SLAC – PUB – 3253
November 1983
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CASIMIR ENERGY OF A SCALAR FIELD WITH A SPACE-DEPENDENT MASS DISTRIBUTION*

HIDEAKI AOYAMA

Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Submitted for publication.

* Work supported by the Department of Energy, contract DE – AC03 – 76SF00515

ABSTRACT

The Casimir energy is evaluated for a free scalar field that has a mass term $m^2(x_1)$, depending on one space coordinate x_1. The formalism for evaluating the Casimir energy is developed for the case of $m^2(x_1)$ finite everywhere in d-dimensional space-time. The case with $m^2(x_1) = m_0^2\theta(L/2 - |x_1|) + m_\infty^2\theta(|x_1| - L/2)$ is explicitly evaluated for any value of m_0 and m_∞ without any approximation. The result consists of volume energy terms, a surface energy term, and a non-leading term. Most of the UV divergences are in the volume energy terms and renormalize the coupling constants of the underlying theory. The surface energy term is finite for $d \leq 4$ and divergent for $d \geq 5$ due to the boundaries being sharp. A closed finite expression is obtained for the non-leading term. Our results are shown to reproduce the known Casimir energies for the limiting cases, $m_0 \to \infty$ and $m_\infty \to \infty$.

1. Introduction

The Casimir energy is a quantum correction of the energy of a finite sized classical object due to its interaction with quantized fields. It usually depends differently on the size and shape of the object than the classical energy, and therefore is potentially important whenever we deal with the dynamics of such objects. Some earlier historical examples include conducting spheres and plates. Casimir[1] derived an attractive force $\propto 1/L^4$ between two neutral conducting plates with distance L, which actually is in agreement with experiment[2]. The van der Waals force between neutral atoms was also explained as a Casimir effect[3]. More recently, the Casimir energy has played a role in the bag model of hadrons[4], where the quark and gluon field confined in a bag provided a quantum energy $\propto 1/(\text{bag radius})$.

There are new classes of finite-sized objects which come into sight in the theory of the very early universe. They include the bubbles nucleated during a first order phase transition, the Higgs field fluctuations during the slow roll-overs, domain walls and cosmic strings. The Casimir energy could be potentially important for their dynamics whenever other quantum corrections are essential. In fact, some cosmological scenarios are based on a Coleman-Weinberg type theory, where one-loop quantum corrections are essential for symmetry breaking. People often treat finite-sized objects such as Higgs fluctuations using only the effective potential. This corresponds to neglecting the kinematic part of the full one-loop correction to the effective action and therefore cannot be regarded as a consistent procedure.

Traditionally the Casimir energy has been calculated for cases when objects act on the quantized field as boundary conditions. In many physically interesting cases, especially for the objects mentioned above, this is not so. Typically, these objects consist of a coherent background field $\phi_c(x)$. It affects its fluctuations ϕ_f through self-coupling $V(\phi_c + \phi_f)$. If ϕ_c is infinite in a region of space, its effect on quantum field could be counted as a boundary condition. However, ϕ_c is usually finite everywhere in the above-mentioned cases. While this finiteness causes some calculational difficulty, it is useful for clarifying the origin of divergences in Casimir energy calculations. Calculations based on boundary conditions suffers from *UV* divergences that are either intrinsic to the theory or due to ϕ_c being infinite in a region of space. When we deal with a ϕ_c that is finite everywhere, we should be able to separate the *UV* divergences intrinsic to the theory and renormalize them away.

This paper deals with these "finite" ϕ_c's. We work at full one loop order and do not use the Feynman graph expansion, or "multiple reflection expansion"[5]. This enables us to keep to same order approximation to the effective potential and thus makes the separation of the *UV* divergences straightforward. We discuss a single scalar field ϕ_f with a mass term $m^2(x_1)$ that depends only on one space coordinate x_1 of d-dimensional space-time. As an underlying theory one can imagine a single scalar field ϕ with a renormalizable self-interaction $V(\phi)$. A coherent background $\phi_c(x_1)$ provides a mass $m^2(x_1) = d^2V(\phi_c)/d\phi_c^2$ to its fluctuation $\phi_f = \phi - \phi_c$. In the next section, we present the formalism and an approximate scheme suitable for $m^2(x_1)$ consisting of a constant central region and outside regions. (Fig. 1). In Section 3, we explicitly evaluate the case with sharp boundaries, $m^2(x_1) = m_0^2\theta(L/2-|x_1|)+m_\infty^2\theta(|x_1|-L/2)$, for arbitrary real values of m_0 and m_∞. Section 4 gives some discussions including the comparison of our results with known special cases.

2. Formalism

Casimir energy E_c is the sum of the zero energies of excitation modes The of the quantized field ϕ_f,

$$E_c = \sum_n \frac{\omega_n}{2} . \tag{2.1}$$

The nth excitation energy ω_n is given by the Klein-Gordon equation,

$$\Big(\omega_n^2 + \Delta - m^2(x)\Big)\phi_f(x) = 0 , \tag{2.2}$$

with an appropriate boundary condition. When $m^2(x)$ depends only one one coordinate x_1, we separate the parallel coordinates $2 \sim d-1$ to obtain

$$E_c = L_p^{d-2} \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int dp_1 \rho(p_1) \frac{\sqrt{p_1^2 + p_p^2}}{2} , \tag{2.3}$$

where L_p denotes the width of space in the parallel directions, which is much bigger than any length scale in $m^2(x_1)$, and $\rho(p_1)$ the density of levels of the separated Klein-Gordon equation,

$$\Big(-\frac{d}{dx_1^2} - m^2(x_1)\Big)\phi_f(x_1) = p_1^2 \phi_f(x_1) . \tag{2.4}$$

In general, $\rho(p_1)$ consists of a continuous spectrum and a discrete spectrum,

$$\rho(p_1) = \rho_c(p_1)\theta(p_1 - m_\infty) + \sum_n \delta(p_1 - p_{(n)}) , \tag{2.5}$$

where $m_\infty = \min\{m(\infty), m(-\infty)\}$. In this section, we discuss techniques to deal with each of the spectrums separately. The close connection between the formulae in each case will be demonstrated with examples in the next section. Hereafter, we consider cases when $m^2(x_1)$ consists of one central flat region of width L, with $m^2(x_1) = m^2(-x_1)$. Extensions to more general cases are straightforward.

(i) Continuous Spectrum

The technique developed here is based on the use of phase shifts, which have been applied to two-dimensional soliton mass correction calculations[6], and on an approximation method[7]. We present them here in a slightly different manner suitable for our application.

In order to obtain ρ_c in (2.5) we first need to solve (2.4) and obtain standing wave solutions. From them, we find the phase shifts, $\delta_e(p_1)$ and $\delta_o(p_1)$ for the even and odd modes, defined in the following asymptotic behavior for $x_1 \to \pm\infty$,

$$\phi_f(x_1) \to \begin{cases} \cos\left(px_1 \pm \dfrac{\delta_e(p_1)}{2}\right) & \text{even modes,} \\ \sin\left(px_1 \pm \dfrac{\delta_o(p_1)}{2}\right) & \text{odd modes,} \end{cases} \tag{2.6}$$

where $p \equiv \sqrt{p_1^2 - m_\infty^2}$. By requiring periodic boundary conditions at $x_1 = \pm L_1/2$ ($L_1 \gg L$), we get a discrete spectrum given by,

$$p\,L_1 + \delta(p_1) = 2\pi n \ , \tag{2.7}$$

for each of the even and odd modes. As $L_1 \to \infty$, the continuous spectrum is obtained with level density dn. Thus the total density is

$$\rho_c(p_1)dp_1 = \frac{L_1}{\pi}\,dp + dG(p_1)\ , \tag{2.8}$$

where

$$G(p_1) \equiv \frac{\delta_o(p_1) + \delta_o(p_1)}{2\pi}\ . \tag{2.9}$$

One should note that for $p = 0$ (2.6) does not uniquely determine δ's, and therefore the above formulae are true only for $p > 0$. This arbitrariness is resolved when one goes back to the finite L_1 formula (2.7). This will be done for the examples in the next section. The L_1-term in (2.8) gives the volume energy of the uniform mass case, $m(x_1) \equiv m_\infty$, since it leads to

$$L_p^{d-2} L_1 \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_0^\infty \frac{dp}{\pi} \frac{\sqrt{p^2 + p_p^2 + m_\infty^2}}{2} . \tag{2.10}$$

The momentum integral in the above is merely a one-loop correction to the effective potential at the constant value of ϕ_c that gives m_∞.

An approximation scheme for the phase shifts has been devised for cases when $m^2(x_1)$ is sufficiently constant in the central region and has relatively well-defined boundaries (at $x_1 = \pm L/2$, see Fig. 1). In such a case, a solution of (2.4) can well be approximated by a free solution for $x_1 \sim 0$. This allows us to construct approximate solutions obeying (2.6) from solutions of (2.4) in the half-space $x_1 > 0$. For $p_1 > \max\{m(0)(\equiv m_0), m_\infty\}$ we define amplitudes $A(p_1)$ and $B(p_1)$ from the following half-space solution,

$$\phi_f(x_1) \sim \begin{cases} e^{ip'(x_1 - L/2)} & \text{for } x_1 \sim 0 , \\ A(p_1) e^{ip(x_1 - L/2)} + B(p_1) e^{-ip(x_1 - L/2)} & \text{for } x_1 \sim \infty , \end{cases} \tag{2.11}$$

where $p' \equiv \sqrt{p_1^2 - m_0^2}$. The exponents of $L/2$ are chosen such that A and B are independent of L and depend only on the characters of the boundaries. After the appropriate reflection, complex conjugation and translation, we find a solution in $x_1 < 0$. By connecting them at $x_1 = 0$, we arrive at solutions in whole space, from which the phase shifts δ_e and δ_o are found. As a result, we have

$$G(p_1) = -\frac{1}{\pi} pL + \frac{1}{\pi} p'L + \frac{1}{\pi} \arg A^2 + \frac{1}{\pi} \arg \left(1 - \left(\frac{B^*}{A}\right)^2 e^{-2ip'L}\right) . \tag{2.12}$$

The branches of arg should be chosen such that $G(p_1)$ is continuous. Due to flux conservation $p' = p(|A|^2 - |B|^2)$ in the half space solution (2.11), we demand

$$\left|\frac{B^*}{A}\right|^2 < 1 . \tag{2.13}$$

Thus the second arg in (2.12) stays in one branch as p_1 varies. We choose it to be $-\pi \sim \pi$. Since $A(p_1) \to 1$ and $B(p_1) \to 0$ as $p_1 \to \infty$, this choice leads to vanishing arg term in that limit. It is also convenient to choose the branches of arg A^2 so that it vanishes in the same limit. When $m_0 > m_\infty$, we need another set of amplitudes $A_\pm(p_1)$ defined as follows

$$\phi_f(x_1) \sim \begin{cases} e^{\pm\rho(x-L/2)} & \text{for } x_1 \sim 0 \,, \\ A_\mp(p_1)e^{ip(x-L/2)} + e.c. & \text{for } x_1 \sim +\infty \,, \end{cases} \tag{2.14}$$

where $\rho \equiv \sqrt{m_0^2 - p_1^2}$. A similar construction gives for $m_0 > p_1 > m_\infty$

$$G(p_1) = -\frac{1}{\pi}pL + \frac{1}{\pi}\ \arg\,(iA_-^2) + \frac{1}{\pi}\ \arg\,\left(1 - \left(\frac{A_+}{A_-}\right)^2 e^{-2\rho L}\right) . \tag{2.15}$$

In this case, we have $|A_+/A_-| = 1$. Thus, the branch of the second arg can be chosen to be $-\pi \sim +\pi$. The phase of arg (iA^2) should be chosen by considering the continuity of $G(p_1)$.

(ii) Discrete Spectrum

The eigenvalue problem for each of the even and odd sectors should reduce to the following type of equation,

$$n = f(p_1) \,, \tag{2.16}$$

which gives a unique solution of p_1 for each integer of n in an appropriate range. We assume that f is a monotonically increasing function of p_1. The level density of this discrete spectrum is given by the following

$$\sum_n \delta(p_1 - p_{(n)}) = \sum_n \delta(n - f(p_1))f'(p_1) \quad (\equiv \rho_d(p_1)) \,, \tag{2.17}$$

where $p_{(n)}$ is the solution of (2.17). The n-summation in the above is done by applying the Poisson formula

$$\sum_{n=-\infty}^{+\infty} F(n) = \sum_{j=-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(k)\, e^{2\pi ijk} dk \ . \tag{2.18}$$

(The fact that (2.18) is a finite sum while (2.19) is not is irrelevant because we always can multiply $F(n)$ by an appropriate cutoff function of n. This does not affect $\rho_d(p_1)$, since for a value of p_1 only one n is important.) As a result, we find that for each sector ρ_d is expressed as follows

$$\rho_d(p_1)dp_1 = d\, g(p_1) \tag{2.19}$$

$$g(p_1) = f(p_1) + \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{1}{j} \sin(2\pi j f(p_1)) \ . \tag{2.20}$$

Applying a summation formula, we find that

$$\begin{aligned} g(p_1) &= f(p_1) + \frac{1}{\pi} \arg \left(1 - e^{-2\pi i f(p_1)}\right) , \\ &= \left[f(p_1)\right] + \frac{1}{2} , \end{aligned} \tag{2.21}$$

where the arg is in the range $-\pi \sim \pi$.

As with to the continuous spectrum, an approximation scheme can be devised. For $x_1 > 0$, we can have a solution that behaves as follows,

$$\phi(x_1) \sim \begin{cases} A' e^{-p(x-L/2)} + e.c. & \text{for } x_1 \sim 0 \ , \\ e^{-\rho'(x-L/2)} & \text{for } x_1 \to \infty \end{cases} \tag{2.22}$$

where $\rho' \equiv \sqrt{p_1^2 - m_\infty^2}$. It is straightforward to show that the f's in (2.16) are given by

$$f_e(p_1) = \frac{1}{2\pi} p' L + \frac{1}{\pi} \arg A' \ , \quad f_o(p_1) = f_e(p_1) + \frac{1}{2} \ , \tag{2.23}$$

for even and odd modes. Following (2.19)-(2.21), we find that the total level density is given by

$$\rho_d(p_1)dp_1 = dG(p_1)\ ,$$

$$\begin{aligned} G(p_1) &= g_e(p_1) + g_o(p_1) \\ &= \frac{1}{\pi}p'L + \frac{1}{\pi}\ \arg\left(iA'^2\right) + \frac{1}{\pi}\ \arg\left(1 - \left(\frac{1}{A'}\right)^4 e^{-2ip'L}\right) . \end{aligned} \tag{2.24}$$

One advantage of our scheme is that the last terms in (2.12), (2.15) and (2.24) are related to each other on the Rieman surface of p_1, allowing a unified treatment for the integration (2.3). Also, the major volume energy terms are already separated as L-terms,

$$\int dp_1\, \rho(p_1) = (L_1 - L)\int_0^\infty \frac{dp}{\pi} + L\int_0^\infty \frac{dp'}{\pi} + (\text{arg terms})\ . \tag{2.25}$$

These points will be demonstrated with examples in the next section.

3. Examples with Sharp Boundaries

In this section we evaluate the cases with sharp boundaries,

$$m^2(x_1) = m_0^2\theta\Big(\frac{L}{2} - |x_1|\Big) + m_\infty^2\theta\Big(|x_1| - \frac{L}{2}\Big)\,, \tag{3.1}$$

for any positive or zero m_0, m_∞, and L. For cases when either m_0 or m_∞ are infinite, the Casimir energies are known. The author has given the results for $m_\infty = 0$.[7] The results obtained in this section will be discussed and compared with them in the next section.

We note that in this case the approximation formulae (2.11)-(2.15) and (2.22)-(2.25) are exact since $m^2(x_1)$ is constant for $x_1 \sim 0$. However, we use them here as a guideline to be followed. The evaluation is done for two separate cases, (i) $m_0 > m_\infty$ and (ii) $m_0 < m_\infty$. Note that the notation is $p_1^2 = p^2 + m_\infty^2 = -\rho'^2 + m_\infty^2 = p' + m_0^2 = -\rho^2 + m_0^2$ and $m^2 \equiv |m_0^2 - m_\infty^2|$.

(i) $m_0 > m_\infty$

In this case the spectrum consists of only the continuous one, $p_1 > m_\infty$. The phase shifts are

$$\delta_e(p_1) = -p\,L + 2\tan^{-1}\Big(\frac{p'}{p}\tan\frac{p'L}{2}\Big)\,, \tag{3.2a}$$

$$\delta_o(p_1) = -p\,L + 2\tan^{-1}\Big(\frac{p}{p'}\tan\frac{p'L}{2}\Big)\,. \tag{3.2b}$$

The above lead to the following expressions that correspond to (2.12) and (2.15),

$$G(p_1) = \begin{cases} \dfrac{1}{\pi}(-p+p')L + \dfrac{1}{\pi}\ \arg\Big(1 - \Big(\dfrac{p-p'}{m}\Big)^4 e^{-2ip'L}\Big) & \\ \qquad \text{for } p_1 > m_0 & \text{(3.3a)} \\ -\dfrac{1}{\pi}p\,L + \Big(\dfrac{1}{2} - \dfrac{2}{\pi}\cos^{-1}\dfrac{p}{m}\Big) + \dfrac{1}{\pi}\ \arg\Big(1 - \Big(\dfrac{p+i\rho}{m}\Big)^4 e^{-2\rho L}\Big) & \\ \qquad \text{for } m_0 > p_1 > m_\infty & \text{(3.3b)} \end{cases}$$

where $m \equiv \sqrt{|m_0^2 - m_\infty^2|}$. We choose $\cos^{-1}$ to take its principal value so that $G(p_1)$ is continuous at $p_1 = m_0$. This $G(p_1)$ has the limit $-\frac{1}{2}$ for $p_1 \to m_\infty$. However, $G(p_1)$ should be understood to be zero at $p_1 = m_\infty$ and sharply falls off to $-\frac{1}{2}$ as $p_1 \to m_\infty + 0$. That is, the level density ρ_c has a $-\frac{1}{2}\delta(p_1 - m_\infty - 0)$, not given in (2.8), which contributes to the integral (2.3). It is straightforward to see that otherwise we encounter a contradiction. Namely, for $L \to 0$, G goes to 0 except for $p_1 \leq m_\infty + m^4L^2/m_\infty$ $(p \lesssim m^2L)$ where $G \sim -\frac{1}{2}$ (see Fig. 2(a)). Thus, (2.8) gives $\rho_c \sim +\frac{1}{2}\,\delta(p_1 - m_\infty - 0)$ as $L \to 0$. This contributes an extra amount other than the pure volume energy piece (2.10) to the Casimir energy. The $-\frac{1}{2}\,\delta$ mentioned above cancels its contribution as $L \to 0$. The existence of this gap at $p_1 \sim m_\infty$ can be understood from the finite L_1 formalism. Imagine that we keep L_1 finite and use the Poisson formula (2.19) and (2.21). The discrete spectrum is given by the derivative of the following

$$G_{L_1}(p_1) = \left[\frac{pL_1}{2\pi} + \frac{\delta_e(p_1)}{2\pi}\right] + \left[\frac{pL_1}{2\pi} + \frac{\delta_o(p_1)}{2\pi}\right] + 1 \equiv \frac{pL_1}{\pi} + F(p_1)\,. \qquad (3.4)$$

For large L_1, $F(p_1)$ rapidly oscillates around $G(p_1)$ with the amplitude ~ 1 and the period $\sim \pi/L_1$, as illustrated in Fig. 2(b). As $L_1 \to \infty$, the integrals including $F(p_1)$ (but not its derivatives) converge to the integrals with $F(p_1)$ replaced by $G(p_1)$. For integrals with dF/dp_1, such as (2.3), F should be replaced by G with the right boundary value $G(m_\infty) = F(m_\infty)$, which in our case if zero (see (2.2) and (3.2)). Therefore, G should be zero at $p_1 = m_\infty$, and for $\Delta p \sim \pi/L_1$ $(\Delta p_1 \sim \pi^2/2m_\infty L_1^2)$ it recovers to its value given by (3.3b).

The p_p-integration in (2.3) can be carried out to yield the following expression,

$$E_c = A \int dp_1\, p_1^{d-1} \rho(p_1)\,, \qquad (3.5)$$

where

$$A \equiv -\frac{\Gamma\left(\frac{1-d}{2}\right)}{2(4\pi)^{\frac{d-1}{2}}}. \qquad (3.6)$$

Let us denote the nontrivial part of the Casimir energy per unit "area" as e_c,

$$E_c = (\text{volume energy}) + L_p^{d-2} e_c \,. \tag{3.7}$$

Substituting (3.3) into (2.8) and partially integrating over p, we find that the contribution of $p_1 > m_0$ to e_c is

$$e_{c+} = A\Big(B_1 + \frac{m_0^{d-1}}{\pi}\tan^{-1}\frac{2}{mL}\Big) - \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_m^\infty dp \, \frac{p}{2\sqrt{p^2+p_p^2+m_\infty^2}} \, \frac{\arg}{\pi} \,, \tag{3.8}$$

where $\arg/\pi$ denotes the second term in (3.3a) and the first two terms comes from the boundary value of G,

$$B_1 = -\frac{1}{2} m_0^{d-1} \,. \tag{3.9}$$

Similarly from $m_0 > p_1 > m_\infty$, we obtain the following

$$e_{c-} = A\Big(D_1 - \frac{m_0^{d-1}}{\pi}\tan^{-1}\frac{2}{mL}\Big) - \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_0^m dp \, \frac{p}{2\sqrt{p^2+p_p^2+m_\infty^2}} \, \frac{\arg}{\pi} \tag{3.10}$$

$$D_1 = -\frac{1}{2} m_\infty^{d-1} + \int_0^m dp \, \frac{2}{\pi} \, \frac{(p^2+m_\infty^2)^{\frac{d-1}{2}}}{\sqrt{m^2-p^2}} \,. \tag{3.11}$$

In the Appendix, we evaluate e_{c+} and e_{c-} separately. For E_c, it is enough to notice that the third terms in (3.8) and (3.10) combine into one integration in the complex p-plane,

$$\frac{-i}{2} \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_{-\infty+i\epsilon}^{\infty+i\epsilon} dp \, \frac{p}{2\sqrt{p^2+p_p^2+m_\infty^2}} \, \ell n\Big(1 - \Big(\frac{p-p'}{m}\Big)^4 e^{-2ip'L}\Big) \,. \tag{3.12}$$

The integral can be rewritten as an integral along the cut on the positive imaginary axis, $\mathrm{Im}\, p (= k) > \sqrt{p_p^2 + m_\infty^2}$. Integrating p_p first, we arrive at the following expression for e_c,

$$e_c = S(m_0, m_\infty) + T_1(m_0, m_\infty, L) \tag{3.13}$$

$$\begin{aligned} S(m_0, m_\infty) &= A(B_1 + D_1) \\ &= A\left(-\frac{m_0^{d-1} + m_\infty^{d-1}}{2} + \int_0^m dp \, \frac{2}{\pi} \frac{(p^2 + m_\infty^2)^{\frac{d-1}{2}}}{\sqrt{m^2 - p^2}} \right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} T_1(m_0, m_\infty, L) &= F \int_{m_0}^{\infty} dk \, k(k^2 - m_\infty^2)^{(d-3)/2} \\ &\times \ell n\left(1 - \left(\frac{\sqrt{k^2 - m^2} - k}{m} \right)^4 e^{-2L\sqrt{k^2+m^2}} \right) \\ F &\equiv \frac{1}{(4\pi)^{\frac{d-2}{2}}} \frac{1}{\Gamma\left(\frac{d-1}{2}\right)} \end{aligned} \tag{3.15}$$

For $L \to \infty$, T_1 decays exponentially (see the Appendix for the derivation and further expansions),

$$T_1 \to -\frac{1}{2} \left(\frac{m_0 - m_\infty}{m_0 + m_\infty} \right)^2 \left(\frac{m_0}{4\pi L} \right)^{\frac{d-1}{2}} e^{-2m_0 L} + \cdots \tag{3.16}$$

The S-term is called the "surface" energy term, since it remains finite for $L \to \infty$ and is proportional to the "area" L_p^{d-2}.

(ii) $m_0 < m_\infty$

In this case, we have a discrete spectrum ($m_0 < p < m_\infty$) in addition to the continuous spectrum ($m_\infty < p$). For a continuous spectrum (3.2)-(3.3a) applies. The discrete spectrum is given by the following eigenvalue equations,

$$p' \tan \frac{p'L}{2} = \rho' , \qquad \frac{1}{p'} \tan \frac{p'L}{2} = -\frac{1}{\rho'} . \tag{3.17}$$

These equations are equivalent to the following,

$$n_e = \frac{1}{2\pi}\left(p'L - 2\cos^{-1} \frac{p'}{m} \right), \qquad n_o = \frac{1}{2\pi}\left(p'L - 2\cos^{-1} \frac{p'}{m} + \pi \right). \tag{3.18}$$

Corresponding to (2.24), we obtain,

$$G(p_1) = \frac{1}{\pi} p' L + \left(\frac{1}{2} - \frac{2}{\pi} \cos^{-1} \frac{p'}{m}\right) + \frac{1}{\pi} \arg \left(1 - \left(\frac{p' + i\rho'}{m}\right)^4 e^{-2ip'L}\right) . \quad (3.19)$$

We choose the overall phase of arg such that $G(m_0) = 0$. The $G(p_1)$ given by (3.4) and (3.19) has a discontinuity 1/2 at $p_1 = m_\infty$,

$$G(m_\infty - 0) = G(m_\infty + 0) + \frac{1}{2} . \quad (3.20)$$

For the same reason as in case (i), this discontinuity should contribute to the spectrum. That is, $\rho(p_1)$ has an extra $-\frac{1}{2}\,\delta(p_1 - m_\infty)$ not given by the derivatives of (3.3a) and (3.19).

It is now straightforward to show that the nontrivial part e_c of the Casimir energy has the following contributions from the continuous spectrum and from the discrete spectrum:

$$\begin{aligned} e_{c+} = A\Bigg(B_2 - m_\infty^{d-1}\Big(\left[\frac{Lm}{\pi}\right] - \frac{Lm}{\pi} + \frac{1}{2}\Big)\Bigg) \\ - \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_m^\infty dp' \frac{p'}{2\sqrt{p'^2 + p_p^2 + m_0^2}} \frac{\arg}{\pi} , \end{aligned} \quad (3.21)$$

$$B_2 = -\frac{1}{2} m_\infty^{d-2} , \quad (3.22)$$

$$\begin{aligned} e_{c-} = A\Bigg(D_2 + m_\infty^{d-1}\Big(\left[\frac{Lm}{\pi}\right] - \frac{Lm}{\pi} + \frac{1}{2}\Big)\Bigg) \\ - \int \frac{dp_p^{d-2}}{(2\pi)^{d-2}} \int_0^m dp' \frac{p'}{2\sqrt{p'^2 + p_p^2 + m_0^2}} \frac{\arg}{\pi} , \end{aligned} \quad (3.23)$$

$$D_2 = -\frac{1}{2} m_0^{d-1} + \int_0^m dp' (p'^2 + m_0^2)^{\frac{d-1}{2}} \frac{1}{\sqrt{m^2 - p'^2}} . \quad (3.24)$$

We find that the sum of the above is written as follows

$$e_c = S(m_\infty, m_0) + T_2(m_0, m_\infty, L) \tag{3.25}$$

where S is defined by (3.16) (note that the arguments are exchanged) and T_2 is expressed as an integral along the positive imaginary axis of p',

$$T_2(m_0, m_\infty, L) = F \int_{m_0}^{\infty} dk'k'(k'^2 - m_0^2)^{\frac{d-3}{2}} \ell n\left(1 - \left(\frac{\sqrt{k'^2+m^2} - p_2'}{m}\right)^4 e^{-2Lk'}\right). \tag{3.26}$$

After changing the variables by $k'^2 - m_0^2 = k^2 - m_\infty^2 = q^2$, we find that T_2 and T_1 of (3.16) represent the following same function T in different regions of the parameter space, $m_\infty > m_0$ and $m_0 > m_\infty$,

$$T(m_0, m_\infty, L) = F \int_0^{\infty} dq\, q^{d-2} \ell n\left(1 - \frac{\left(\sqrt{q^2+m_\infty^2} - \sqrt{q^2+m_0^2}\right)^4}{(m_\infty^2 - m_0^2)^2} e^{-2L\sqrt{p+m_0^2}}\right). \tag{3.27}$$

Therefore, the asymptotic behavior of T_2 is the same as (3.16) for $m_0 \neq 0$. Otherwise, it behaves as follows (for the derivation, see the Appendix),

$$T_2(0, m_\infty, L) \to -\frac{\Gamma(d/2)\varsigma(d)}{(4\pi)^{d/2}} \frac{1}{L^{d-1}} + \cdots \tag{3.28}$$

4. Conclusions

We have presented a formalism suitable for the evaluation of the Casimir energies of membrane (or slab)-like configurations in d-dimensional infinite space-time that give different masses, m_0 and m_∞, to a quantized scalar field inside and outside the membrane (Fig. 1). This formalism directly deals with the mode summation and does not allow any arbitrariness such as contact terms. An approximation scheme, which is valid when the mass variation at the center of the membrane is small, has been given. This formalism has been applied to the cases with sharp boundaries, (3.1). The level density $\rho(p_1)$ of the continuous spectrum is found to be a smooth function $dG(p_1)/dp_1$ plus $-\frac{1}{2}\,\delta(p_1-m_\infty)$, minus half of an isolated state of zero asymptotic momentum. The Casimir energy E_c is expressed as a sum of the volume energy, the surface energy S and a thickness dependent term T. These terms have the following properties:

(i) The volume energy is merely a (one-loop correction to the effective potential) $\times$ (the volume). Almost all the UV divergences in the Casimir energy are absorbed into this term and removed by usual renormalizations of the coupling constants of the basic theory.

(ii) The surface energy is found to be unique; it is given by $L_p^{d-2}S(\max\{m_0,m_\infty\},\ \min\{m_0,m_\infty\})$ (see (3.14) and (3.25)). This agrees with the physical intuition that it belongs to the boundary. In other words, a sharp, flat boundary that separates the regions of masses m_1 and m_2 ($m_1 \geq m_2$) has a surface energy per unit area of $\frac{1}{2}\,S(m_1,m_2)$. If one calculates the Casimir energy for a spherical case, one should find this surface energy. The general definition of S is given in (3.15) and is illustrated for $d=4$ in Fig. 3. In particular[7]

$$S(m,0) = C(d)m^{d-1}\ , \tag{4.1}$$

$$C(d) = \frac{2}{(4\pi)^{(d+1)/2}}\,\Gamma\Big(\frac{1-d}{2}\Big)\left[\frac{\pi}{2} - B\left(\frac{1}{2}\,,\frac{d}{2}\right)\right], \tag{4.2}$$

$$
= \begin{cases} \dfrac{1}{\pi} - \dfrac{1}{4} \simeq 0.0683 & \text{for d=2} \\ \dfrac{\ell n\, 4 - 1}{16\pi} \simeq 0.00769 & \text{for d=3} \\ \dfrac{1}{9\pi^2}\Big(\dfrac{3\pi}{8} - 1\Big) \simeq 0.00201 & \text{for d=4} \ . \end{cases} \tag{4.3}
$$

In general, S is finite for $d < 5$ and divergent for $d \geq 5$.* This divergence is due to the boundaries being sharp.[7] A finite thickness, Δ, of the boundaries should render S finite of order of $\ell n\, \Delta$ for $d = 5$ and Δ^{5-d} for $d > 5$.

(iii) the T-term is given by (3.27). It is finite for any d as long as $L \neq 0$. For $L = 0$, we have

$$
T(L = 0) = -S \ , \tag{4.4}
$$

which agrees with $E_c(L = 0) = 0$.† (see Fig. 4). This can also be shown directly for $m_0 = 0$ or $m_\infty = 0$. Therefore $T(L = 0)$ is divergent for $d \geq 5$ because of the sharp boundary. This may become clear by considering the small L behavior of T as given in the Appendix, (A.17),

$$
T = \begin{cases} * \ \ell n L + \cdots & \text{for d} = 5 \ , \\ * \ \dfrac{1}{L^{d-5}} + \cdots & \text{for d} \geq 6 \ . \end{cases} \tag{4.5}
$$

That is, T diverges to $-\infty$ as $L \to 0$. The divergence of S corresponds to this behavior of T through the formal relation (4.4). When we have "mild" boundaries of thickness Δ, the mass function $m^2(x_1)$ goes to a constant function m_∞ smoothly for $L \to 0$. Therefore, T would be cut off at $L \sim \Delta$ to yield finite

---

* Due to the dimensional regularization, the divergence in S does not show for $d = 7, 9, \ldots$. This merely is an artifact. These divergences become clear when we discuss S in relation to T in (iii).

† One might still suspect that S has some arbitrariness since it is independent of L. However, this relation shows that S is well defined and also physically relevant.

values, $\sim \ell n\,\Delta$ for $d = 5$ and $\sim \Delta^{5-d}$ for $d > 5$. This agrees with the expected behavior of S for finite Δ.

Finally, we shall show that our results for general m_0 and m_∞ reproduce some known results for special cases. The case $m_0 \to 0$ is almost trivial[7]. In this case, since the space is divided into two regions without any tunneling between them, the Casimir energy should be independent of L. Our T term satisfies this property; it simply vanishes (see (3.16)). For $m_\infty \to 0$, the Casimir energy is calculated by Hayes[8] for $d = 2$ and by Ambjørn and Wolfram[9] for arbitrary d. Correcting some minor errors and changing their notation to ours, Ambjørn and Wolfram's result (their (2.17) and (2.18)) reads as follows

$$\begin{aligned}\frac{E_c}{L^{d-2}} = &-\frac{\Gamma\left(-\frac{d}{2}\right)}{2(4\pi)^{\frac{d}{2}}}\, m_0^d L + \frac{\Gamma(\frac{1-d}{2})}{4(4\pi)^{(d-1)/2}} m_0^{d-1} \\ &+ \frac{1}{(4\pi)^d \Gamma(d/2)}\, \frac{1}{L^{d-1}} \int_0^\infty dt\, \ell n\left(1 - e^{-2\sqrt{t^2+(m_0 L)^2}}\right).\end{aligned} \tag{4.6}$$

The first term is the volume energy. The second term, the surface energy term, corresponds to our S in (3.25), which is the following

$$+\frac{\Gamma(\frac{1-d}{2})}{4(4\pi)^{\frac{d-1}{2}}} \left(m_\infty^{d-1} + m_0^{d-1} - \int_0^m dp\, \frac{4}{\pi}\, \frac{(p^2+m_0^2)^{\frac{d-1}{2}}}{\sqrt{m^2-p^2}} \right). \tag{4.7}$$

In the dimensional regularization scheme, the first and third term disappear for $m_\infty \to \infty$, because in this limit their only well-defined values in the analytic d-plane are zero. This way, our surface energy reproduces theirs. For the third term, it is easy to see that our T as given in (3.27) has a well-defined limit for $m_\infty \to \infty$; it is exactly the third term in (4.6).

APPENDIX

In this appendix, we derive several expressions that are useful for discussion in the main body of this paper.

(i) First, we discuss the behavior of the e_{c+}'s and the e_{c-}'s for $m_0 > m_\infty$ case, (3.9) and (3.11). The following expansion formula is useful,

$$\arg(1-\alpha) = \frac{-i}{2}\ell n \frac{1-\alpha}{1-\alpha^*} = \sum_{j=1}^{\infty}\frac{1}{j}\mathrm{Im}(\alpha^{*j}) \qquad \text{for } |\alpha| < 1\,. \tag{A.1}$$

For $L \to \infty$, the terms other than B_1 in (3.9) decay in powers of $1/L$. For the third term, this is seen by applying (A.1) to the integrand,

$$\frac{\arg}{\pi} = \frac{1}{\pi}\sum_{j=1}^{\infty}\left(\frac{p-p'}{m}\right)^{4j}\sin 2jp'L\,. \tag{A.2}$$

By carrying out the trivial p_p integral and changing the integration variable p to p', we arrive at the following expressions

$$-\frac{d-1}{\pi}A\sum_{j=1}^{\infty}\frac{1}{j}\int_0^{\infty} dp' f_j(p')\sin 2jp'L\,, \tag{A.3}$$

$$f_j(p') \equiv p'(p'^2+m_0^2)^{\frac{d-3}{2}}\left(\frac{p-p'}{m}\right)^{4j}\,. \tag{A.4}$$

When we expand $f_j(p')$ in powers of p', each term can be integrated and re-summed. As a result, we obtain the following

$$e_{c+} = A\,B_1 + Am_0^{d-1}\left[\frac{2}{\pi mL} - \left(\frac{8}{3\pi} + \frac{d-1}{\pi}\varsigma(3)\frac{m^2}{m_0^2}\right)\frac{1}{(mL)^2}\right.$$
$$\left. + \left(\frac{32}{5\pi} + \left(8\varsigma(3) - \frac{\varsigma(3)}{2}\right)\frac{m^2}{m_0^2} + \frac{3}{2}(d-3)\varsigma(3)\frac{m^4}{m_0^4}\right)\frac{1}{(mL)^5} + O\left(\frac{1}{L^7}\right)\right]\,. \tag{A.5}$$

A similar evaluation for e_{c-} of (3.11) results in a power series expansion in $1/Lm$,

$$e_{c-} = AD_1 + O\Big(\frac{1}{Lm}\Big) . \tag{A.6}$$

Calculation of the first three coefficients (i.e. up to the $1/(mL)^5$ term) shows that they are the opposite of those in (A.5). In fact, all the coefficients in the power series expansions in (A.5) and (A.6), except for the constant term, are opposite of each other and yield an exponential damping as $L \to \infty$, (3.16). Note that each coefficient is divergent due to A, which comes from the integration of the parallel momentum dp^{d-1}. The divergences in the constant terms AB_1 and AD_1 cancel each other for $d < 5$. The other divergences cancel each other completely and yield the finite expression (3.15).

(ii) For the second case, $m_\infty > m_0$, the $e_{c\pm}$'s are evaluated similarly. For e_{c+} of (3.21), the only difference is that the lower limit of the integration is now m instead of 0 in (A.3). The expansion is obtained by expanding f_j in $p' - m$. The result is

$$\begin{aligned} e_{c+} = AB_2 + Am_\infty^{d-1}\Big[&\alpha_1(Lm) + (d-1)\frac{m^2}{m_\infty^2}\alpha_2(Lm)\frac{1}{mL} \\ &- (d-1)\frac{m^2}{m_\infty^2}\alpha_3(Lm)\frac{1}{(mL)^{3/2}} + O\Big(\frac{1}{L^2}\Big)\Big] , \end{aligned} \tag{A.7}$$

where the α's are of order 1, periodic in Lm (period π), and have vanishing average,

$$\begin{aligned} &\alpha_1(x) \equiv y - \frac{1}{2} , \qquad \alpha_2(x) \equiv \frac{\pi}{2}\Big(\Big(y - \frac{1}{2}\Big)^2 - \frac{1}{3}\Big) , \\ &\alpha_3(x) \equiv 2\sqrt{2}\pi\varsigma\Big(-\frac{1}{2}, 1 - y\Big) \qquad \Big(y \equiv \frac{x}{\pi} - \Big[\frac{x}{\pi}\Big]\Big) . \end{aligned} \tag{A.8}$$

(The generalized zeta function $\varsigma(z, a)$ is defined by $\sum_{n=0}^{\infty}(a+n)^{-z}$) As in the $m_0 > m_\infty$ case, the expansion of e_{c-} yields

$$e_{c-} = AD_2 - Am_\infty^{d-1}\Big[\alpha_1(Lm) - \cdots\Big] , \tag{A.9}$$

i.e. the series in (A.7) and (A.8) cancel each other. The α_s given (A.8) has a discontinuity of -1 as Lm increases and passes an integer times π. This corresponds to the appearance of a new discrete level. At this point, e_{c^-} of (A.9) increases by Am_∞^{d-1} and e_{c^+} of (A.7) decreases by the same amount. The sum e_c however is a smooth function given by (3.26).

(iii) Next, we discuss the behavior of the function T, which is an increasing function of L defined in (3.27). For large L, it can be evaluated by expanding the ℓn as

$$T = -F \sum_{j-1}^{\infty} \frac{1}{j} \int_{m_0}^{\infty} dk' g_j(k')\, e^{-2jLk'} \,. \qquad (A.10)$$

After expanding the g_j's in $k' - m_0$, we find that the jth term contributes to T as follows,

$$-\frac{1}{2j^{\frac{d-1}{2}}} \left(\frac{m_0 - m_\infty}{m_0 + m_\infty}\right)^{2j} \left(\frac{m_0}{4\pi L}\right)^{\frac{d-1}{2}} e^{-2jm_0L}(1+\cdots)\,, \qquad (A.11)$$

which is valid for $m_0 \neq 0$. For $m_0 = 0$, we have

$$-\frac{\Gamma\left(\frac{d}{2}\right)}{(4\pi)^{d/2}} \frac{1}{j^d} \frac{1}{L^{d-1}} (1+\cdots)\,. \qquad (A.12)$$

Therefore, for $m_0 = 0$ the leading contribution comes from all j, while for $m_0 \neq 0$, the $j = 0$ term is the major term. The resulting asymptotic behavior of T, including some higher order terms, is given by the following,

$$
T = \begin{cases} -\frac{1}{2}\left(\frac{m_0 - m_\infty}{m_0 + m_\infty}\right)^2 \left(\frac{m_0}{4\pi L}\right)^{\frac{d-1}{2}} e^{-2m_0 L} \\ \times \begin{cases} \left(1 + (d-1)\left(\frac{d+1}{16m_0} - \frac{1}{m_\infty}\right)\frac{1}{L} + O\left(\frac{1}{L^2}\right)\right) \\ \qquad \text{for } m_0 \neq 0, m_\infty \neq 0, \\ \left(1 - \frac{\Gamma(d/2)}{\Gamma\left(\frac{d-1}{2}\right)} \frac{4}{\sqrt{m_0 L}} + (d-1)\frac{4}{m_0 L} + O\left(\frac{1}{L^{3/2}}\right)\right) \\ \qquad \text{for } m_0 \neq 0, m_\infty = 0, \end{cases} \\ -\frac{\Gamma(d/2)\varsigma(d)}{(4\pi)^{d/2}}\left(\frac{1}{L^{d-1}} - \frac{2(d-1)}{m_\infty L} + \frac{2d(d-1)}{(m_\infty L)^2} + O\left(\frac{1}{(m_\infty L)^3}\right)\right) \\ \qquad \text{for } m_0 = 0, m_\infty \neq 0\ . \end{cases} \tag{A.13}
$$

The behavior of T for $L \sim 0$ is obtained by considering the high "momentum" limit. The integrand of T behaves as $\sim q^{d-6}e^{-2Lq}$ for large q. Therefore, $T(0)$ is divergent for $d \geq 5$. This agrees with the behavior of S through (4.4). For $d \leq 3$. $T(L)$ behaves as follows for small L,

$$
T(L) = -S + T'(0)L + \cdots \tag{A.14}
$$

where

$$
T'(0) = \frac{\Gamma\left(-\frac{d}{2}\right)}{8(4\pi)^{d/2}}\left\{(d-2)m_\infty^d - dm_0^2 m_\infty^{d-2} + 2m_0^d\right\}\ . \tag{A.15}
$$

The above $T'(0)$ have an IR divergence for $d = 2$, $m_\infty = 0$, in which case

$$
T(L) = -S - \frac{1}{2\pi} L\,\ell n\, L + \cdots \tag{A.16}
$$

For $d \geq 4$, we have UV divergences in $T'(0)$. Consequently, $T(L)$ behaves as follows

$$T(L) = \begin{cases} -S - \dfrac{(m_0^2 - m_\infty^2)^2}{32\pi^2} L \,\ell n\, L + \cdots & \text{for } d = 4 \\ \dfrac{(m_0^2 - m_\infty^2)^2}{256\pi^2} \ell n\, L + \cdots & \text{for } d = 5 \\ \dfrac{\Gamma\left(\frac{d-4}{2}\right)(m_0^2 - m_\infty^2)^2}{4(4\pi)^{d/2}(d-3)(d-5)} \dfrac{1}{L^{d-5}} + \cdots & \text{for } d \geq 6 \end{cases} \tag{A.17}$$

ACKNOWLEDGEMENTS

The author would like to thank Dr. Michael Peskin for helpful discussions. He also thanks Drs. Marvin Weinstein and A. Hosoya for comments and Dr. Jan Ambjørn for explaining the work of Ref. 9. Dr. Jonathan Bagger advised me of English grammar.

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FIGURE CAPTIONS

1. Three separate regions are considered in which the scalar field has different masses m_0 and m_∞. They are separated by an infinite "plane" boundary, where character is arbitrary for Section 2 and sharp in Section 3.

2. (a) $G(p_1)$ for a small L.

   (b) The oscillating function $F(p_1)$ defined in (3.4) for a small L.

3. The surface energy $S(m_1, m_2)$ for $d = 4$.

4. The e_c for $d = 4$, $m_0 = 0$.

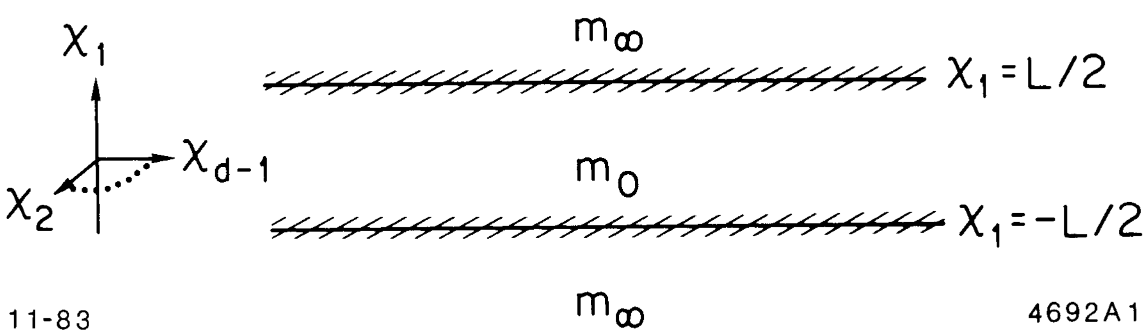

Fig. 1

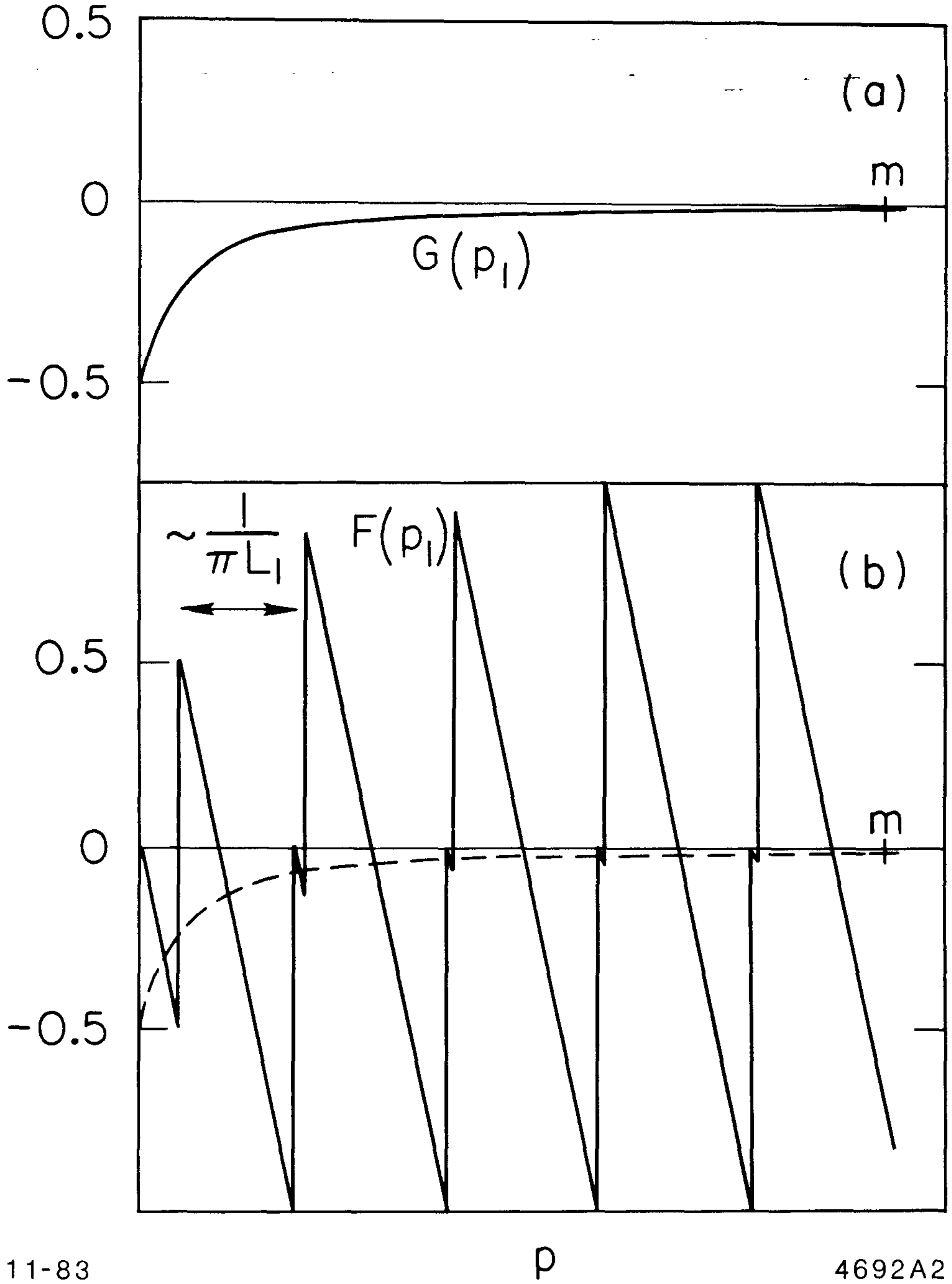

Fig. 2

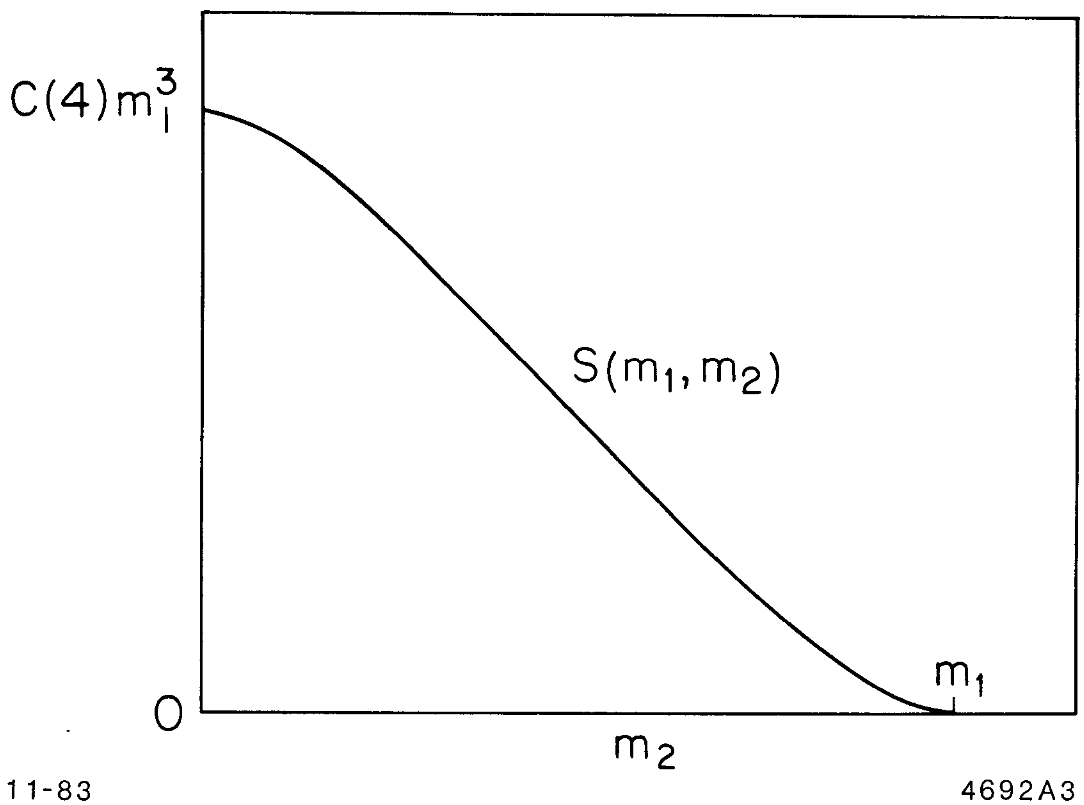

Fig. 3

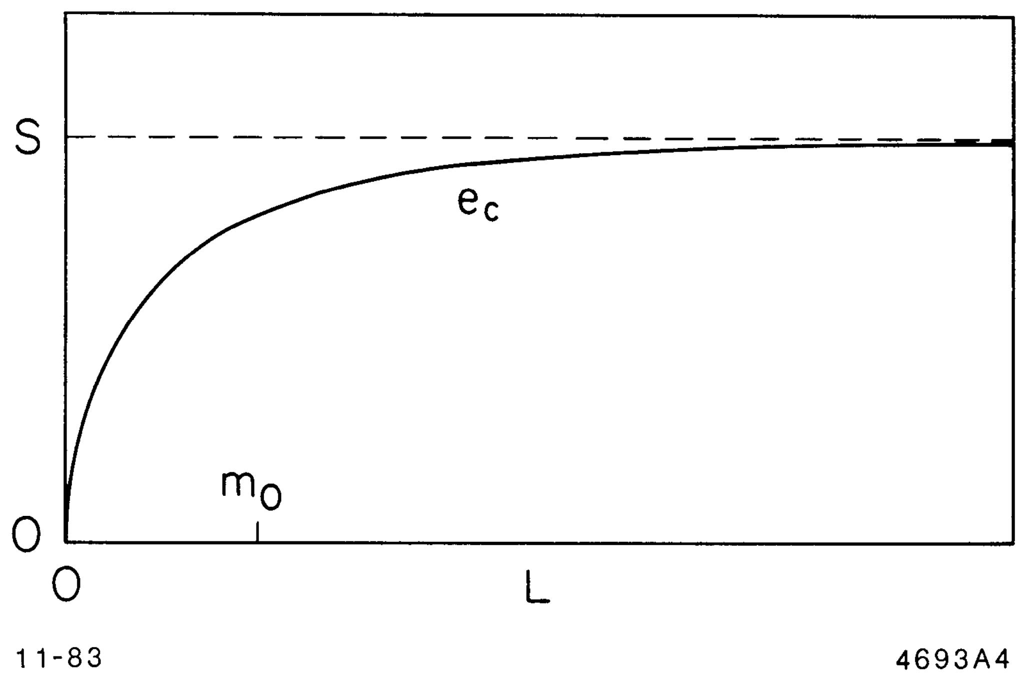

Fig. 4