves carry color charges and interact with each other. So the color charge of a quark is no longer located at a definite place in space, it is diffusely spread out due to gluon emission and absorption. At small spatial distance (for large Q^2), only a small part of color charge acts effectively, and the effective coupling constant thus appears weaker as the momentum Q^2 is increased. On the other hand, at large spatial distance (for small Q^2), the effective charge should be much stronger, which may lead to confinement property. This gives an intuitive physical picture of the momentum dependence of the effective charge in QCD. From this viewpoint the running coupling constant $\bar{g}^2(Q^2)$ was studied using the color dielectric function (e.g. Ref. 10), which plays an antiscreening role in QCD.

In this paper the running coupling constant $\bar{g}^2(Q^2)$ in QCD is investigated using the general principle of renormalization and renormalization group theory. From the formal expression of the color dielectric function, an exact functional relation has been obtained.

$$\frac{1}{\bar{g}^2} = b\, \ell n \frac{Q^2}{M^2} - a\, \ell n\, \bar{g}^2 + f(\bar{g}^2)$$

where $b = -\beta_0$ and $a = \beta_1/\beta_0$, β_0 and β_1 correspond respectively to the one–and two–loop renormalization group coefficients. The function $f(\bar{g})^2$ possesses nonperturbative feature, it characterizes all higher order contributions. This functional relation had been studied in ghost–free gauge[11] ($z_1 = z_3$), in which the running coupling constant is defined as the renormalized transverse gluon propagator. In our theory the definition of the running coupling constant is directly from the renormalization group theory and the gauge is not specified.

The theory is consistent with the perturbative QCD. When $\bar{g}^2 \to 0$, the function $f(\bar{g}^2)$ vanishes guaranteeing the asymptotic freedom behavior. Using the functional relation of $\bar{g}^2$ and $t = \frac{1}{2} \ell n \frac{Q^2}{\mu^2}$, the analytic properties of the function $\ell n \frac{\bar{g}^2(t)}{g^2}$ in the complex plane of t has been studied, from which we obtain the dispersion relation for $\ell n\, \bar{g}^2(t)$. We find that in the infrared region (for small Q^2)

$$f(\bar{g}^2) = c\, \ell n\, \bar{g}^2, \quad c > a$$

and in the infrared region

$$\bar{g}^2(Q^2) = \left(\frac{M^2}{Q^2}\right)^k, \quad k = \frac{b}{c-a} > 0$$

The corresponding Callan–Symanzik function for large $\bar{g}^2(Q^2)$is

$$\beta(\bar{g}^2) = -2\, k\, \bar{g}^2$$

This momentum dependence of the running coupling constant $\bar{g}(Q^2)$ in case $k = 1$ is consistent with the confinement solution (or self-consistent ansatz) of Schwinger–Dyson equation for $Q^2/M^2 \ll 1$ in Landau gauge[3] and in axial gauge.[5–7] The case

$k < 1$ was studied in dipersion relation theory.[8] The unknown index k in our theory should be further determined by the dynamic structure of non–perturbative QCD.

In this paper, it is also shown the in the infrared region for different renormalization schemes the corresponding running coupling constants $\bar{g}^2$ and $\bar{g}'^2$ are related by the scale transformation (or scale mapping):

$$\bar{g}'^2 = \lambda \; \bar{g}^2$$

where λ is an arbitrary constant. This new invariant property is essential to the renormalization group theory in QCD for small Q^2 and plays an important role in studying the infrared behavior of QCD.

The theoretical structure and formulation proposed in this paper is also applicable to other quantum gauge field theory.

2. A Formal Theory of Running Coupling Constant

In renormalization group theory the running coupling constant or the effective charge $\bar{g}(t)$ is introduced by the defining equation

$$\frac{d \; \bar{g}(t)}{d \; t} = \beta_c\Big(\bar{g}(t)\Big) \tag{1}$$

with the boundary condition

$$\bar{g}(0) = g \tag{2}$$

Here $\beta_c(\bar{g})$ is the Callan–Symanzik function

$$t = \frac{1}{2} \; \ell n \frac{Q^2}{\mu^2}, \qquad Q^2 = -q^2 \tag{3}$$

μ^2 is the renormalization point and g is the renormalized coupling constant. For convenience equation (1) is usually written as

$$\frac{d \, \bar{g}^2(t)}{d \; t} = \beta\Big(\bar{g}^2(t)\Big) \tag{4}$$

where $\beta(\bar{g}^2)$ is given by

$$\beta(\bar{g}^2) = 2\,\bar{g}\,\beta_c(\bar{g}) \tag{5}$$

The solution of equation (4) with boundary condition (2) can be formally expressed as

$$t = \int_{g^2}^{\bar{g}^2(t)} \beta^{-1}(x)dx = \Psi\Big(\bar{g}^2(t)\Big) - \Psi(g^2) \tag{6}$$

For a given renormalization scheme $\Psi\Big(\bar{g}^2(t)\Big)$ should be a unique function of $\bar{g}^2(t)$.

An important property directly deduced from equation (4) is that the derivative of any function of $\bar{g}^2(t)$ with respect to t is also a function of $\bar{g}^2(t)$:

$$\frac{d\phi(\bar{g}^2)}{d\,t} = \phi'(\bar{g}^2)\beta(\bar{g}^2) \tag{7}$$

It can be also proved that if the derivative of a function $f(t, g^2)$ with respect to t is a function of $\bar{g}^2(t)$ alone, then the general expression for $f(t, g^2)$ should be

$$f(t, g^2) = f_1(\bar{g}^2) + k_1 t + k_2 \tag{8}$$

where k_1 and k_2 are constants, k_1 is independent of g^2 and $f_1(\bar{g}^2)$ is a function of $\bar{g}^2$ alone.

Let us denote the ratio of the bare coupling constant g_0^2 to $\bar{g}^2(t)$ by $\mathcal{E}(Q^2)$ and the ratio of g^2 to $\bar{g}^2(t)$ by $K(t)$

$$\bar{g}^2(t) = \frac{g_0^2}{\mathcal{E}(Q^2)} \tag{9}$$

and

$$\bar{g}^2(t) = \frac{g^2}{K(t)}, \qquad K(0) = 1 \tag{10}$$

From (2), (9) and (10) we have

$$g^2 = \frac{g_0^2}{\mathcal{E}(\mu^2)} \tag{11}$$

and

$$K(t) = \frac{\mathcal{E}(Q^2)}{\mathcal{E}(\mu^2)} \tag{12}$$

The function $\mathcal{E}(Q^2)$ in (9) may be looked upon as the color dielectric function[10] in momentum space, and the function $K(t)$ expressed by (12) may be regarded as the relative color dielectric function. In the Axial gauge $\bar{g}^2(t)/g_0^2 = \mathcal{E}(Q^2)^{-1}$ could be expressed by the gluon propagator $d(Q^2) = 1/1 - g_0^2\pi(Q^2)$, where $\pi(Q^2)$ is the gluon vacuum polarization function, and $\pi(Q^2)$ could be determined further in terms of the gluon propagator and vertex function through Schwinger–Dyson equation. In this paper we shall not specify the gauge and shall not be concerned with the dynamical expression for $\mathcal{E}(Q^2)$ and $K(t)$.

Differentiating (10) with respect to t and using (4) we find

$$-\frac{1}{\bar{g}^4(t)}\,\beta\Big(\bar{g}^2(t)\Big) = \frac{1}{g^2}\,\frac{dK(t)}{d\,t} \tag{13}$$

Since the left hand side at (13) is a function of $\bar{g}^2(t)$ alone, it requires the expression $\frac{1}{g^2}\,\frac{dK(t)}{dt}$ at the right hand side to be a function of $\bar{g}^2(t)$. From this requirement and (8) we find that the general expression for the function $K(t)$ should be

$$\frac{1}{g^2}K(t) = 2bt + F\Big(\bar{g}^2(t)\Big) + A \tag{14}$$

where $F(\bar{g}^2)$ is a function of $\bar{g}^2(t)$ alone, A and b are two constants. Using the boundary condition $K(0) = 1$ and $\bar{g}^2(0) = g^2$, we have

$$A = \frac{1}{g^2} - F(g^2)$$

Substituting A into (14) we find the general expression for $K(t)$:

$$K(t) = 1 + 2bg^2t + g^2\Big[F[\bar{g}^2(t)] - F(g^2)\Big] \tag{15}$$

Then from (10) and (15) we obtain the relation between $\bar{g}^2(t), g^2$ and t

$$\frac{1}{\bar{g}^2(t)} = \frac{1}{g^2} + 2bt + \Big[F(\bar{g}^2) - F(g^2)\Big], \tag{16}$$

and the function $\Psi(\bar{g}^2)$ in (6) should be

$$\Psi(\bar{g}^2) = \frac{1}{2b}\left[\frac{1}{\bar{g}^2} - F(\bar{g}^2)\right] \tag{17}$$

Using the multiplicative renormalization relation

$$g = (Z_3)^{3/2}(Z_1)^{-1/2}\, g_0$$

and (11) we find

$$\mathcal{E}(\mu^2) = (Z_3)^{-3} Z_1 \tag{18}$$

where Z_3 is the gluon wavefunction renormalization constant and Z_1 is the triple gluon vertex renormalization constant. If we use the cut–off procedure, Z_3 and Z_1 will be expressed as the function of μ^2, g^2 and the ultraviolet cut–off Λ^2. Then it follows from (18) that $\mathcal{E}(\mu^2)$ is also a function of μ^2, g^2 and Λ^2, and $\mathcal{E}(Q^2)$ should be a function of Q^2, μ^2, g^2 and Λ^2. Taking notice of the fact that $\mathcal{E}(Q^2)$ is dimensionless, the expression (12) can be written in a more detailed form

$$K(t, g^2) \;=\; \frac{g^2}{\bar{g}^2(t, g^2)} \;=\; \frac{\mathcal{E}(r, s, g^2)}{\mathcal{E}(s, s, g^2)} \tag{19}$$

where r and s are two independent and dimensionless variables which are defined as

$$r = \frac{1}{2}\,\ell n\frac{Q^2}{\Lambda^2}, \qquad s = \frac{1}{2}\,\ell n\frac{\mu^2}{\Lambda^2} \tag{20}$$

and t is related to r and s by

$$t = r - s \tag{21}$$

Since the running coupling constant $\bar{g}^2(t, g^2)$is independent of Λ^2, the right hand side at (19) should be independent of Λ^2. To study the consequence of this requirement, differentiating (19) with respect to $\ell n\Lambda^2$ and using (20), we have

$$\mathcal{E}(s, s, g^2)\left[\frac{\partial \mathcal{E}(r, s, g^2)}{\partial r} + \frac{\partial \mathcal{E}(r, s, g^2)}{\partial s}\right] - \mathcal{E}(r, s, g^2)\ \frac{\partial \mathcal{E}(s, s, g^2)}{\partial s} = 0 \tag{22}$$

Defining

$$\phi(r, s, g^2) = \ell n\ \mathcal{E}(r, s, g^2) \tag{23}$$

(19) and (22) can then be rewritten as

$$\phi(r, s, g^2) - \phi(s, s, g^2) = \ \ell n\ \frac{g^2}{\bar{g}^2(t, g^2)} \equiv A(t, g^2) \tag{24}$$

$$\frac{\partial \phi(r, s, g^2)}{\partial r} + \frac{\partial \phi(s, s, g^2)}{\partial s} = \frac{d\phi(s, s, g^2)}{ds} \equiv B(s, g^2) \tag{25}$$

From (24) and (25) we see that the expression in the left hand side of (24) is required to be a function of t, and the expression in the left hand side at (25) is required to be a function of s. It is not difficult to find that the general form of the function $\phi(r, s, g^2)$ which satisfies the conditions (24) and (25) simultaneously should be

$$\phi(r, s, g^2) = 2k' r + \chi(t, g^2) + \eta(s, g^2) \tag{26}$$

where k' is a constant and $\chi(t, g^2) = \chi(r - s, g^2)$. Substituting (26) into (24) and using (21) we have

$$\ell n \frac{g^2}{\bar{g}^2(t)} = 2k' t + \chi(t, g^2) - \chi(0, g^2) \tag{27}$$

Differentiating (27) with respect to t and using (4) we obtain

$$-\frac{1}{\bar{g}^2(t)}\beta\left[\bar{g}^2(t)\right] = 2k' + \frac{d\chi(t,g^2)}{dt} \tag{28}$$

This equation requires that $d\chi(t,g^2)/dt$ is a function of $\bar{g}^2(t)$. Thus from (8) we have

$$\chi(t,g^2) = 2k''t + G\left[\bar{g}^2(t)\right] + D \tag{29}$$

where k'' and D are constants. Substituting (29) into (27) we find

$$\ell n\frac{g^2}{\bar{g}^2(t)} = 2kt + G(\bar{g}^2) - G(g^2), \qquad k = k' + k'' \tag{30}$$

From (6) and (30) we obtain

$$\Psi(\bar{g}^2) = -\frac{1}{2k}\left[\ell n\,\bar{g}^2 + G(\bar{g}^2)\right] \tag{31}$$

For a given renormalization scheme the function $\Psi(\bar{g}^2)$ in (6) should have a unique form. Comparing (17) and (31) we find that the function $F(\bar{g}^2)$ in (17) must contain a term $\ell n\,\bar{g}^2$ and the function $G(\bar{g}^2)$ in (31) must contain a term $1/\bar{g}^2$. This means that $F(\bar{g}^2)$ and $G(\bar{g}^2)$ must take the following forms respectively

$$\begin{aligned} F(\bar{g}^2) &= -a\ \ell n\,\bar{g}^2 + f(\bar{g}^2) \\ G(\bar{g}^2) &= \frac{1}{a}\left[\frac{1}{\bar{g}^2} - f(\bar{g}^2)\right] \end{aligned} \tag{32}$$

where $a = -b/k$. These lead to a unique form of $\Psi(\bar{g}^2)$:

$$\Psi(\bar{g}^2) = \frac{1}{2b}\left[\frac{1}{\bar{g}^2} + a\ \ell n\,\bar{g}^2 - f(\bar{g}^2)\right] \tag{33}$$

From (17) and (33) we obtain the implicit relation between $\bar{g}^2(t), g^2$ and t:

$$\frac{1}{\bar{g}^2(t)} = \frac{1}{g^2} + 2bt - a\ \ell n\frac{\bar{g}^2(t)}{g^2} + \left[f(\bar{g}^2) - f(g^2)\right] \tag{34}$$

This result is deduced from the general principle of renormalization and renormalization group theory, which is independent of gauge and is exact for all values of Q^2. Therefore it gives a nonperturbative description of the momentum dependence of the running coupling constant $\bar{g}^2(t)$. The expression (34) had been found in ghost free gauge ($Z_1 = Z_3$) by means of the relationship between $\bar{g}^2(t)$ and gluon propagator with some assumption[11] and also used to study the higher order calculations in perturbative QCD.[12]

Differentiating (34) with respect to t and using (4), we obtain the non–perturbative expression for $\beta(\bar{g}^2)$:

$$\beta(\bar{g}^2) = \frac{-2b\,\bar{g}^4}{1 - a\ \bar{g}^2 + \bar{g}^4\, f'(\bar{g}^2)} \tag{35}$$

If we define a mass parameter

$$M^2(\mu^2, g^2) = \mu^2 e^{-\frac{1}{b}\left[\frac{1}{g^2} + a\ \ell n\ g^2 - f(g^2)\right]} \tag{36}$$

i.e.,

$$b\ \ell n \frac{\mu^2}{M^2} = \frac{1}{g^2} + a\ \ell n\ g^2 - f(g^2), \tag{37}$$

then the expression (34) can be reduced to a simple form

$$\frac{1}{\bar{g}^2(t_M)} = 2bt_M - a\ \ell n\ g^2(t_M) + f(\bar{g}^2) \tag{38}$$

where

$$t_M = \frac{1}{2} \ell n \frac{Q^2}{M^2} \tag{39}$$

The mass M^2 expressed by (36) gives a general definition for dynamic mass, which is used to scale the momentum dependence of the running coupling constant, and is the nonperturbative generalization of the expression given in Ref. (13). It can be proved from (35) and (36) that M^2 satisfies the renormalization group equation

$$\mu\frac{\partial M^2}{\partial\mu}+\beta(g^2)\frac{\partial M^2}{\partial g^2}=0. \tag{40}$$

3. Comparison with Perturbative Theory

In perturbative theory the Callan–Symanzik function can be expressed by the following power series

$$\beta_c(\bar{g})=\sum_{k=0}^{\infty}\beta_0\,\bar{g}^{2k+3} \tag{41}$$

From (5) and (41) we have

$$\beta(\bar{g}^2)=\sum_{k=0}^{\infty}2\beta_0(\bar{g}^2)^{k+2} \tag{42}$$

To study the relationship between the nonperturbative theory (35) and the perturbative theory (42) we write (42) in the form

$$\beta(\bar{g}^2)=2\beta_0\,\bar{g}^4\,u(\bar{g}^2) \tag{43}$$

where

$$u(\bar{g}^2)=\sum_{k=0}^{\infty}b_k(\bar{g}^2)^k \tag{44}$$

and

$$b_k=\frac{\beta_k}{\beta_0},\qquad (b_0=1) \tag{45}$$

Using the formula of the expansion of an inverse function, we have

$$\frac{1}{u(\bar{g}^2)}=\sum_{k=0}^{\infty}a_k(\bar{g}^2)^k \tag{46}$$

where the coefficient a_k are determined by [12,14]

$$a_0 = 1, \quad a_1 = -b_1, \quad a_2 = b_1^2 - b_2, \quad a_3 = (-1)^3(b_1^3 - 2b_1b_2 + b_3), \quad \ldots \tag{47}$$

The general expression for a_n is[14]

$$a_n = \sum \frac{m!}{\alpha_1!\,\alpha_2!\,\ldots\,\alpha_\ell!}(-1)^m\,(b_1)^{\alpha_1}\,(b_2)^{\alpha_2}\,\ldots\,(b_\ell)^{\alpha_\ell} \tag{48}$$

with

$$\begin{cases} \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_\ell = m \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + \ell\alpha_\ell = n \end{cases}$$

The above series hold true only in its region of convergence, which will be discussed later.

Substituting (46) into (43), the perturbative expression for $\beta(\bar{g}^2)$ can be expressed as

$$\beta(\bar{g}^2) = \frac{2\beta_0\,\bar{g}^4}{\sum_{k=0}^{\infty} a_k(\bar{g}^2)^k} = \frac{2\beta_0\,\bar{g}^4}{1 + a_1\,\bar{g}^2 + \bar{g}^4\sum_{k=2}^{\infty} a_k(\bar{g}^2)^k} \tag{49}$$

comparing (49) with the nonperturbative expression (35) we find

$$b = -\beta_0, \qquad a = -a_1 = b_1 = \frac{\beta_1}{\beta_0} \tag{50}$$

$$f'(\bar{g}^2) = \sum_{k=1}^{\infty} c_k(\bar{g}^2)^{k-1} \tag{51}$$

where

$$c_k = a_{k+1}$$

It is well-known that when flavor number $n_f = 4$ or 6, we have

$$\beta_0 = \frac{g^4}{4\pi^2}(-11 + \frac{2}{3}n_f) < 0$$

and

$$\beta_1 = \frac{g^6}{(2\pi)^4}(-102 + \frac{38}{3}n_f) < 0$$

Then from (50) we find that the constants a and b in (34) and (35) are both positive

$$b > 0 \quad , \quad a > 0 \tag{52}$$

where b and a are determined by one and two loop contributions in QCD.

From (51) we have

$$f(\bar{g}^2) = \sum_{k=1}^{\infty} \frac{c_k}{k}(\bar{g}^2)^k \tag{53}$$

where

$$\begin{gathered} c_1 = b_1^2 - b_2 \quad , \quad c_2 = -(b_1^2 - 2b_1 b_2 + b_3), \; ... \\ c_k = \sum \frac{m!}{\alpha_1!\alpha_2! \, ...\alpha_\ell!}(-1)^m \; b_1^{\alpha_1} \; b_2^{\alpha_2} \; ... \; b_\ell^{\alpha_\ell} \end{gathered} \tag{54}$$

with

$$\alpha_1 + \alpha_2 + ... + \alpha_\ell = m$$

$$\alpha_1 + 2\alpha_2 + ... + \ell\alpha_\ell = 2k + 2$$

Therefore the function $f(\bar{g}^2)$ characterizes all the higher order contributions in QCD. The series (53) should be understood as the Taylor's expansion of the nonperturbative function in (34) for small $\bar{g}^2$.

From (53) we have

$$|f(\bar{g}^2)| \leq \sum_{k=1}^{\infty} \frac{|c_k|}{k}(\bar{g}^2)^k \tag{55}$$

We notice that each coefficient β_k in the series (42) must be finite, otherwise the perturbative expression for $\beta(\bar{g}^2)$ would be meaningless. Then the coefficients $b_k = \beta_k/\beta_0$ should be also finite. Let the largest absolute value of coefficient b_k be B:

$$Sup \, |b_k| = B \quad , \quad B > 0 \tag{56}$$

from (54) it can be proved that

$$|C_k| \leq B(1+B)^k \tag{57}$$

Substituting (57) into (55) we find

$$|C_k| \leq B \sum_{k=1}^{\infty} \frac{1}{k}(1+B)^k(\bar{g}^2)^k \tag{58}$$

This series is convergent when $\bar{g}^2 < \frac{1}{1+B}$, and the function $|f(\bar{g}^2)|$ satisfies the inequality

$$|f(\bar{g}^2)| \leq -B\, \ell n[1-(1+B)\,\bar{g}^2] = B\, \ell n \frac{1}{1-(1+B)\,\bar{g}^2} \tag{59}$$

which means when $\bar{g}^2 < \frac{1}{1+B}$, $B\, \ell n\, \frac{1}{1-(1+B)\bar{g}^2}$ is the upper bound function of $f(\bar{g}^2)$. From (58) we know that

$$f(\bar{g}^2) \to 0 \quad , \quad \text{when } \bar{g}^2 \to 0 \tag{60}$$

The property of $f(\bar{g}^2)$ for small $\bar{g}^2$ (60) and the constant $b > 0$ are of importance, because they guarantee that QCD is an asymptotically free theory for large Q^2. To see this we express (34) as

$$1 = 2bt_M\,\bar{g}^2 - a\,\bar{g}^2\,\ell n\,\bar{g}^2 + \bar{g}^2\, f(\bar{g}^2) \tag{61}$$

Using the formula

$$\bar{g}^2\,\ell n\,\bar{g}^2 \to 0, \qquad \text{when } \bar{g}^2 \to 0$$

and (60)

$$\bar{g}^2\, f(\bar{g}^2) \to 0, \qquad \text{when } \bar{g}^2 \to 0$$

we find that for small $\bar{g}^2$ (61) becomes $1 = 2bt_M\bar{g}^2$, which leads to the well-known asymptotic freedom formula

$$\bar{g}^2 = \frac{1}{2bt_M} = \frac{1}{b\,\ell n\frac{Q^2}{M^2}}, \qquad b > 0, \tag{62}$$

which shows that small $\bar{g}^2$ corresponds to $Q^2 \gg M^2$.

Since for small $\bar{g}^2$, $a|\ell n\,\bar{g}^2| \gg |f(\bar{g}^2)|$ (for small $\bar{g}^2$, $f(\bar{g}^2) \simeq c_1\bar{g}^2$), dropping higher term $f(\bar{g}^2)$ in eq. (38), we obtain expression up to two loops

$$\ell n\frac{Q^2}{M^2} = \frac{1}{b}\Big[\frac{1}{\bar{g}^2} + a\;\ell n\;\bar{g}^2\Big] \tag{63}$$

This means that if we choose the renormalization scheme in which g^2 is sufficiently small, the dynamic mass M^2 determined by (37) can be expressed as

$$b\;\ell n\frac{\mu^2}{M^2} = \frac{1}{g^2} + a\;\ell n\;g^2 \tag{64}$$

Iterating the expression of (64) we obtain the renormalized coupling constant

$$\frac{1}{g^2} = b\;\ell n\frac{\mu^2}{M^2} + a\;\ell n\;\ell n\frac{\mu^2}{M^2} \tag{65}$$

which depends only on the one– and two–loop renormalization group coefficients.[15] But (64) is more rigorous then (65).

4. The Infrared Behavior of $\bar{g}^2(t)$

In order to study the infrared behavior of the running coupling constant $\bar{g}^2(t)$, we write (34) as

$$\frac{1}{2}\ell n\frac{Q^2}{M^2} = \frac{1}{2b}\Big[\frac{1}{\bar{g}^2} + a\;\ell n\,\bar{g}^2 - f(\bar{g}^2)\Big] = \Psi(\bar{g}^2) \tag{66}$$

where $a > 0, b > 0$. From (66) we find that when Q^2 decreases from $+\infty$ to 0, the function $\Psi(\bar{g}^2)$ decreases continuously from $+\infty$ to $-\infty$. We know from (62) that when $Q^2 \to \infty, \bar{g}^2 \to 0$, this gives $\bar{g}^2 \to 0$, $\Psi(\bar{g}^2) \to +\infty$. Suppose that $\Psi(\bar{g}^2)$ is a continuous function of $\bar{g}^2$ in the interval $+\infty > \bar{g}^2 > 0$, then $\Psi(\bar{g}^2)$ is finite for finite $\bar{g}^2$. Excluding the case $\bar{g}^2 \to 0$ which corresponds to $\Psi(\bar{g}^2) \to +\infty$ [see (60) and (66)], there is only one possibility for the value of $\bar{g}^2$ for $\Psi(\bar{g}^2) \to -\infty$, this is $\bar{g}^2 \to +\infty$, when $Q^2 \to 0$.

Therefore in the infrared limit $Q^2/M^2 \ll 1, \bar{g}^2$ is large, and (66) can be expressed as

$$b\ \ell n \frac{Q^2}{M^2} = a\ \ell n\, \bar{g}^2 - f(\bar{g}^2) \tag{67}$$

Since when $Q^2/M^2 \ll 1, \ell n(Q^2/M^2) \ll 0$, from (67) we have the inequality

$$f(\bar{g}^2) > a\ \ell n\, \bar{g}^2 , \qquad \text{for large } \bar{g}^2 \tag{68}$$

Moreover, from (35) we find that for large $\bar{g}^2$ (when $\bar{g}^2 \gg \frac{1}{a}$), $\beta(\bar{g}^2)$ takes the form

$$\beta(\bar{g}^2) = \frac{-2b\,\bar{g}^2}{\bar{g}^2 f'(\bar{g}^2) - a} \qquad (a > 0, b > 0) \tag{69}$$

which tells us that the infrared behaviour of $\beta(\bar{g}^2)$ is determined by the property of the function $\bar{g}^2 f'(\bar{g}^2)$ for large $\bar{g}^2$.

We continue to study the behavior of $\bar{g}^2(t)$ by means of the dispersion relation theory. To do this, we define a function

$$T(t) = \ell n \frac{\bar{g}^2(t)}{g^2} \tag{70}$$

and study its analytic properties in complex plane of t. From (34) the function $T(t)$ can be written as

$$T(t) = \frac{2b}{a} t + \frac{1}{a} V(t) \tag{71}$$

where

$$V(t) = [f(\bar{g}^2) - f(g^2)] - \left[\frac{1}{\bar{g}^2} - \frac{1}{g^2}\right] \tag{72}$$

Since when $Q^2 = \mu^2$, $t = 0$ and $\bar{g}^2 = g^2$, we have

$$V(0) = 0 \tag{73}$$

Using (4) and (73), in the neighborhood of the $t = 0$. The Taylor expansion of $V(t)$ gives

$$V(t) = h\ t \tag{74}$$

where h is a finite constant

$$h = \left[f'(g^2) + \frac{1}{g^4}\right]\beta(g^2) \tag{75}$$

Then in the neighborhood of $t = 0$, (71) can be expressed in the form

$$T(t) = \frac{2b + d}{a}\ t \tag{76}$$

Therefore in the neighborhood of $t = 0$, $T(t)$ is a regular function of t.

For Q^2 is large enough, $t_M = t = \frac{1}{2}\ell n Q^2$, substituting (62) into (70), we have

$$T(t) = -\ell n\ t, \quad \text{for large } t \tag{77}$$

This means that the function $T(t)$ possesses a branch cut along the real axis of t and a branch point is at $t = +\infty$. Since $T(t)$ is a regular function in the neighborhood of $t = 0$, the other branch point, say $t = \alpha$, must be on the positive real axis, that is $\alpha > 0$.

Now we consider a function defined by

$$G(t) = \frac{T(t)}{t^2} = \frac{1}{t^2}\ell n\ \frac{\bar{g}^2(t)}{g^2} \tag{78}$$

Using (70), (76) and (77), we find that this function has the following singularities and asymptotic properties:

(i) A branch cut along the positive real axis of t from α to $+\infty$. The branch point $\alpha > 0$ and $G(\alpha)$ is finite.

(ii) A simple pole at $t = 0$, which is not lying on the branch cut and the residue of $G(t)$ corresponding to this pole is

$$r_0 = \frac{zb + h}{a} \tag{79}$$

where the constant h is defined by (75).

(iii) $|t|G(t) \to 0$, when $|t| \to \infty$.

(We suppose that there is no branch cut along the negative real axis of t.)

These properties are of importance to derive the dispersion relation for $G(t)$. We take a contour C that excludes the branch cut and the pole of $G(t)$ in the usual way in the theory of complex variables so that we can apply Cauchy's integral theorem:

$$G(t) = \frac{1}{2\pi i} \int_C \frac{G(t')dt'}{t' - t} \tag{80}$$

Since $|t|G(t) \to 0$, when $|t| \to \infty$, the integral around the large contour vanishes. Then (80) gives the dispersion relation for $G(t)$:

$$G(t) = \frac{r_0}{t} + \frac{1}{\pi} \int_\alpha^\infty \frac{A(t')dt'}{t'^2(t' - t)} \tag{81}$$

where $A(t)$ is the difference of the discontinuity of $K(t)$ along the cut and is given by

$$A(t) = \frac{1}{2i}[K(t + i\epsilon) - K(t - i\epsilon)] \tag{82}$$

It is well-known that if $K(t)$ satisfies the condition

$$K^*(t^*) = K(t)$$

then $A(t)$ is the imaginary part of $K(t)$;

$$A(t) = Im\ K(t)$$

In this paper we shall not concern with this condition.

Substituting (78) into (81), we have the following dispersion relation for $\bar{g}^2(t)$:

$$\ell n\ \frac{\bar{g}^2(t)}{g^2} = r_0\ t + \frac{t^2}{\pi}\int_{\alpha}^{\infty} \frac{A(t')dt'}{t'^2(t'-t)} \tag{83}$$

We notice that for all value of t off the cut the denominator $(t' - t)$ in the integral of the above expression is never zero and therefore the integral is well defined in this region. In the infrared limit $Q^2 \ll \mu^2$, $t \ll 0$ $(t \to -\infty)$, t is far away from the cut, in this case (83) becomes

$$\ell n\ \frac{\bar{g}^2(t)}{g^2} = r_0\ t + \frac{t}{\pi}\int_{\alpha}^{\infty} \frac{A(t')dt'}{t'^2} \quad \text{for}\ Q^2 \ll \mu^2 \tag{84}$$

If we define a constant

$$k = -\left(r_0 + \frac{1}{\pi}\int_{\alpha}^{\infty} \frac{A(t')dt'}{t'^2}\right), \tag{85}$$

then from (3) and (84), we find that $\bar{g}^2$ has the following infrared behavior

$$\bar{g}^2(Q^2) = g^2\left(\frac{\mu^2}{Q^2}\right)^k, \qquad \text{for}\ Q^2 \ll \mu^2 \tag{86}$$

We have argued at the beginning of this section that when $Q^2 \to 0$, $\bar{g}^2 \to +\infty$, therefore k must be a positive constant:

$$k > 0 \tag{87}$$

Using (35), (75) and (79), it can be proved that

$$r_0 = -\frac{2bg^2}{1 - ag^2 + g^4 f'(g^2)} = \frac{1}{g^2}\beta(g^2) \tag{88}$$

Usually, we always choose the renormalization point that $\beta(g^2)$ is negative, therefore

$$r_0 < 0$$

Then from (85) and (87), we have the inequality

$$\frac{1}{\pi}\int_{\alpha}^{\infty} \frac{A(t')dt'}{t'^2} < \frac{1}{g^2}|\beta(g^2)| \tag{89}$$

In the infrared limit, taking into account of (68), (34) can be written as

$$f(\bar{g}^2) = a\ell n\ \frac{\bar{g}^2}{g^2} + b\ \ell n\ \left(\frac{\mu^2}{Q^2}\right)$$

Substituting (86) into above equation we find that for large $\bar{g}^2$

$$f(\bar{g}^2) = c\ \ell n\ \bar{g}^2 \tag{90}$$

where

$$c = a + \frac{b}{k} > 0 \tag{91}$$

Using (68) and (90), we have $c > a$, then from (91) we see again that $k > 0$.

From (69) and (90), we obtain for large $\bar{g}^2$

$$\beta(\bar{g}^2) = -\frac{2b}{c-a}\ \bar{g}^2 = -k\,\bar{g}^2 \tag{92}$$

which tells us in the infrared limit, $\bar{g}^2(t)$ satisfies the equation

$$\frac{d\,\bar{g}^2(t)}{dt} = -k\ \bar{g}^2(t) \tag{93}$$

We notice that equation (93) is invariant under the scale transformation

$$\bar{g}'^2 = \lambda\ \bar{g}^2 \tag{94}$$

where λ is an arbitrary constant. This transformation is essential to the renormalization group theory in QCD for small Q^2.

To study the meaning of the scale transformation (93). Let $x = \bar{g}^2$ and $y = \bar{g}'^2$ be two running coupling constants corresponding to two different renormalization schemes respectively and they are related by the scale transformation

$$y = \lambda x \tag{94'}$$

It is well–known that in renormalization group theory x and y should satisfy the relation[16]

$$\frac{dy}{\bar{\beta}(y)} = \frac{dx}{\beta(x)} \tag{95}$$

where $\bar{\beta}(y)$ and $\beta(x)$ are the Callan-Symanzik functions corresponding to two different renormalization schemes respectivey. In our theory (see eq. (35))

$$\beta(x) = \frac{-2bx^2}{1 - ax + x^2 f'(x)}$$

and

$$\bar{\beta}(y) = \frac{-2by^2}{1 - ay + y^2 \bar{f}'(y)}$$

where $f(x)$ and $\bar{f}(y)$, in general are two different functions.

For small Q^2, x and y are large, $\beta(x)$ and $\bar{\beta}(y)$ can be written as

$$\beta(x) = \frac{-2bx}{xf'(x) - a} \quad \text{and} \quad \bar{\beta}(y) = \frac{-2by}{y\bar{f}'(y) - a} \tag{96}$$

Substituting (96) into (95), we have

$$\frac{d\ell n\ y}{d\ell n\ x} = \frac{xf'(x) - a}{y\bar{f}'(y) - a} \tag{97}$$

Since for the scale transformation (94)′

$$\frac{d\ell n\ y}{d\ell n\ x} = 1$$

then from (97) we obtain an equation

$$x f'(x) = y \bar{f}'(y) \tag{98}$$

Substituting (94)′ into (98), we find

$$x f'(x) = \lambda x \bar{f}'(\lambda x) \quad \text{for large } x$$

this equation can be expressed as

$$x f'(x) = \frac{d\bar{f}(\lambda x)}{d\ell n\ \lambda} \tag{99}$$

The left hand side of (99) is a function of x alone, it requires that the derivative of $\bar{f}(\lambda x)$ with respect to $\ell n\ \lambda$ is to be independent of λ. The only solution for $\bar{f}(\lambda x)$ satisfies this requirement is

$$\bar{f}(\lambda x) = c\ell n\ x + c\ell n\ \lambda$$

where c is a constant. Substituting it into (99), we have

$$x f'(x) = c \qquad \text{for large } x \tag{100}$$

which leads to

$$f(x) = c\ell n x \qquad \text{for large } x$$

From (98) and (100), we have the analogous equation

$$y \bar{f}'(y) = c$$

and

$$\bar{f}(y) = c\ell n\ y \qquad \text{for large } y$$

we see that $\bar{f}(y)$ and $f(x)$ have the same form and the constant c is also the same. This means if in the infrared limit two running coupling constants in different renormalization schemes are related by the scale transformation (94), the unknown function $f(\bar{g}^2)$ in our theory must take the unique form (90).

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