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PAULI STRUCTURE OF THE INTERCLUSTER INTERACTIONS*

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ABSTRACT

The "fish-bone" structure of the intercluster interaction introduced by Schmid is derived from first principles. It is shown that this structure does not depend on details of the cluster ground states, nor on the details of the microscopic interaction, and is therefore a fundamental symmetry of interactions between composites. The derivation is based on a recently proposed microscopic method of calculating Pauli-corrected intercluster interactions. The relation between the present approach and the Schmid model is examined.

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The Schmid or fish-bone model (FBM)[1] was a major progress in the understanding of the forces between nuclear fragments subsequent to the analysis of exchange kernels of the resonating group theory (RGT)[2] by Tang and collaborators[3] and the introduction of the orthogonality condition model (OCM)[4] by Saito in 1969. In Ref. (1), it was found by numerical studies and physical insight that the RGM exchange kernels can be split into a dominant and a residual part, where the dominant part of the exchange interaction in the representation of the norm kernel eigenstates is characterized by a fish-bone-like symmetry reflected in the matrix M_{ij}:

$$(\underline{M}_{ij}) = \begin{pmatrix} \lambda_1 - \lambda_1 - \lambda_1 - \lambda_1 & . & . & . \\ | & & & \\ \lambda_1 \quad \lambda_2 - \lambda_2 - \lambda_2 & . & . & . \\ | \quad | & & & \\ \lambda_1 \quad \lambda_2 \quad \lambda_3 - \lambda_3 & . & . & . \\ | \quad | \quad | & & & \\ \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 & . & . & . \\ . \quad . \quad . \quad . & . & . & . \\ . \quad . \quad . \quad . & . & . & . \end{pmatrix} \tag{1}$$

with λ_i being the eigenvalues of the RGM norm kernel K. A physical interpretation of the fish-bone symmetry was given in terms of the reflection properties of a Fermi sphere of relative motion states with a diffuse surface.[5] In the past two years, FBM has been applied very successfully in various reactions[6,7,8] and extended to the three-cluser system.[9,10] The important feature of the FBM is that it does not violate the Pauli exclusion principle even if the residual interaction is neglected, and its principal advantage over the basic RGM theory is its simplicity.

It has been argued[11] that the discrete ambiguity of the potential depth in the conventional optical models[12] may be related to the

ambiguity in the choice of considering a partly Pauli-forbidden state to be either fully Pauli-forbidden, or fully Pauli-allowed.

Although there are several indications[1,11,13] that the fish-bone symmetry as an approximate symmetry of the effective interaction of tightly bound clusters of fermions is more fundamental than it may appear from the introduction given in Ref. (1), there has so far been no microscopic proof of the validity of this assertion. In this paper, we show from basic principles that the fish-bone structure is indeed a general symmetry of the intercluster interaction. Our derivation is based on a new microscopic method of calculating the Pauli-corrected cluster interactions.[14] This method overcomes many difficulties of the RGM, OCM and the projection method.[15] Because of the simplicity of its interactions, it can be extended to realistic cluster ground states without much effort. This method has been successfully applied in the investigation[16] of the influence of the Pauli exclusion principle on two- and three-cluster resonances.

In the two-cluster, single-channel, no distortion case, the fish-bone model is defined by Eq. (1).

$$\Big\{T + V_D - E_r - \sum_{ij} |u_i\rangle\langle u_i|T + V_D - E_r|u_j\rangle\, M_{ij}\, \langle u_j| - \sum_{ij} |u_i\rangle\, \langle u_i|\, T + V_D - E_r|u_j\rangle\, M^*_{ij}\, \langle u_j|\Big\}\chi = 0 \qquad (2)$$

The potential V_D is the direct potential of the resonating group theory. The two double sums represent the effect of the Pauli exclusion principle. The first one is the dominant part, and the second one arises from certain residual interactions. The expansion basis $\{u_i\}$ is given by the eigenstates of the resonating group norm operator K. The

matrix M is given by Eq. (1). The matrix M* can be found only by numerical comparison with resonating group exchange kernel for specific reaction and nucleon-nucleon interactions.

One objection which could be raised against the introduction of FBM in Ref. (1) is that the splitting in M + M* is somewhat arbitrary and not based on a rigorous derivation, the more so, since the matrix elements corresponding to the partly Pauli-forbidden states of M and M* for $\alpha - \alpha$ interactions (with potential parameters chosen as in Ref. (17)) are of the same order.

In all applications of the FBM, the residual interaction term was neglected and its absence compensated by a slight adjustment of the direct potential V_D. One thus arrives at the optical version of FBM

$$\left(T + V_{opt} - E_r - \sum_{ij} |u_i\rangle \langle u_i| T + V_{opt} - E_r |u_j\rangle M_{ij} \langle u_j|\right)\chi = 0 \quad . \tag{3}$$

In case of the $n - \alpha$ system, it was shown[18] directly that Eq. (3) reproduces almost exactly not only the on-shell properties of the RGM equation but also its off-shell properties as well. This indicates again that the fish-bone symmetry is a characteristic symmetry of the RGM exchange kernels.

In order to prove that the fish-bone symmetry given by the matrix M reflects a fundamental property of intercluster interaction, we have to recall the Pauli-corrected Schrödinger equation for two clusters derived in Ref. (14). It reads

$$\Lambda(T + V_D - E_r)\,\Lambda\chi = 0, \tag{4}$$

where

$$\Lambda = 1 - \Gamma$$

and

$$\Gamma = \sum_n |f_n\rangle \langle f_n| \quad .$$

The vectors f_n are cluster-projections of microscopic Pauli-forbidden states N of the two-cluster system:

$$f_n(\vec{r}) = \langle N_n | \phi(1)\ \phi(2)\vec{r}\rangle \quad . \tag{5}$$

$\phi(i)$ is the internal ground state wave function of the i-th cluster. It was pointed out in Ref. (14) that in contrast to the set $\{N_n\}$, $\{f_n\}$ is not necessarily an orthonormal set. We can represent the operator Γ by its spectral decomposition:

$$\Gamma = \sum |u'_n\rangle\ \lambda'_n\ \langle u'_n| \quad , \quad \text{with } 1 \geq \lambda'_n \geq \lambda'_{n+1} \quad . \tag{6}$$

From the derivation given in Ref. (14) and Eq. (5), it follows that $\lambda'_n \geq 1$. Using the representation given in (6), we can evaluate Eq. (4) and obtain

$$(T + V_D - E_r - \sum_{ij} |u'_i\rangle \langle u'_i|T + V_D - E_r|u'_j\rangle\ O'_{ij}\ \langle u'_j|)\chi = 0 \quad , \tag{7}$$

where

$$O'_{ij} = \lambda'_i + \lambda'_j - \lambda'_i\lambda'_j \quad . \tag{8}$$

The matrix O′ has the following properties:

1) $O'_{ij} = 1$ if $\lambda'_i = 1$ or $\lambda'_j = 1$,

2) $O'_{ij} \xrightarrow[i \to \infty]{} \lambda'_j$,

3) $O'_{ij} \xrightarrow[j \to \infty]{} \lambda'_i$,

Together with the fact that O′ is a symmetric matrix, the fish-bone-like structure of the matrix O′ is established. The elements above (below) the diagonal and close to it are larger than the rest of the elements on the same row (column) and above (below) the diagonal: $O'_{ij} > O'_{i(j+1)}$ for $j \geq i$, and $O'_{ij} > O'_{(i+1)j}$ for $i \geq j$. We therefore call O′ the

"realistic" fish-bone structure. It will be seen later that this property leads in the use of α - α scattering to a smaller residual interaction for the matrix O′ than for the matrix M when compared to the RGM potential exchange kernel.

In order to make this comparison, we consider as an example the $\ell = 0$ α - α scattering, and use Gaussian functions for the alpha cluster internal wave function $\phi(i)$. According to RGM, we have then two Pauli-forbidden states ($\lambda_1 = \lambda_2 = 1$) and infinite series of partly Pauli-forbidden states with the decreasing weights 1/4, 1/16, 1/64 ... , given by the corresponding eigenvalues λ_3, λ_4, λ_5 ... . Then we can directly construct the microscopic Pauli-forbidden states

$$N_i = \phi(1)\,\phi(2)\,u_i \quad .$$

It follows that $u_i \equiv f_i \equiv u'_i$ and $\lambda_i = \lambda'_i$, for $i = 1,2$. According to the derivation given in Ref. (14) the fully antisymmetrized microscopic wave function Ψ can be expressed in two ways

$$\Psi = \mathcal{A}\{\phi(1)\phi(2)\chi\} = \mathcal{A}\bar{\Psi} \quad \text{or} \quad \Psi = (1\!\!1 - \sum_i |N_i\rangle\langle N_i|)\bar{\Psi} \quad ,$$

where $\mathcal{A}$ denotes the antisymmetrization operator. From this we obtain

$$\langle\phi(1)\phi(2)\vec{r}|\,\mathcal{A}\,|\phi(1)\phi(2)\vec{r}'\rangle = \langle\phi(1)\phi(2)\vec{r}|\,1\!\!1 - \sum_i |N_i\rangle\langle N_i|\,|\phi(1)\phi(2)\vec{r}'\rangle \quad ,$$

and therefore the remaining part Γ' of the operator Γ can be expressed in the RGM approximation by

$K' = \sum_{i=3} |u_i\rangle\,\lambda_i\,\langle u_i|$. In this case $O_{ij} = \lambda_i + \lambda_j - \lambda_i\lambda_j$.

The matrix M and O for the α - α system are given in Tables 1 and 2. The matrix elements for the residual interaction M^*_{ij} for $\ell = 0$ RGM equation with parameters used in Ref. (17) are adopted from Ref. (1) and displayed in Table 3. The corresponding matrix of the residual

interaction for the matrix O and for the same interaction is given in Table 4. One observes that the residual interaction using the matrix O is even smaller than in the case of M. However, we do not wish to dwell on this comparison, since we have used the same approximations as in the RGM. The difference mainly originates in the fact that in case of Eq. (4) the partly Pauli-forgidden states are partially projected out because of the operator Λ on the left-hand side of Eq. (4). We finally mention that we can make the interaction given by Eq. (7) energy-independent by an off-shell transformation:

$$\mathcal{T} = 1 - \sum_i |u'_i\rangle \left[1 - \sqrt{1 - O'_{ij}}\right] \langle u'_i| \quad , \tag{9}$$

and

$$\mathcal{T}^{-1} = 1 + \sum_i |u'_i\rangle \frac{1 - \sqrt{1 - O'_{ii}}}{\sqrt{1 - O'_{ii}}} \langle u'_i| \quad , \tag{10}$$

where the summation includes only the partly Pauli-forbidden inter-cluster states. This transformation is a simple generalization of the original off-shell transformation given by Schmid[1]. Then for $\bar{\chi} = \mathcal{T}|\chi\rangle$ we obtain from Eq. 7

$$\left(T + V_D - E_r - \sum_i |u'_i\rangle \langle u'_i| T + V_D - \epsilon_i |u'_j\rangle \bar{O}'_{ij} \langle u_j|\right)\bar{\chi} = 0 \tag{11}$$

where ϵ_i is some large number if $\bar{O}'_{ii} = 1$,[1,15] $\epsilon_i = 0$ if $\bar{O}'_{ii} < 1$, and

$$\bar{O}'_{ij} = \begin{cases} 1 & \text{if} \quad O'_{ij} = 1 \\ 1 - \dfrac{1 - O'_{ij}}{\sqrt{1 - O'_{ii}}\ \sqrt{1 - O'_{jj}}} \end{cases} \tag{12}$$

The significance of the off-shell transformation $\mathcal{T}$ and of the general structure of Eq. (11) was discussed by Schmid and his collaborators in several papers.[1,7,8,10]

In summary, we have shown rigorously that the fish-bone symmetry found by Schmid is a characteristic symmetry of interactions between nuclear fragments. This symmetry is independent of the microscopic interactions of the constituents or the details of the internal cluster states. Moreover, it is valid for all approximations which take the Pauli exclusion principle fully into account. We thus conclude that the discovery of this symmetry by Schmid was significant progress with implications reaching beyond the scope of the resonating group theory and the methods used in the theory of nuclear fragments.

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Table 1. The matrix M_{ij} of $\ell = 0$ α-α scattering adapted from Ref. (1). The first two rows and the first two columns correspond to the Pauli-forbidden intercluster states. The matrix elements not equal one correspond to partly Pauli-forbidden intercluster states.

1	1	1	1	1	1
1	1	1	1	1	1
1	1	0.2500	0.2500	0.250	0.250
1	1	0.2500	0.0625	0.0625	0.0625
1	1	0.2500	0.0625	0.015625	0.015625
1	1	0.2500	0.0625	0.015625	0.003906

Table 2. The matrix O_{ij} of $\ell = 0$ $\alpha - \alpha$ scattering (see also caption of Table 1).

1	1	1	1	1	1
1	1	1	1	1	1
1	1	0.4375	0.296875	0.261719	0.252930
1	1	0.296875	0.121094	0.0771484	0.066162
1	1	0.261719	0.0771484	0.031006	0.019470
1	1	0.252930	0.066162	0.019470	0.007797

Table 3. The matrix M^*_{ij} of $\ell = 0$ $\alpha - \alpha$ scattering. This table corresponds to Table 1 in Ref. (1). The first two rows and first two columns correspond to the Pauli-forbidden intercluster states. The matrix elements not equal zero correspond to partly Pauli-forbidden intercluster states.

0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.21	0.16	0.15	0.14
0.0	0.0	0.16	0.22	0.18	0.16
0.0	0.0	0.15	0.18	0.17	0.15
0.0	0.0	0.14	0.16	0.15	0.13

Table 4. The matrix O^*_{ij} of $\ell = 0$ $\alpha - \alpha$ scattering. Potential parameters are chosen as in Refs. (1,17) (see also caption of Table 3).

0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.02	0.11	0.14	0.14
0.0	0.0	0.11	0.16	0.17	0.16
0.0	0.0	0.14	0.17	0.15	0.15
0.0	0.0	0.14	0.16	0.15	0.13