NONPERTURBATIVE RENORMALIZATION WITH LATTICE REGULARIZATION[*]

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ABSTRACT

Starting from the correlation functions of a particular lattice theory a sequence of theories is defined in a consequent way. The determination of the sequence of parameters is based on a certain set of functions which has to exhibit an attractive fixed point and further specific properties. Possible functional dependences are studied with respect to the requirements found. It turns out that there is one standard dependence which is necessary to allow the prescription of physical values at different scales. Then the consequences of this dependence are investigated. The surprising result is that the features related to renormalization such as running coupling constant, Λ parameter, renormalization group equations and properties of β and γ functions follow in a remarkably transparent and detailed way. Examples are discussed in the light of these developments and a remark on the nature of the limit is made.

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I. INTRODUCTION

The nonperturbative treatment of quantum gauge theories is essential for a number of well known physical questions. To realize it, so far the only promising way is to use the lattice formulation.[1] This formulation provides a particular regularization of quantum field theory which has the unique feature not to rely on any perturbation expansion.

The general requirements for a nonperturbative formulation of renormalization are known. The parameters of the theory must become functions of the ultraviolet cutoff in such a way that physical quantities get finite limits. To be able to achieve this one needs a suitable critical point of the underlying lattice theory. In the analysis of the Monte Carlo data of gauge theories the quantity commonly used to fix the scheme is the string tension.[2] Recently Creutz[3] introduced the interesting alternative of considering certain ratios of functions at a finite length scale. The ideas involved in this work suggest to start a systematic study of the subject. To do this is the aim of the present paper.

In the present first step it appears appropriate to consider the set of correlation functions to be given, as is the situation in the analysis of the Monte Carlo data. The systematic investigation then should not only provide a better basis for the analysis of such data but also faciliatate the task of finding properties of correlation functions analytically, allowing to concentrate more precisely on what is actually needed. This is important, apart from the limitation of numerical studies by the available lattice sizes, because in general one wants to get analytical answers to fundamental questions.

Mathematically a prerequisite of any limit is the definition of a sequence. This here means to specify a sequence of theories solely by

using the given set of lattice correlation functions, which is well
defined in the action formulation of a particular theory. The introduction
of this sequence is the topic of Sec. II.

A central problem is the determination of the dependence of the
parameters on the numbering of the sequence. It is based on the prescrip-
tion of the value of a physical quantity as limit of a sequence of functions
which must be possible in a consistent way. The requirements for the
prescription as well as for the used set of functions are investigated
in Sec. III.

The next step is to study possible functional dependences which can
meet the requirements found. In Sec. IV this is done and leads to one
standard dependence which is acceptable.

Given the standard dependence the consequences for physical quantities,
i.e., for features as running coupling constant, Λ parameter and β function
are of interest. This is worked out in Sec. V.

In Sec. VI, then, the properties of correlation functions are
investigated on the basis of the standard dependence. Renormalized func-
tions are introduced, the renormalization group equations derived and the
structure of the occurring quantities is studied.

Current examples of physical quantities are discussed in the light
of the obtained results in Sec. VII, and remarks, in particular on the
nature of the limit, are made.

II. SEQUENCE OF THEORIES

The correlation functions of a lattice theory do not really involve
a length. The factors corresponding to canonical dimensions of fields
can readily be scaled out. Then the functions have the form $\Gamma (n_1,\dots,n_s;$

$g_1, \ldots, g_r; N_1, \ldots, N_d)$. The integers n_σ are variables characteristic of the particular function. They are allowed here not only to specify distances between points but also, for example, the two extensions of a rectangular loop. The g_ρ are the parameters of the theory, thus standing for bare couplings and bare masses. The finite lattice size is described by the integers N_δ.

The limit ultimately needs to let $N_\delta \to \infty$ as well as $n_\sigma \to \infty$. For the present purpose it appears appropriate to perform the (less problematic) limit $N_\delta \to \infty$ first, holding the n_σ and the g_ρ fixed. Then the set of functions of form $\Gamma(n, g)$ remains to be considered, where n stands for all components n_σ and g for all g_ρ. To describe the limit $n_\sigma \to \infty$ a sequence of theories numbered by $\nu = 1, 2, \ldots$ is introduced letting n and g depend on ν, such that one has sets of functions of form $\Gamma(n(\nu), g(\nu))$.

The connection to physical length variables can be established by a mapping $n \to x$ which relates the underlying length unit b to a change of n_σ by ν, thus generating finer subdivisions for higher ν. This means that $x = a(\nu) \, n(\nu)$ where

$$a(\nu) = \frac{b}{\nu} \tag{2.1}$$

and $n(\nu) = \nu \, n(1)$. Conversely, the simplest definition of the mapping $x \to n(\nu)$ then is

$$n(\nu) = \text{int} \left(\frac{\nu x}{b} \right) \tag{2.2}$$

where for real α the function $\text{int}(\alpha)$ equals the largest integer which does not exceed α, and in (2.2) this is meant for the components.

In (2.2) for integer x_σ/b one has again $n(\nu) = \nu \, n(1)$ where $n(1) = x/b$. For rational x_σ/b this holds for a suitable subset of ν. In any case one has

$$n(\nu) \to \nu \, \frac{x}{b} \qquad \text{for} \qquad \nu \to \infty \qquad . \qquad (2.3)$$

Therefore in the following the presentation can be simplified by using

$$n(\nu) = \nu \, n(1) \qquad , \qquad n(1) = \frac{x}{b} \qquad (2.4)$$

throughout, tacitly understanding this in the indicated sense.

III. PARAMETER DETERMINATION

To determine the dependence of g on ν, which is studied here for one component g, a physical quantity is required to have a prescribed limit. To get the sequence related to such a quantity, a suitable family of functions $P(\nu,g)$ is constructed or extracted from the functions of form $\Gamma(n(\nu),g)$ where now arbitrary values of g are considered. The condition

$$P\big(\nu,g(\nu)\big) \to P_0 \qquad \text{for} \qquad \nu \to \infty \qquad (3.1)$$

then leads to the subset $g(\nu)$ of the values of g.

To find necessary properties of the family $P(\nu,g)$ one has to note that according to the construction ν is multiplied by x. Thus, to preserve the dependence on x in the limit, the dependence on g must be such that it can combine with ν to give a finite factor. This means that a value $g = g_0$ has to exist giving independently of ν

$$\lim_{g \to g_0} P(\nu,g) = C \qquad , \qquad (3.2)$$

i.e., that the family $P(\nu,g)$ must have a fixed point. A simple example is shown in Fig. 1.

The easiest way to realize (3.1) is to require equality,

$$P\big(\nu,g(\nu)\big) = P_0 \qquad , \qquad (3.3)$$

which corresponds to what is usually imposed in conventional continuum approaches. From (3.3) and the given family $P(\nu,g)$, the values $g(\nu)$ follow immediately.

A look at Fig. 1 now shows that to have steeper curves for larger ν is necessary in order that

$$g(\nu) \to g_0 \qquad \text{for} \qquad \nu \to \infty \quad . \tag{3.4}$$

This also holds if the slope of the functions is negative. Thus, in order that the fixed point can be attractive, one has the condition

$$\left| \frac{\partial P(\nu+1,g)}{\partial g} \right| > \left| \frac{\partial P(\nu,g)}{\partial g} \right| \tag{3.5}$$

for g approaching g_0.

If the family $P(\nu,g)$ behaves suitably (as, e.g., in the example of Fig. 1) the limit (3.4) can be from $g(\nu) > g_0$ or from $g(\nu) < g_0$. The considerations here apply to each of these cases separately.

If x is scaled by λ one gets $P(\lambda\nu,g)$ instead of $P(\nu,g)$, the limit of which still must be a physical quantity. One then has

$$P\big(\lambda\nu,g(\nu)\big) \to Q(\lambda) \qquad \text{for} \qquad \nu \to \infty \tag{3.6}$$

with $g(\nu)$ as determined before and with $Q(1) \equiv P_0$ according to (3.1). The sequences occurring in (3.6) are illustrated in Fig. 2. The generalization from integer λ to rational ones by restricting to an appropriate subset of ν is straightforward. Thus finally one can approach any positive real value of λ. In the following the use of λ is tacitly understood in this sense.

The curves drawn in Fig. 2 for various λ, so far to guide the eye, can be substantiated. Polynomial interpolation gives for any finite number of points a unique polynomial, i.e., in the limit a unique, infinitely differentiable function. Thus one arrives at smooth curves for all λ given by

$$P = \pi(\alpha,\lambda) \quad , \quad g = \gamma(\alpha,\lambda) \quad . \tag{3.7}$$

This allows one to view the procedure in the alternative way, that for

$\lambda = 1$ a curve (3.7) is prescribed which satisfies the condition

$$\pi(\alpha,\lambda) \to Q(\lambda) \quad , \quad \gamma(\alpha,\lambda) \to g_{\delta} \quad \text{for} \quad \alpha \to \alpha_{\delta} \tag{3.8}$$

and then curves (3.7) for $\lambda \neq 1$ are determined in the indicated manner,

which, under suitable conditions, should have the property (3.8). Then

(3.3) corresponds to the rather special choice $\pi(\alpha,1) = $ const, a behavior

which in general does not persist for $\lambda \neq 1$. It is, however, obvious now

how one can start equally well from a more general curve. Actually only

an equivalence class of curves satisfying (3.8) enters.

It is seen that in (3.7) the choice $\gamma(\alpha,g) = $ const is not possible

because it does not allow the prescription of Q according to (3.8). This

reflects that the ν dependence of g is necessary.

Consistency requires that one can start from any value λ in (3.7)

getting the same result, i.e., that one can prescribe any curve as the

one on the basis of which the other ones are obtained. In order that

this is possible, specific properties of the family $P(\nu,g)$ in addition to

those already mentioned are needed. This can be seen by analyzing the

construction in Fig. 2. As illustrated in Fig. 3, depending on the

distances, focusing and defocusing of the curves (3.7) for $\lambda \neq 1$ can

occur. Apart from giving reasonable results this can lead to two extreme

cases. One of these is that $\pi(\alpha,\lambda) \to Q(1)$ for $\alpha \to \alpha_{\delta}$ independently of λ.

The other one is that $\pi(\alpha,\lambda)$ tends to the same maximal value for all

$\lambda > 0$ and to the same minimal one for all $\lambda < 0$. Both of these extreme

cases are not acceptable because they do not allow consistency of pre-

scriptions at different scales.

IV. POSSIBLE FUNCTIONAL DEPENDENCES

The first dependence to be studied with respect to the conditions found in Sec. III is

$$P(\nu,g) = F\big(\nu f(g)\big) \tag{4.1}$$

where f has the property

$$f(g) \to 0 \quad \text{for} \quad g \to g_{\emptyset} \quad . \tag{4.2}$$

The approach in (4.2) can be from above or from below, depending on the particular function f. In addition it is required that functions h and H, with

$$h\big(f(g)\big) = g \tag{4.3}$$

and

$$H\big(F(u)\big) = u \quad , \tag{4.4}$$

respectively, exist.

It is seen that (3.2) is satisfied with $C = \lim\limits_{u \to 0} F(u)$, and (3.5) according to

$$\left| \frac{\partial P(\nu,g)}{\partial g} \right| = \left| F'\big(\nu f(g)\big) f'(g) \right| \nu \quad , \tag{4.5}$$

where F' and f', because of the existence of H and h, do not vanish.

Imposing (3.3) for an arbitrary scale λ_0, from (4.1) and (4.4) one obtains

$$\nu f\big(g(\nu)\big) = H\big(Q(\lambda_0)\big) \tag{4.6}$$

which is finite for $g \neq g_{\emptyset}$ since f is monotone. Using (4.3) one then gets

$$g(\nu) = h\left(\frac{H\big(Q(\lambda_0)\big)}{\nu} \right) \tag{4.7}$$

for which, because of $h(f) \to g_{\emptyset}$ for $f \to 0$, in fact (3.4) holds. From (3.6), (4.1) and (4.6) it follows for the physical quantity at scale λ that

$$Q(\lambda) = F\left(\frac{\lambda}{\lambda_0} \ H\big(Q(\lambda_0)\big)\right) \qquad . \qquad (4.8)$$

By (4.4) this leads to

$$\frac{H\big(Q(\lambda)\big)}{\lambda} = \frac{H\big(Q(\lambda_0)\big)}{\lambda_0} \qquad , \qquad (4.9)$$

i.e., to a constant ratio. Therefore, (4.8) does not depend on λ_0 and one has the desired consistency.

A second possibility is a "weaker" dependence of form

$$P(\nu,g) = F\big((\ell n \nu)f(g)\big) \qquad (4.10)$$

with F and f as before. Again (3.2) and (3.5) are satisfied. With analogous steps as for (4.1), instead of (4.8) one now gets

$$Q(\lambda) = \lim_{\nu \to \infty} F\left[\frac{\ell n(\lambda \nu)}{\ell n(\lambda_0 \nu)} \ H\big(Q(\lambda_0)\big)\right] = Q(\lambda_0) \qquad . \qquad (4.11)$$

This is an example of one of the extreme cases discussed in Sec. III, which is not acceptable.

A third possibility is a "stronger" dependence of type

$$P(\nu,g) = F\big(e^{\alpha \nu}f(g)\big) \qquad \text{with} \qquad \alpha > 0 \qquad , \qquad (4.12)$$

with F and f as before. Equations (3.2) and (3.5) hold again. Instead of (4.8) one now obtains

$$Q(\lambda) = \lim_{\nu \to \infty} F\left(e^{\alpha \nu(\lambda - \lambda_0)} \ H\big(Q(\lambda_0)\big)\right) = \begin{cases} F(\infty) \\ Q(\lambda_0) \\ F(0) \end{cases} \quad \text{for} \quad \begin{cases} \lambda > \lambda_0 \\ \lambda = \lambda_0 \\ \lambda < \lambda_0 \end{cases} \quad (4.13)$$

This is an example of the other extreme case envisaged in Sec. III, which is not acceptable.

A slightly different view of the situation is obtained by noting that (4.10) can be considered as a dependence on ν^f and (4.12) as one on $\nu - \tilde{f}$ (where $\tilde{f} = -(1/\alpha)\ell nf$). Therefore, the three possibilities studied

so far have been those based on the fundamental operations of multiplication, exponentiation and subtraction, and it has turned out that only the multiplication allows consistency.

To see the general manner of realization one has to note that the dependences of the physical quantities in the theory on their variables must not be wiped out in the limit. Therefore, these variables necessarily enter in the combination

$$x_\sigma \, \nu \, f_\sigma(g) \qquad , \qquad (4.14)$$

where all f_σ in the theory are asymptotically proportional, i.e.,

$$\frac{f_\sigma(g)}{f_\rho(g)} \to c_{\sigma\rho} \qquad \text{for} \qquad g \to g_0 \qquad \text{with} \qquad 0 < c_{\sigma\rho} < \infty \qquad . \qquad (4.15)$$

A possible residual dependence on g, entering separately, must be such that it does not spoil the limit. By a redefinition of functions, the asymptotically proportional set of f_σ may be replaced by an asymptotically equal one (for which all $c_{\sigma\rho} = 1$ in (4.15)). The types of functions which can occur are seen to be asymptotically equivalent to the form (4.1), which from now on will be called the standard dependence.

V. PROPERTIES FROM THE STANDARD DEPENDENCE

Equations (4.1) thru (4.9) give the definition and some of the properties of the standard dependence. By (4.9) a parameter R of the theory shows up for which

$$R = \frac{H\big(Q(\lambda_0)\big)}{\lambda_0} \qquad (5.1)$$

holds. Then (4.8) reads

$$Q(\lambda) = F(\lambda R) \qquad . \qquad (5.2)$$

R corresponds to a generalization of the Λ parameter in conventional approaches. $Q(\lambda)$ is to be considered as a function of the renormalized coupling constant, the definition of which leaves considerable freedom. As discussed by Creutz[3] the only requirement is that one gets the correct weak coupling behavior (which in his example leads to a second power).

Equation (4.8) can be straightforwardly generalized to the one for an arbitrary physical quantity given by

$$P_a(\nu, g) = F_a\big(\nu f(g)\big) \qquad , \tag{5.3}$$

which (still using (4.1) to determine g) leads to

$$Q_a(\lambda) = F_a\left(\frac{\lambda}{\lambda_0} H\big(Q(\lambda_0)\big)\right) = F_a(\lambda R) \qquad . \tag{5.4}$$

The independence of (5.4) of the choice of λ_0 is the invariance under renormalization group transformations.

If the description is changed to functions $\widetilde{F}$ and $\widetilde{g}$ in (4.1) such that

$$F\big(\nu f(g)\big) = \widetilde{F}\big(\nu \widetilde{f}(g)\big) \qquad , \tag{5.5}$$

f and $\widetilde{f}$ must be asymptotically proportional (cf. Sec. IV). For

$$\widetilde{f} = cf \tag{5.6}$$

one obtains $\widetilde{F}(cu) = F(u)$, $\widetilde{H} = cH$ and therefore

$$\widetilde{R} = cR \qquad , \qquad Q(\lambda) = F(\lambda R) = \widetilde{F}(\lambda \widetilde{R}) \qquad . \tag{5.7}$$

The change in (5.4) is analogous.

If a different physical quantity, given by $F_b(\nu f(g))$, is used to determine the ν dependence of g, instead of (5.1) one gets in self-evident notation

$$R_b = \frac{H_b\big(Q_b(\lambda_0)\big)}{\lambda_0} \qquad , \tag{5.8}$$

and instead of (5.4) one has

$$Q_a(\lambda) = F_a(\lambda R_b) \qquad . \tag{5.9}$$

Consistency of (5.4) and (5.9) then requires

$$R_b = R \qquad _\circ \tag{5.10}$$

From (5.10) with (4.4) one obtains

$$Q_b(\lambda_0) = F_b\left(\frac{\lambda_0}{\lambda} H\big(Q(\lambda)\big)\right) \tag{5.11}$$

for the relation between different functions of the renormalized coupling.

For $g(\nu)$ one gets from (4.7) and (4.9)

$$g(\nu) = h\left(\frac{\lambda_0}{\lambda} \frac{H\big(Q(\lambda)\big)}{\nu}\right) \qquad , \tag{5.12}$$

or equivalently, using (5.1),

$$g(\nu) = h\left(\frac{\lambda_0 R}{\nu}\right) \qquad . \tag{5.13}$$

(5.13) shows that the curves given by (3.7) and (3.8) now specialize to

$$P = \pi(\alpha,\lambda) = Q(\lambda) \qquad \text{for all } \alpha \qquad ,$$
$$g = \gamma(\alpha,\lambda) = h(\alpha\lambda) \qquad \text{with } \alpha_{\flat} = 0 \qquad . \tag{5.14}$$

(5.12) gives the relation between the bare coupling g at scale λ_0 and a

function Q of the renormalized coupling at scale λ.

A β function can be defined in the present framework by

$$\beta = -\lambda \frac{dQ(\lambda)}{d\lambda} \qquad . \tag{5.15}$$

To evaluate (5.15) one has to note that $F(H(y)) = y$ implies

$F'(H(y)) H'(y) = 1$. Then (5.2) gives

$$\beta = -\frac{H(Q)}{H'(Q)} = \left(-\frac{d}{dQ} \ln H(Q)\right)^{-1} \qquad . \tag{5.16}$$

It is explicit here that β depends only on $Q(\lambda)$. In addition it turns out that β has to have the specific functional structure given by (5.16).

From (3.2), (4.1) and (4.2) one obtains

$$F(u) \rightarrow C \qquad \text{for} \qquad u \rightarrow 0 \qquad . \qquad (5.17)$$

By (5.2) this leads to

$$Q(\lambda) \rightarrow C \qquad \text{for} \qquad \lambda \rightarrow 0 \qquad , \qquad (5.18)$$

i.e., to $\lambda = 0$ at the fixed point value of the function family. It is further to be noted that, because of $H(Q(\lambda)) = \lambda R$,

$$H\big(Q(\lambda)\big) \rightarrow 0 \qquad \text{for} \qquad \lambda \rightarrow 0 \qquad (5.19)$$

holds. Since, due to the existence of F, one has $H' \neq 0$, it follows from (5.16) and (5.19) that

$$\beta \rightarrow 0 \qquad \text{for} \qquad \lambda \rightarrow 0 \qquad , \qquad (5.20)$$

i.e., that there is a fixed point of (5.15) at $\lambda = 0$.

VI.   PROPERTIES OF CORRELATION FUNCTIONS

After investigating the limit for physical quantities now the correlation functions themselves are studied on the basis of the standard dependence. The conditions on the limit are relaxed requiring only the form

$$\Gamma\big(n(\nu), g(\nu)\big) \rightarrow \rho\big(n(\nu), g(\nu)\big) \, F_\alpha\big(\nu f\big(g(\nu)\big)\big) \qquad , \qquad (6.1)$$

where F_α is a function of the type already introduced by (5.3), and where ρ is a contribution which does not have this behavior.

The first manner to get rid of the ρ is to form appropriate ratios of correlation functions in which the occurring ρ factors cancel. In this way one obviously arrives at physical quantities. Examples of this type are the ratios constructed by Creutz[2,3] for Wilson loops, where perimeter length and number of corners are chosen in such a way that divergent factors divide out.

The second possibility occurs if the ρ factors carry no x dependence, i.e., if

$$\Gamma\big(n(\nu),g(\nu)\big) \rightarrow \rho\big(g(\nu)\big) \; F_\alpha\big(\nu f\big(g(\nu)\big)\big) \quad , \tag{6.2}$$

in which case the situation corresponds to that of conventional multiplicative renormalization. Then, using a suitable function $Z(\nu)$, one can consider the limiting behavior

$$Z(\nu) \; \Gamma\big(n(\nu),g(\nu)\big) \rightarrow \phi\big(\nu f\big(g(\nu)\big)\big) F_\alpha\big(\nu f\big(g(\nu)\big)\big) \tag{6.3}$$

which implies that one must have

$$Z(\nu) \; \rho\big(g(\nu)\big) \rightarrow \phi\big(\nu f\big(g(\nu)\big)\big) \quad . \tag{6.4}$$

$Z(\nu)$ obviously corresponds to an adequate power of the wave function renormalization constant.

For the realization of (6.4), because of (4.3), one can first put

$$\rho(g) = \rho\big(h\big(f(g)\big)\big) = \hat\rho\big(f(g)\big) \quad . \tag{6.5}$$

Then it is obvious that in order to get the property $Z(\nu)\hat\rho(f) = \phi(\nu f)$, one needs power behavior, i.e.,

$$\hat\rho(f) = \mathscr{A}f^\delta \quad , \quad Z(\nu) = \mathscr{B}\nu^\delta \tag{6.6}$$

and

$$\phi(u) = \mathscr{C}u^\delta \quad . \tag{6.7}$$

Starting from the lattice formulation not involving a length as indicated in Sec. II, δ in (6.6) and (6.7) is the total dimension of the correlation function.

The usual subdivision of δ in a canonical and an anomalous part arises because for correlation functions of products of fields one has

$$\tilde\Gamma = \Big(\frac{\nu}{b}\Big)^d \Gamma \tag{6.8}$$

where $\tilde\Gamma$ is the function in the formulation involving a length, Γ the presently used one, and d the canonical dimension. It is seen that $\tilde\Gamma$ has

to be multiplied by $\tilde{Z} = \tilde{\mathcal{B}} \, \nu^{\delta-d}$, where $\delta - d$ is the anomalous dimension, to give (6.3).

Inserting (4.6) one obtains from (6.3) the renormalized correlation function at scale λ

$$G\left(\lambda;Q(\lambda_0),\lambda_0\right) = \phi\left(H\left(Q(\lambda_0)\right)\right) \, F_a\left(\frac{\lambda}{\lambda_0} \, H\left(Q(\lambda_0)\right)\right) \tag{6.9}$$

or equivalently, using (5.1),

$$G\left(\lambda;Q(\lambda_0),\lambda_0\right) = \phi(\lambda_0 R) \, F_a(\lambda R) \quad . \tag{6.10}$$

This follows similarly as (4.8) or (5.4). ϕ has, however, no λ dependence because it does not depend on x. It is seen that ϕ and therefore G is not independent of the choice of λ_0, i.e., not invariant under renormalization group transformations.

From (6.10) one obtains

$$\lambda_0 \, \frac{d}{d\lambda_0} \, (\phi F_a) = \gamma \phi F_a \tag{6.11}$$

where

$$\gamma = \lambda_0 \, \frac{d \ln \phi}{d\lambda_0} \quad . \tag{6.12}$$

(To get the conventional form of the definition of γ one has to insert (6.4) into (6.12) and to keep ρ fixed.) According to (6.9), (6.11) and (5.15) one now has

$$\left(\lambda_0 \, \frac{\partial}{\partial\lambda_0} - \beta \, \frac{\partial}{\partial Q} - \gamma\right) G(\lambda;Q,\lambda_0) = 0 \quad . \tag{6.13}$$

Thus the renormalization group equations follow in a straightforward way.

By using (5.15), (6.12) can be cast into the form

$$\gamma = -\beta \, \frac{d \ln \phi}{dQ} \tag{6.14}$$

from which it is obvious that γ depends on Q only. Equation (6.14) can be evaluated exploiting the dependence $\phi(H(Q))$ and (5.16) which gives

$$\gamma = H \frac{d \ln \phi}{dH} \qquad . \tag{6.15}$$

Inserting (6.7) into (6.15) one sees that

$$\gamma = \delta \qquad , \tag{6.16}$$

i.e., that γ is, in fact, the total dimension.

The relation between correlation functions at different scales can be derived by using $H(Q(\alpha\lambda_0)) = \alpha H(Q(\lambda_0))$ which holds according to (4.9). From (6.9) it then follows that

$$G\left(\alpha\lambda;Q(\lambda_0),\lambda_0\right) = G\left(\lambda;Q(\alpha\lambda_0),\lambda_0\right) \frac{\phi\left(H(Q(\lambda_0))\right)}{\phi\left(H(Q(\alpha\lambda_0))\right)} \qquad \circ \tag{6.17}$$

The last factor in (6.17) because of (6.14) may be written as

$$\frac{\phi\left(H(Q(\lambda_0))\right)}{\phi\left(H(Q(\alpha\lambda_0))\right)} = \exp\left[\int_{Q(\lambda_0)}^{Q(\alpha\lambda_0)} du \; \frac{\gamma(u)}{\beta(u)}\right] \tag{6.18}$$

which leads to the form occurring in conventional approaches.

In concluding this section it is to be noted that instead of inserting the explicitly derived relation (4.6) into (6.3) to obtain (6.9), one could also in an implicit manner base the scheme on the F_α of a function of form (6.2). This would be closer to what is done in conventional procedures.

VII. EXAMPLES AND REMARKS

The functions introduced by Creutz[3] to get physical quantities in the present formulation read

$$P(\nu,g) = 1 - \frac{\Gamma(2\nu,2\nu;g) \; \Gamma(\nu,\nu;g)}{\left(\Gamma(2\nu,\nu;g)\right)^2} \tag{7.1}$$

where $\Gamma(n_1,n_2;g)$ are the correlation functions of rectangular Wilson loops. The functions F and G of Creutz are given by (7.1) for $\nu = 1$ and $\nu = 2$. His staircase construction generates sequences of values $g_s(2^\ell)$ and P_ℓ with $\ell = 0,1,2,\ldots$ by determining $g_s(2^\ell)$ from

$$P\left(2, g_s(2^\ell)\right) = P_{\ell-1} \tag{7.2}$$

and P_ℓ from

$$P\left(1, g_s(2^\ell)\right) = P_\ell \tag{7.3}$$

after imposing P_0 in the sense of (3.3). In the case of the standard dependence one obtains

$$P_\ell = Q(2^{-\ell}) \quad , \quad g_s(\nu) = g(\nu) \quad . \tag{7.4}$$

Otherwise the P_ℓ for $\ell > 0$ differ from $Q(2^{-\ell})$. An estimate of these differences needs the knowledge of the functions $P(\nu, g)$ for larger ν.

The use of correlation length and string tension also fits in the present scheme. In these cases the fact is exploited that for large n one expects a factorization of the correlation functions which in the present formulation is given by (6.1) with F_a having the forms

$$F_\zeta\left(\nu f_\zeta(g(\nu))\right) = \exp\left(-\frac{n(\nu)}{\zeta(g(\nu))}\right) \tag{7.5}$$

and

$$F_\sigma\left(\nu f_\sigma(g(\nu))\right) = \exp\left(-\sigma(g(\nu)) \, n_1(\nu) \, n_2(\nu)\right) \quad , \tag{7.6}$$

respectively. Noting (2.4) it is seen that one has, in fact, standard dependence with $f_\zeta(g) = 1/\zeta(g)$ and $f_\sigma(g) = \sqrt{\sigma(g)}$. According to (4.6) and (5.1) one now gets the correlation length b/R_ζ and the string tension $(R_\sigma/b)^2$. Within the same theory, f_ζ and f_σ must be asymptotically proportional, i.e., satisfy (4.15). Then one has $R_\sigma/R_\rho = c_{\sigma\rho}$ reflecting the description dependence caused by the use of different functions f.

From the present point of view the application of (7.1) and (7.6) in the analysis of Monte Carlo data looks as follows. In (7.1) a function related to a physical quantity with small $n_\mu(1)$ is constructed and ν has to be large enough in order that one gets standard dependence. For (7.6)

$n_\mu(\nu)$ has to be large enough such that a physical quantity separates in standard form. Because of (2.4), given a maximal numerically available loop size, the limitation in both procedures is essentially the same.

Finally it is to be remembered that, given a sequence, mathematically any limit needs the specification of a topology in which it is to be understood. For physical quantities one expects the limit of ordinary numbers. For the renormalized correlation functions, which according to conventional continuum theory should be distributions, it is possible to introduce the related weak topology on the basis of the present formulation. For this purpose, using a test function $\psi(x)$, (6.3) with (6.9) is written as

$$
\sum_{n(\nu)} (a(\nu))^S \; Z(\nu) \; \Gamma(n(\nu),g(\nu)) \; \psi\big(x(n(\nu))\big)
$$

$$
\rightarrow \int dx^S \; G(\lambda;Q,\lambda_0) \; \psi(x) \qquad .
$$

(7.7)

The l.h.s. of (7.7) defines the action of a sequence of distributions on the test function. Then one can exploit the theorem that if the l.h.s. converges for all test functions, the limit defines again a distribution.

It is to be noted that this concept can be extended to gauge invariant functions provided one considers the limit for classes of such functions which can be described by a fixed and finite number of variables. An example of this is the class of rectangular Wilson loops with two variables. The possibility to treat particular forms of loops and strings in this manner appears important in view of the fundamental role of gauge invariant functions which emerged in a recent analysis[4] of gauge fixing.

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FIGURE CAPTIONS

Fig. 1. Example of a family of functions $P(\nu,g)$.

Fig. 2. Illustration of consistency considerations with functions $P(\nu,g)$ for $\nu = 1,2,\ldots$ and curves $\pi(\alpha,\lambda)$, $\gamma(\alpha,\lambda)$ for $\lambda = 2,1,1/2$.

Fig. 3. Defocusing by a larger distance between $P(4,g)$ and $P(2,g)$ in the construction illustrated in Fig. 2.
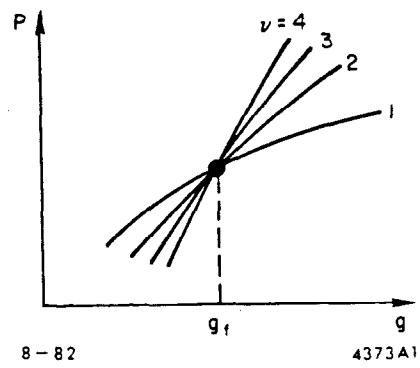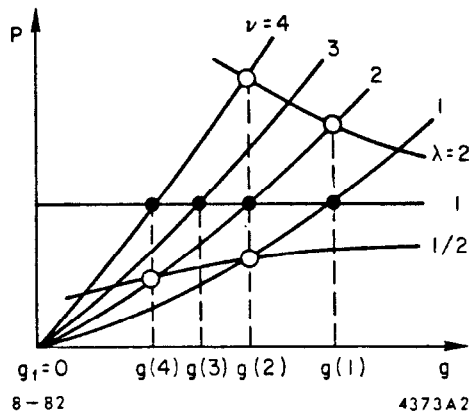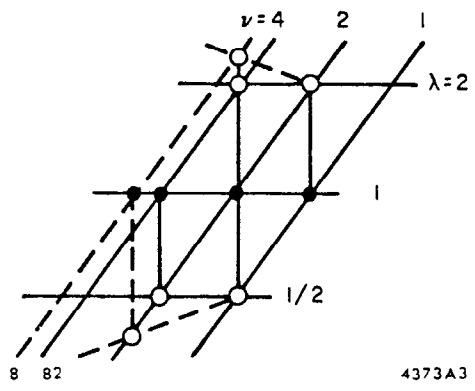
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Fig. 1

Fig. 2

Fig. 3