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GENERAL INFRARED AND ULTRA-VIOLET PROPERTIES OF THE GLUON PROPAGATOR IN AXIAL GAUGE*

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ABSTRACT

Some general properties of the gluon propagator are studied in the axial gauge with emphasis on its infrared and ultra-violet behaviour. In space-like gauges we derive a Lehmann representation and consequently a Kallen sum rule for Z_3. We investigate in some detail the apparent inconsistency between $Z_3 > 1$ (antiscreening and the origin of asymptotic freedom) and the positivity of the Hilbert space due to the absence of ghosts which suggests $Z_3 < 1$. We show that in time-like gauges there is no Lehmann representation so that the propagator is no longer analytic in its virtual mass parameter. General positivity coupled with these results restricts certain IR singularities from being any worse than they are in QED so consequently it is not possible to use the IR behaviour of the propagator in this gauge (whether time-like or space-like) as a signal for confinement if the Wilson loop is used as the criterion. We also show that a judicious use of the renormalization group involving a variation of the axial gauge parameter allows one to drive the theory into the IR in such a way that the effective coupling constant is the asymptotically free one! We use this result to show that the presence of non-perturbative contributions of the generic form e^{-c/g^2} associated with instantons leads to a <u>power</u> law correction to the IR behaviour of the propagator.

1. INTRODUCTION

Although non-Abelian gauge theories and, in particular, quantum chromodynamics (QCD) have now come of age there still remain many unanswered basic questions.[1] Most of these inevitably concern the issues of confinement and the nature of the physical spectrum. It is generally believed that non-perturbative effects such as instantons play a central role in understanding these problems. The best known aspect of QCD is its large momentum behaviour which can be calculated reliably since the theory is asymptotically free. It is the purpose of the present paper to modestly investigate some of these general problems by studying the simplest non-trivial object in the theory, namely the gluon propagator. Although this is a gauge-dependent object, its properties presumably reflect the general physical features of the complete theory.

Our original motivation for this work was to understand a result of Oehme and Zimmerman[2] who claimed that QCD was not a consistent theory unless there were a minimum number of flavors. This result was based on a contradiction between the positivity of the gluon spectral function and the constraints of asymptotic freedom. Although this calculation was performed in a covariant gauge where the Hilbert space contains ghost states, a positivity constraint was derived by projecting onto a positive definite subspace. In the axial gauge,[3] on the other hand, the Hilbert space is positive definite from the outset and so any subtlety due to the projection is avoided.[4] We were thus motivated to study the propagator in this gauge to see whether a similar contradiction arose and investigate its origins. This gauge is, in any case, an attractive gauge to work in since it is conceivable that its positive definiteness could lead to results that are not transparent in other gauges. Furthermore, the gauge parameter is an arbitrary four-vector n_μ which gives rise to a new kinematic variable $q_L (\equiv n\cdot q$, say), which does not occur in covariant gauges. Rather than being a nuisance the presence of q_L could conceivably be exploited by allowing it to vary. It is well known that the price paid for a positive definite Hilbert space (no ghosts) is to sacrifice Lorentz invariance and this manifests itself as an unphysical gauge singularity. Because the treatment of this singularity plays a crucial role in solving the "paradox" we derive a prescription for treating it by performing a gauge transformation from a covariant propagator to its axial gauge counterpart.

The phenomenon of asymptotic freedom is often described as the anti-screening of the colour charge, by analogy with the screening of electric charge that occurs in quantum electrodynamics (QED). Thus, it is the polarization of the vacuum that is responsible for screening. Indeed there exists a rigourous argument due to Kallen showing that the magnitude of the physical charge (e) is always less than the bare one (e_0).[5] The argument is based on the Lehmann sum rule for the photon renormalization constant (Z_3):

$$Z_3 = 1 - \int_0^\infty \rho_1(q^2)dq^2$$

The identification $Z_3 = (e/e_0)^2$ and the positivity of the photon spectral function ρ_1, both guaranteed by gauge invariance, then leads to a proof of screening ($Z_3 < 1$).[6] In modern parlance, screening is intimately related to the fact that QED possesses an infrared (IR) stable fixed point at $e^2 = 0$; $Z_3 < 1$ simply corresponds to a positive

β-function.[1] In QCD the situation is more complicated because in an arbitrary gauge $Z_3 \neq (g/g_0)^2$ nor is the relevant spectral function positive definite. Nevertheless asymptotic freedom (i.e. an ultra-violet stable fixed point at $g^2 = 0$) is a gauge invariant concept corresponding to $(g/g_0)^2 > 1$. The argument of Ref. 2 seems to suggest that in a positive sector of a covariant Hilbert space a conventional Lehmann sum rule leads, via Kallen's argument, to a violation of this. In an axial gauge, however, the problem is even more transparent since positivity in the complete Hilbert space is guaranteed by the absence of ghosts.[4] Furthermore in this gauge the divergent piece of Z_3 [$\equiv (Z_3)_{div}$] is gauge invariant and related to $(g/g_0)^2$ so we have precisely the conditions required for Kallen's theorem to be valid; consequently $(Z_3)_{div} < 1$, corresponding to screening rather than anti-screening! In this case, however, the solution to the paradox is known; it was pointed out by Frenkel and Taylor[7] that if one takes into account the subtle gauge singularity of the axial gauge then in perturbation theory $(Z_3)_{div} > 1$ as it must. This suggests that there may well be a similar subtlety in the covariant case that allows anti-screening to develop.

Because of the subtle nature of this result we present a somewhat more general and detailed version of the argument given in Ref. 7. In Section 2 we give the general definitions of the propagator and its spectral function relevant to the axial gauge. In Section 3 the analogue of the Lehmann representation is derived. We show that the conventional form is valid only in space-like gauges; put slightly differently, this says that the gluon propagator is an analytic function of q^2 at fixed q_L <u>only if $n^2 < 0$.</u> In such gauges the canonical commutation relations then lead in the usual way to the standard sum rule, with the analogue to ρ_1 now being a function of q^2 <u>and</u> $q^2{}_L$. In Section 4 we study the positivity constraints on the spectral functions that follow from the ghost-free nature of the gauge. Again we find that the situation depends critically on whether n^2 is space-like or time-like. In space-like gauges we find that $\rho_1(q^2,q^2{}_L)/q^2{}_L \geq 0$ whereas there is no such constraint when $n^2 > 0$. As already remarked this appears disastrous since it naively leads to a contradiction with asymptotic freedom. The crucial observation of Frenkel and Taylor[7] was that in the usual interpretation of $q^{-2}{}_L$ this singularity is <u>not</u> a positive definite quantity. The conventional prescription is as a principal value and this is <u>not</u> a positive definite concept. Thus, $\rho_1(q^2,q^2{}_L) \geq 0$ provided $q^2{}_L \neq 0$; however, when $q^2{}_L = 0$ it is in fact negative. Now, on purely dimensional grounds, the limit $q^2{}_L \to 0$ corresponds to a $q^2 \to \infty$, the region relevent to asymptotic freedom. Thus it is not surprising that the nature of the gauge singularity at $q^2{}_L = 0$ provides the solution to the screening paradox. Indeed this observation allows us to generalize the argument of Frenkel and Taylor and make a Kallen-like proof that (in pure QCD) $(Z_3)_{div} > 1$ independent of perturbation theory.

At the other end of the energy scale we encounter the problem of the IR behaviour of the propagator. As already remarked this is expected to involve non-perturbative aspects of the theory. We can make use of the freedom to vary q_L to learn about this region, for, just as $q^2{}_L \to 0$ corresponds to $q^2 \to \infty$, so $q^2{}_L \to \infty$ corresponds to $q^2 \to 0$. Indeed a judicious use of the renormalization group allows us to relate this region to the same region but with an asymptotically free coupling constant. Unlike the ultra-violet region, however, small coupling here does not necessarily drive the theory into its perturbative regime. In fact we show that the presence of an instanton-like non-perturbative term[8] of the form e^{-c/g^2} leads to a power change in the IR behaviour, in marked contrast to ordinary perturbation theory which usually only induces logarithmic corrections.[1] Thus, if a highly singular IR behaviour of the gluon propagator is interpreted as a signal for confinement, then this suggests that instantons are, indeed, the physical origin. In a recent paper[9] we have, in fact, shown that, in spite of its gauge dependence, the singular nature of the gluon propagator can be used as a criterion for confinement. More specifically, we proved that if the $g_{\mu\nu}$ term in the propagator is more singular than $1/q^2$ in any one gauge then, for pure QCD, the gauge invariant Wilson loop[10] behaves like e^{-A}, where A is the area of an asymptotically large loop. The propagator need not be singular in all gauges; all that is required is that it be highly singular in at least one gauge. Since the area law is generally accepted as the criterion for confinement in pure QCD, a demonstration that the $g_{\mu\nu}$ term be highly singular in some gauge is therefore tantamount to a proof of confinement. There has been extensive work in the axial gauge on the IR behaviour of the propagator particularly by Baker, Ball and Zachariasen.[11] They attempt to solve a truncated form of the Schwinger-Dyson equations by looking for a self-consistent solution and claim that this leads to a $1/q^4$ behaviour. Their truncation involves keeping only one invariant piece of the $g^{\mu\nu}$ term [the Abelian-like piece][12] and it is this that is singular; there, of course, remains the possibility that such a behaviour for the full $g^{\mu\nu}$ term does not survive in the full theory. Unfortunately this appears to be the case. Indeed, it can be shown that the positivity of the relevant spectral functions together with the proven analyticity properties make it impossible for the $g^{\mu\nu}$ term to be more singular than $1/q^2$ in the IR. This behaviour is obviously not sufficient to infer an area law for the Wilson loop and we therefore conclude that it is not possible to prove confinement (at least via the Wilson loop) from the IR behavior of the gluon propagator in the axial gauge. It is, of course, conceivable that confinement could be related to the truncated propagator via some criterion other than the Wilson loop so in this sense the work of Ref. 11 may well be relevant to the confinement issue. It should be noted that there appears to be no such constraint in other gauges since positivity no longer remains valid.[13] Ironically, positivity does not constrain the singularity structure of the truncated propagator considered in Ref. 11 and this is, in fact, allowed to develop a highly singular behaviour. The point is that this is only one piece that contributes to the confinement criterion. This argument has the further consequence that our renormalization group discussion on the role of instantons is not strictly relevant to the confinement problem. Nevertheless the general conclusion that instantons induce IR power singularities remains valid and is presumably related to the singular behaviour discovered in Ref. 11.

2. DEFINITION AND GENERAL FORM OF THE PROPAGATOR IN THE AXIAL GAUGE

In the axial gauge defined by $n^\mu A_\mu = 0$ the conventional generating functional may be written in a particularly simple form[4,14] namely

$$W[J] = \frac{\int \mathscr{D} A^a{}_\mu \; e^{i\int [\mathscr{L} + J^a{}_\mu A^\mu{}_a] d^4x} \; \delta[n^\mu A^a{}_\mu]}{\int \mathscr{D} A^a{}_\mu \; e^{i\int \mathscr{L} d^4x} \; \delta[n^\mu A^a{}_\mu]} \tag{1}$$

where $\mathscr{L} \equiv 1/4 \, (F_{\mu\nu})^2$, $F_{\mu\nu}$ being the standard non-Abelian field tensor:

$$F^a{}_{\mu\nu} \equiv \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + g f^a{}_{bc} A^b{}_\mu A^c{}_\nu$$

In this gauge the theory requires no ghosts so that the gluon Hilbert space, $\mathscr{H}_G$, is positive definite. In terms of (1) the gluon propagator is defined to be

$$\tilde{D}_{\mu\nu}{}^{ab}(x - y) \equiv - \frac{\delta^2 W[0]}{\delta J^a{}_\mu(x) \; \delta J^b{}_\nu(y)} \tag{2}$$

This can be related in the usual way to a time-ordered product of the fields:

$$\tilde{D}_{\mu\nu}{}^{ab}(x - y) = \langle 0 | T[A^a{}_\mu(x) \; A^b{}_\nu(y)] | 0 \rangle \tag{3}$$

The normalization inherent in (1) [i.e. dividing by "$\langle 0|0\rangle$"] will conventionally be suppressed; note, however, that its presence tells us that the vacuum state used in (3) is the vacuum appropriate to the gluon Hilbert space, $\mathscr{H}_G$. This is not, in general, to be identified with the physical vacuum state; we shall return to this point below in Section 3.

The momentum space representation of the propagator will be denoted by

$$D_{\mu\nu}{}^{ab}(q) \equiv \int d^4x \; e^{iq\cdot x} \; \tilde{D}_{\mu\nu}{}^{ab}(x) \tag{4}$$

Below we shall derive a Lehmann representation for this object and relate it to its spectral function defined by

$$\rho_{\mu\nu}{}^{ab}(q) \equiv \int d^4x\ e^{iq\cdot x} \langle 0|[A^a{}_\mu(x), A^b{}_\nu(0)]|0\rangle \tag{5}$$

These objects are, of course, gauge dependent and therefore their general Lorentz decomposition varies from gauge to gauge. In the axial gauge, for example, the most general form for $\rho_{\mu\nu}$ is given by

$$\rho_{\mu\nu}{}^{ab} = -\rho_1{}^{ab}\left[g_{\mu\nu} - \frac{q_\mu n_\nu + n_\mu q_\nu}{n\cdot q} + \frac{n^2 q_\mu q_\nu}{(n\cdot q)^2}\right]$$

$$+\ \rho_2{}^{ab}\left[g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right] \tag{6}$$

The scalar functions ρ_i depend on the invariants q^2, $n\cdot q$ and n^2; there is, of course, an analogous decomposition for $D_{\mu\nu}$ leading to two corresponding scalar functions $D_{1,2}$. Below we shall elaborate on the variable dependence of these functions when discussing the renormalization group equations. Since their dependence on the colour indices is trivial ($\propto \delta_{ab}$) we henceforth drop these indices.

Before deriving the Lehmann representation we should say a few words about the essential difference in structure between the Abelian (QED) and non-Abelian versions of the propagator. The Abelian piece of the field tensor is $\bar{F}^a{}_{\mu\nu} \equiv \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu$ so $F^a{}_{\mu\nu} = \bar{F}^a{}_{\mu\nu} + g f_{abc} A^b{}_\mu A^c{}_\nu$. Now, consider the correlation function

$$\int d^4x\ e^{iq\cdot x} \langle 0|[\bar{F}^a{}_{\mu\lambda}(x), \bar{F}^b{}_{\rho\nu}(0)]|0\rangle \tag{7}$$

which can be trivially related to $\rho_{\mu\nu}$ of Eq. (5). It is straightforward to check that its sole dependence on the gauge parameter n_μ is through the $n_\mu n_\nu$ term in Eq. (6). However, in QED, $\bar{F}_{\mu\nu}$ is gauge invariant (i.e., it is independent of n_μ) and so ρ_2 (the coefficient of $n_\mu n_\nu$) must vanish identically. Furthermore, this also shows that in this case ρ_1 can only depend on q^2 and indeed, below, we shall show that it is basically gauge invariant reflecting the gauge invariance of the classical long-range 1/r Coulomb potential. Note, however, that this argument immediately breaks down for QCD since $\tilde{F}_{\mu\nu}$ is no longer gauge invariant and thus $\rho_2 \neq 0$. In fact, this shows that ρ_2 is directly sensitive to the presence of the non-linear triple-gluon coupling and vanishes only in the free field limit, namely when $g \to 0$. Thus, an approximation in which ρ_2 is neglected as in Ref. 11 is inevitably an inconsistent one for the non-Abelian theory; this does not, of course, necessarily invalidate results on the IR behavior of ρ_1.

3. THE SPECTRAL REPRESENTATION

In this section we shall derive a spectral representation for $D_{\mu\nu}$ in the axial gauge. The fact that the ρ_i depend on $n \cdot q$ as well as q^2 introduces some minor complications which lead to a restriction to space-like n^2. Before beginning the derivation we need to introduce a notation for the decomposition of an arbitrary vector B_μ into its longitudinal (L) and transverse (T) parts with respect to n_μ: we define

$$B^\mu{}_L \equiv \left(\frac{n \cdot B}{n^2}\right) n^\mu$$

and

$$B^\mu{}_T \equiv B^\mu - B^\mu{}_L$$

Thus $B^2{}_L = (n \cdot B)^2/n^2$ and $B^2 = B^2{}_L + B^2{}_T$. Furthermore $n \cdot B_T = B_T \cdot B_L = 0$.

Consider the correlation function

$$\rho^+{}_{\mu\nu}(q,n) \equiv \int d^4x \; e^{iq\cdot x} \langle 0|A_\mu(x) \; A_\nu(0)|0\rangle \tag{8}$$

We shall assume that there exists a complete set of states $|N\rangle$ which spans $\mathscr{H}_G$ (one of which is the vacuum state $|0\rangle$). This set of states is not, in general, to be identified with the physical spectrum $|N_p\rangle$ which is complete in the physical Hilbert space $\mathscr{H}_p$; in general $\mathscr{H}_p$ may or may not overlap with $\mathscr{H}_G$. Note that the physical states $|N_p\rangle$ are not complete in $\mathscr{H}_G$ (and vice-versa; the "unphysical" states $|N\rangle$ are not complete - in fact, they are expected to be over-complete - in $\mathscr{H}_p$). In pure QCD, where one presumes colour confinement, the $|N_p\rangle$ states which span $\mathscr{H}_p$ will be colour singlets and will also lie in $\mathscr{H}_G$ and therefore overlap a sub-set of the set $|N\rangle$.

Introducing then the complete set $|N\rangle$ in (8) leads to

$$\rho^+{}_{\mu\nu}(q,n) = \sum_n \langle 0|A_\mu|N\rangle \langle N|A_\nu|0\rangle \; (2\pi)^4\delta^{(4)}(q - p_N) \tag{9}$$

with the restriction that $q_0 = p^0 \geq 0$. Now, from Eq. (5) $\rho_{\mu\nu}$ clearly satisfies a crossing property

$$\rho_{\mu\nu}(q,n) = - \rho_{\mu\nu}(-q,n) \tag{10}$$

Furthermore it can be trivially related to $\rho^+_{\mu\nu}$:

$$\rho_{\mu\nu}(q,n) = \rho^+_{\mu\nu}(q,n) - \rho^+_{\mu\nu}(-q,n) \tag{11}$$

The inverse of this reads

$$\rho^+_{\mu\nu}(q,n) = \theta(q_0)\ \rho_{\mu\nu}(q,n) \tag{12}$$

or, inverting the transform,

$$\langle 0|A_\mu(x)A_\nu(0)|0\rangle = \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\, \theta(q_0)\ \rho_{\mu\nu}(q,n) \tag{13a}$$

$$= \int \frac{d^4q}{(2\pi)^4}\, e^{-iq\cdot x}\, \theta(q_0) \int_{-\infty}^{\infty} dy_L\, e^{iq_L\cdot y_L}\ \tilde{\rho}_{\mu\nu}(q^2,y_L) \tag{13b}$$

where we have introduced the longitudinal Fourier transform, $\tilde{\rho}_{\mu\nu}(q^2,y_L)$ of $\rho_{\mu\nu}(q^2,q_L)$.[15] In terms of the canonical advanced (retarded) Green's functions

$$\Delta^{\pm}_{\mu\nu}(x,\mu^2) = \pm \int \frac{d^4q}{(2\pi)^4}\, e^{\mp iq\cdot x}\, \theta(q_0)\ \delta(q^2 - \mu^2) \tag{14}$$

we can express Eq. (13) as

$$\langle 0|A_\nu(0)A_\mu(x)|0\rangle = \int_0^{\infty} dq^2 \int_{-\infty}^{\infty} dy_L\ \tilde{\rho}_{\mu\nu}(q^2,y_L)\ \Delta^+(x-y_L,q^2) \tag{15a}$$

The crossing property, (10), leads to a crossed version of (15a):

$$\langle 0|A_\nu(0)A_\mu(x)|0\rangle = \int_0^{\infty} dq^2 \int_{-\infty}^{\infty} dy_L\ \tilde{\rho}_{\mu\nu}(q^2,y_L)\ \Delta^-(-x+y_L,q^2) \tag{15b}$$

We have thus far been a trifle cavalier about the tensor nature of $\rho_{\mu\nu}$ in the integrands of the above equations. In the coordinate representation the tensor decomposition of $\rho_{\mu\nu}$ in Eqs. (15) leads to space-time derivatives ∂_μ acting on the $\Delta^\pm$ functions. Now, the full propagator $D_{\mu\nu}$ is related to the time-ordered product of the fields and this operation does not commute with the time derivatives arising from the time components of $\rho_{\mu\nu}$. To proceed further we therefore restrict the discussion to the purely spatial components of (15) and consider

$$\langle 0|T[A_i(x)A_j(0)]|0\rangle = \int_0^\infty dq^2 \int_{-\infty}^\infty dy_L\, \tilde{\rho}_{ij}(q^2,y_L) \times [\theta(x_0)\Delta^+(x-y_L,q^2) + \theta(-x_0)\Delta^-(x-y_L,q^2)] \quad (16)$$

In conventional derivations of the Lehmann representation as, for example, in a covariant gauge[5,6], the analogue of $\rho_{\mu\nu}$ does not depend on y_L and so $\rho_{\mu\nu}$ is proportional to $\delta(y_L)$. The resulting combination of Green's functions in the square brackets of Eq. (16) is then precisely the standard Feynman function $\Delta_F(x,\mu^2)$ whose transform is the canonical $(q^2-\mu^2+i\epsilon)^{-1}$ singularity. This then leads to the conventional Lehmann spectral representation. The above argument clearly breaks down in axial gauge due to the presence of y_L; in general, there is no simple representation for the quantity in the square brackets of Eq. (16). On the other hand, when n_μ is space-like, y_L is space-like and causality allows us to replace $\theta(x_0)$ by $\theta(x_0-y_L)$. In that case one can indeed proceed as before and derive

$$\langle 0|T[A_i(x)A_j(0)]|0\rangle = \int_0^\infty dq^2 \int_{-\infty}^\infty dy_L\, \tilde{\rho}_{ij}(q^2,y_L)\, \Delta_F(x-y_L,q^2) \quad (17)$$

In momentum space this reduces to the standard Lehmann form:

$$D_i(q^2,q_L) = \int_0^\infty \frac{\rho_i(q'^2,q_L)}{q'^2 - q^2}\, dq'^2 \quad (18a)$$

in which q_L acts as a fixed parameter. It should be emphasized that (18a) is valid only when n_μ is space-like ($n^2 < 0$); our derivation fails in time-like gauges ($n^2 > 0$) since one can then no longer replace $\theta(x_0)$ by $\theta(x_0-y_L)$.

This representation is in the form of a standard dispersion relation which expresses the analyticity of the D_i as a function of q^2 when q_L is kept fixed. Indeed the conventional Lehmann representation can be derived directly from analyticity considerations which follow from the causal nature of the commutator in Eq. (5)[6]. From this point of view it is only natural to ask why such a proof breaks down when n^2 is time-like. The standard way of "proving" analyticity is to first establish analyticity in $q_0{}^2$ for fixed $\bar{q}^2$. To do so one notes that a typical factor in the integrand of Eq. (4) is of the form $e^{iq_0x_0}\,\theta(x_0)$ multiplied by a factor that vanishes outside of the light cone. Thus the integral over x is expected to converge provided $\mathrm{Im}q_0 > 0$ thereby defining an analytic function of q_0 in the upper half-plane. A similar argument obviously connects the vanishing of the commutator outside of the backward light cone with analyticity in the lower half-plane. The two regions can be connected since there is a region along the real axis defined by $q_0{}^2 < \bar{q}^2$ where D has no discontinuity. We can therefore write a standard dispersion relation in $q_0{}^2$ at fixed $\bar{q}^2$:

$$D_i(q_0{}^2,\bar{q}^2) = \int_{-\infty}^{\infty} \frac{\rho_i(q'_0{}^2,\bar{q}^2)dq'_0{}^2}{q'_0{}^2 - q_0{}^2} \tag{18b}$$

[The usual $i\epsilon$ prescription is to be understood.] A similar representation can be proven in terms of a different component of q_μ in place of q_0 provided the component is time-like; this simply reflects the possibility of replacing $\theta(x_0)$ in (16), for example, by $\theta(n\cdot x)$ provided $n^2 > 0$. If we are in a space-like gauge we can make a change of variables $q_0'^2 = q^2 + q'^2$ to derive (18a). If, however, we are in a time-like gauge this is not possible since then the integration variable is basically the gauge variable q'_L; thus, for example, when q_0^2 is varied, both q^2 and q^2_L necessarily vary so one cannot infer a dispersion relation in q^2 at *fixed* q_L.[16] Nevertheless, it should be emphasized that even in time-like gauges Eq. (18b) is expected to hold and, indeed this can be confirmed directly from Eq. (16).[17]

4. THE LEHMANN SUM RULE AND THE EQUATIONS OF MOTION

From Eq. (15) we can immediately derive a representation for the commutator[18] [see Eq. (5)]:

$$\langle 0|[A_i(x),A_j(0)]|0\rangle = \int \frac{d^4q}{(2\pi)^4} e^{-iq\cdot x} \rho_{ij}(q)$$

$$= \int_0^\infty dq^2 \int_{-\infty}^\infty dy_L \, \tilde{\rho}_{ij}(q^2,y_L) \, \Delta(x-y_L,\mu^2) \tag{19}$$

As before we restrict the indices to be space-like in order to avoid introducing kinematical time derivatives into the integrand via the time components of $\rho_{\mu\nu}$. Let us now set $x_0 = 0$; if y_L is space-like then $\Delta(x-y_L,\mu^2)|_{x_0=0} = 0$ showing that the imposition of the canonical commutation relations

$$[A_i(x),A_j(0)] \, \delta(x_0) = 0 \tag{20}$$

is consistent in space-like gauges. On the other hand, for time-like gauges (where y_L is time-like) setting $x_0 = 0$ does <u>not</u> imply that $\Delta(x-y_L,\mu^2)|_{x_0=0} = 0$; it would therefore be inconsistent to impose Eq. (20) as a canonical commutation relation in these gauges!

A sum rule[5,6] can be derived from the observation that in space-like gauges ($n^2 \leq 0$) $\partial_0\Delta(x-y_L,\mu^2)|_{x_0=0} = i\delta^{(3)}(x-y_L) = i\delta^{(2)}(\bar{x}_T)\delta(x_L-y_L)$. From (19) this leads to

$$\int d^4x \, e^{iq\cdot x} \langle 0|[\partial_0 A_i(x),A_j(0)] \, \delta(x_0)|0\rangle = i \int_0^\infty dq^2 \rho_{ij}(q^2,q_L) \tag{21}$$

That this is <u>not</u> valid in time-like gauges is, of course, intimately related to the fact that the Lehmann representation, Eq. (18a), is not valid in these gauges. Indeed it is not difficult to check that the left-hand-side of (21) is none other than $-i \lim_{q^2\to\infty} q^2 D_{ij}(q^2,q_L)$ which is consistent with Eq. (18a) only when $n^2 < 0$.

In order to extract information from (21) we need to explore the value of the equal time canonical commutation relation which follows from the equations of motion.[7] The canonically conjugate momentum to A_μ is $\Pi^a{}_\mu = F^a{}_{\mu 0}$; since $\Pi^a{}_0 = 0$, $A^a{}_0$ cannot be used as a dynamical variable and must be eliminated prior to quantization. The equations of motion are

$$\partial_\mu F^{\mu\nu}{}_a = gf_{abc}F^b{}_{\mu\nu}A^c{}_\mu \tag{22}$$

Note that this contains Gauss' law

$$\partial_i E^a{}_i = gf^a{}_{bc}{}^b{}_i A^c{}_i \tag{23}$$

where $E^a{}_i \equiv F^a{}_{i0}$ (the colour electric field) and this must be viewed as an equation of constraint since it does not involve time derivatives. Its main role here is to allow $E^a{}_L$ to be expressed in terms of $E^a{}_T$: explicitly

$$\partial_L E^a{}_L = -\partial_T \cdot E^a{}_T + gf^a{}_{bc}E^b{}_T \cdot A^c{}_T \tag{24}$$

(provided n^2 is space-like).

From the definition of $\Pi^a{}_\mu$ one can immediately derive that

$$\partial_L A^a{}_0 = E^a{}_L \tag{25}$$

Thus, in space-like gauges, $A^a{}_0$ can straightforwardly be eliminated in favour of transverse components for which canonical commutation relations are valid. Thus we impose the following commutation relations:

$$[A^\mu{}_T(x), A^\nu{}_T(0)]\,\delta(x_0) = [E^\mu{}_T(x), E^\nu{}_T(0)]\,\delta(x_0) = 0 \tag{26}$$

and

$$[A^\mu{}_T(x), E^\nu{}_T(0)]\,\delta(x_0) = i\left[g_{\mu\nu} - \frac{n_\mu n_\nu}{n^2}\right]\delta^{(4)}(x) \tag{27}$$

A rather long and tedious calculation employing the equations of motion then leads to the additional relation for the transverse components

$$\int d^4x\, e^{iq\cdot x} \langle 0|[\partial_0 A^a{}_i(x), A^b{}_j(0)]\,\delta(x_o)|0\rangle$$

$$= i\delta_{ab}\left(\delta_{ij} - \frac{q_i q_j}{q^2{}_L}\right) + ig^2\, C\int_{-\infty}^{\infty} dx_L x_L \epsilon(x_L)\, e^{iq_L x_L} \langle 0|A^a{}_i(x_L,0)A^b{}_j(0)|0\rangle \tag{28}$$

where $C = f_{abc}f^{abc}/N$. This is what is required to derive the sum rules from (21): one easily sees that

$$\int_0^{\infty} dq^2\ \rho_1(q^2,q_L) = 1 \tag{29a}$$

and

$$\int_0^{\infty} dq^2\ \rho_2(q^2,q_L) = L(q_L) \tag{29b}$$

where

$$\delta_{ij}\ \delta_{ab}\ L(q_L) = -g\cdot\ C\int_{-\infty}^{\infty} dx_L x_L \epsilon(x_L)\ e^{iq_L\cdot x_L} \langle 0|A^a{}_i(x_L,0)A^b{}_j(0)|0\rangle \quad .$$

The technical origin of the terms $x_L\epsilon(x_L)$ arises from eliminating the extra degrees of freedom via Eq. (24) and (25) [i.e. from $\partial^{-2}{}_L\delta(x_L)$]. Their presence is intimately related to the so-called principal value prescription for dealing with the $q^{-2}{}_L$ singularities in the axial gauge which we shall discuss in some detail below.

It is interesting to note that the conventional form of the sum rule for ρ_1 has survived with q_L acting simply as a parameter; the existence of canonical commutation relations guarantees the constancy of the right-hand-side. On the other hand, the sum rule for ρ_2 is considerably more dynamical in structure; the right-hand-side is not only q_L dependent but is directly sensitive to the triple-gluon coupling.[19] This is in agreement with our gauge argument at the end of Section 2 where we also concluded that $\rho_2 = 0$ when $g \to 0$.

The conventional Lehmann sum rule forms the basis for a proof that QED screens the charge; i.e. that the "physical" charge is necessarily smaller than the "bare" one. Crucial in the standard proof[5,6] is the positivity of ρ_1. We now turn to a discussion of this in QCD.

5. POSITIVITY CONSTRAINTS ON THE SPECTRAL FUNCTIONS

In QED the positivity of the photon spectral function plays a crucial role in proving that the charge is screened.[5] In this section we shall derive some analogous constraints for the ρ_i in axial gauge and examine how they give rise to anti -screening in QCD. Below we shall further examine their consequences for the infrared structure of the theory. To derive the constraints we first form the following object:[20]

$$p^\mu \rho^+{}_{\mu\nu} p^\nu = \sum_N \left|\langle 0|p\cdot A|N\rangle\right|^2 (2\pi)^4 \delta^{(4)}(q - p_N) \tag{30}$$

where p_μ is an arbitrary four-vector. In covariant gauges the Hilbert space is not positive definite and the set $|N\rangle$ contains states with negative norm; the presence of such ghosts leaves the sign of (30) indeterminate. To circumvent this problem Oehme and Zimmerman[2] projected the fields onto a positive definite part of the Hilbert space in order to define positive definite ρ_i. They then claim that anti-screening (i.e. asymptotic freedom) is inconsistent with such positivity and that only the additional presence of a certain minimum number of fermion flavours (10) can circumvent this problem. In the axial gauge on the other hand there are no ghosts so that (30) is indeed positive definite. We can read off from it the consequent positivity properties of the ρ_i and examine the possible conflict with asymptotic freedom.

Combining Eqs. (5) and (30) gives

$$-\rho_1\left[p^2 - \frac{2(n\cdot p)(p\cdot q)}{(n\cdot q)} + \frac{n^2(p\cdot q)^2}{(n\cdot q)^2}\right] + \rho_2\left[p^2 - \frac{(n\cdot p)^2}{n^2}\right] \geq 0 \tag{31}$$

There are three non-trivial possibilities for p: (a) perpendicular to n (i.e. $n\cdot p = 0$), (b) perpendicular to q (i.e. $p\cdot q = 0$) and (c) parallel to q (the case when p is parallel to n makes $p^{\mu}\rho^{+}{}_{\mu\nu}p^{\nu} \equiv 0$). We shall also employ the kinematical constraint that $q^2 \geq 0$; (for $q^2 \leq 0$, $\rho^{+}{}_{\mu\nu} \equiv 0$). Let us consider the possibilities one at a time:

a) $n \cdot p = 0$: this gives

$$-\rho_1 n^2 \frac{(p\cdot q)^2}{(n\cdot q)^2} + p^2(\rho_2 - \rho_1) \geq 0 \tag{32}$$

Suppose we now set $p^2=0$ in order to learn about ρ_1: then

$$-n^2 \frac{(p\cdot q)^2}{(n\cdot q)^2} \rho_1(q^2,q_L) \geq 0 \tag{33}$$

If $n^2 > 0$ then since $n\cdot p = 0$, $p_\mu \equiv 0$ and the above is trivially satisfied; on the other hand, if $n^2 < 0$, p_μ, though light-like need not vanish and we can deduce that $\rho_1(q^2,q_L) \geq 0$. This, of course, presumes that $(n\cdot q)^{-2} \geq 0$ which is violated by the usual principal value prescription. We shall return to this point in some detail below. Note, incidentally, that since $q^2 \geq 0$ it is impossible to set $p^2(n\cdot p)^2 = -n^2(p\cdot q)^2$ in (32) in order to deduce any information concerning the positivity of ρ_2, regardless of the sign of n^2.

b) $p\cdot q = 0$: this gives

$$(\rho_2 - \rho_1)p^2 - \frac{(n\cdot p)^2}{n^2} \rho_2 \geq 0 \tag{34}$$

Because q^2 is time-like (≥ 0) we are not permitted to set $p^2 = 0$ here and isolate ρ_2 in (34); in fact, p^2 is necessarily space-like ($p^2 \leq 0$). We are permitted however to set $n\cdot p = 0$ regardless of the sign of n^2 and deduce that $\rho_1 \geq \rho_2$. Note, incidentally, that we can set $n^2p^2 = (n\cdot p)^2$ here provided $n^2 < 0$ and reconfirm that $\rho_1 \geq 0$; this is a special case however since the condition also implies $n\cdot q = 0$, which we shall study below.

c) p = q: this gives

$$q^2\left[1 - \frac{n^2q^2}{(n\cdot q)^2}\right]\left[\rho_1 - \rho_2 \frac{(n\cdot q)^2}{n^2q^2}\right] \geq 0 \tag{35}$$

Since $q^2 \geq 0$ this requires

$$\rho_1 - \rho_2 \frac{(n\cdot q)^2}{n^2q^2} \geq 0 \tag{36}$$

regardless of sign (n^2). For $n^2 < 0$ this is equivalent to

$$\rho_2 \geq \frac{n^2q^2}{(n\cdot q)^2} \rho_1 \tag{37}$$

Let us summarize these results:

A. $n^2 > 0$: Neither ρ_1 nor ρ_2 have definite sign, as in a covariant gauge. Nevertheless the following restrictions are valid

$$\rho_1 \geq \rho_2 \tag{38a}$$

and

$$\frac{n^2q^2}{(n\cdot q)^2} \rho_1 \geq \rho_2 \tag{38b}$$

B. $n^2 < 0$: Again ρ_2 has no definite sign, but it is restricted to the following range:

$$\rho_1 \geq \rho_2 \geq \frac{n^2q^2}{(n\cdot q)^2} \rho_1 \tag{39a}$$

Furthermore

$$\rho_1 \geq 0 \tag{39a}$$

Note that (39a) allows ρ_2 to be zero, as it is in QED[21]

Finally, it should be emphasized that the point $n\cdot q = 0$ is excluded from this discussion since it is a singular point of the gauge and must be dealt with separately. Indeed, as Frenkel and Taylor[7] pointed out, it is precisely this gauge singularity that allows anti-screening to co-exist with the positivity of ρ_1 in axial gauges. Because the gauge singularity is a special point we shall discuss its origins and interpretations in some detail in the following section.

6. THE GAUGE SINGULARITY

The basic origin of the axial gauge singularity resides in the fact that the breaking of Lorentz invariance along one particular direction (n_μ) induces a fake long-range force and thereby an associated fake infinite energy. In two-dimensions this force is, in fact, real and is the reason such theories confine. However in higher dimensions such a force is illusory and has no explicit physics associated with it. The essential features of the singularity have basically nothing to do with the non-Abelian character of the theory so, for simplicity, we shall talk mainly in terms of QED. We shall examine the character of the singularity from that of the gauge nature of the theory as well as from the equations of motion.

In a covariant gauge, the spurious gauge singularities can be dealt with on an identical footing to real dynamical singularities with the additional proviso that the spin-statistics connection be relaxed; hence the concept of ghosts.[14] We can, in principle, thus gain an understanding of how to deal with the non-covariant q_L singularities in the axial gauge by starting in a covariant gauge and making a gauge transformation. For example, suppose we are given the spectral function in an arbitrary covariant gauge; denote it by $\rho^c{}_{\mu\nu}$. Its most general form is

$$\rho^c{}_{\mu\nu}(q) = -\rho^c(q^2) \left[g_{\mu\nu} - \beta(q^2) \frac{q_\mu q_\nu}{q^2}\right] \tag{40}$$

where $\rho^c(q^2)$ and $\beta(q^2)$ are arbitrary functions of q^2. We can obtain the axial gauge value of this, Eq. (6), by the gauge transformation

$$A_\mu = A^c{}_\mu - \partial_\mu \Lambda \tag{41}$$

where $\Lambda(x)$ is an arbitrary function of x. Since $n^\mu A_\mu = 0$,

$$\partial_L \Lambda = A^c{}_L \tag{42}$$

where $\partial_L \equiv n^\mu \partial_\mu$ and $A^c{}_L \equiv n^\mu A_\mu$. If we naively replace ∂_L by iq_L in momentum space then we find that $\rho_1(q^2, q_L, n^2) = \rho^c(q^2)$, independent of $\beta(q^2)$, and $\rho_2 = 0$ (as it must in QED). Indeed it is clear that no pure Abelian gauge transformation can change the coefficient of $g_{\mu\nu}$ and so even in non-covariant gauges, ρ_1 is a function of q^2 only. This is equivalent to the statement that Z_3 is gauge invariant ensuring not only that the notion of screening in QED is gauge invariant but so in Coulomb's law![6]

The naive replacement of ∂_L by iq_L ignores long-range surface terms and it is just these that are the source of the technical problem. To see how they arise we note that the general solution to Eq. (42) can be written as[22]

$$\Lambda(x_L, \bar{x}_T) = \int_{-\infty}^{\infty} A^c{}_L(x'_L, \bar{x}_T)\, G(x_L - x'_L, \bar{x}_T)\, dx'_L + \Lambda_0(\bar{x}_T)$$

where

$$G(x_L, \bar{x}_T) \equiv \epsilon(x_L) + \frac{1}{2}\left[\frac{1 - c(\bar{x}_T)}{1 + c(\bar{x}_T)}\right] ,$$

$$\Lambda_0(\bar{x}_T) \equiv \frac{\Lambda(-\infty, \bar{x}_T) + c(\bar{x}_T)\, \Lambda(\infty, \bar{x}_T)}{1 + c(\bar{x}_T)} \tag{43}$$

and $c(\bar{x}_T)$ is an <u>arbitrary</u> function. Notice that since

$$\Lambda(\infty, \bar{x}_T) - \Lambda(-\infty, \bar{x}_T) = \int_{-\infty}^{\infty} A^c{}_L(x_L, \bar{x}_T) dx_L \tag{44}$$

we cannot simultaneously make Λ vanish for $x_L = \pm\infty$, without also making $\rho(q^2)$ vanish. Denoting longitudinal Fourier transforms by a tilda we obtain from (43):

$$\tilde{A}(q_L,\bar{x}_T) = \tilde{A}^c{}_L(q_L,\bar{x}_T) \left[P \frac{1}{q_L} + K\pi\, \delta(q_L)\right] \tag{45}$$

where $K \equiv \Lambda(\infty,\bar{x}_T) + \Lambda(-\infty,\bar{x}_T)$. We have here followed the generalized function approach to the interpretation of the Fourier transform of singular functions[23] ;thus P represents the principal value prescription. If one carefully follows through the gauge transformation of $\rho_{\mu\nu}$ including the surface contributions one finds that it is indeed consistent to use the naive expression provided the $1/q_L$ singularities are interpreted according to the square bracket in (45). Thus,

$$\frac{1}{q_L} \to P \frac{1}{q_L} + K\pi\, \delta(q_L) \tag{46}$$

One still has the freedom to impose boundary conditions on Λ; the conventional choice[24] is $K = 0$, corresponding to $\Lambda(\infty,\bar{x}_T) = -\Lambda(-\infty,\bar{x}_T)$. This, of course, is just the standard principal value prescription. Notice that (46) can be represented as

$$\frac{1}{q_L} = \lim_{\epsilon\to 0} \frac{q_L + K\epsilon}{q^2{}_L + \epsilon^2} \tag{47}$$

and its derivative as

$$\frac{1}{q^2{}_L} = \lim_{\epsilon\to 0} \frac{q^2{}_L + 2q_L K\epsilon - \epsilon^2}{(q^2{}_L + \epsilon^2)^2}$$

$$= \lim_{\epsilon\to 0} \frac{(q_L + K\epsilon)^2 - \epsilon^2(1 + K^2)}{(q^2{}_L + \epsilon^2)^2} \tag{48}$$

Thus we see that, not only is the conventional principal value prescription (i.e. $K = 0$) for $1/q^2{}_L$ not positive definite, but neither is any allowable generalization ($K \neq 0$). This is important because it guarantees that no alternative prescription for dealing with the gauge singularity (i.e. $K \neq 0$) can lead to a definite sign for $1/q^2{}_L$. Thus, as discussed in the following section, the origin of asymptotic freedom does not depend on a specific prescription for the singularity. Note, also, that $1/q^n{}_L$ has no imaginary part regardless of the magnitude of K and so makes no contribution to unitarity thus preserving the ghost-free nature of the gauge.

Finally, we should point out that the gauge singularity is of course manifest in the equations of motion[22] and was, in fact, explicitly dealt with when deriving the commutators pertinent to the sum rules. The presence of $x_L\epsilon(x_L)$ in Eq. (28) corresponds precisely to the use of the standard principal value prescription. It originates there from the need to eliminate $A^a{}_0$ as in Eqs. (24) and (25):

$$\partial^2{}_L A_0 = \partial_L E_L = -D_T \cdot A_T \qquad (49)$$

These equations are precisely of the type satisfied by Λ and therefore have the same consequences and interpretation.

7. A PROOF OF ASYMPTOTIC FREEDOM AND ANTI-SCREENING

In this section we return to a study of the consequences of the sum rules, (29), exploiting the positivity constraints derived in Section 5 while keeping in mind the problem of the gauge singularity discussed in Section 6. Let us first recall the standard proof of screening in QED.[5,6] We have already shown that, in QED, $\rho_2 = 0$ and that ρ_1 is a function of q^2 only. Separating out the one-photon contribution to the unitarity sum isolates Z_3, the (gauge invariant) charge renormalization constant:[25]

$$Z_3 = 1 - \int_{4m^2}^{\infty} dq^2 \; \rho_1(q^2) \tag{50}$$

Here $\underline{m}$ is the electron mass. Since $\rho_1 \geq 0$ it is clear from this equation that Z_3 must be less than unity. Thus the physical charge is necessarily less than the "bare" charge and we have the phenomenon of screening. Physically this is due to the virtual creation of charged e^+e^- pairs which screen the "bare" charge.

In QCD there are some crucial differences. First, in axial gauge, ρ_1, though still positive, depends on q_L as well as q^2. The analogue to Eq. (50) thus reads

$$Z_3(q_L) = 1 - \int_{0}^{\infty} dq^2 \; \rho_1(q^2,q_L) \tag{51}$$

Notice that the second major difference with QED: the continuum contribution begins at $q^2 = 0$ since a gluon can create virtual massless gluon pairs. Indeed, isolating the one-gluon contribution is, in this sense, arbitrary and misleading since all thresholds pile up at $q^2 = 0$. The only reason for doing this is to isolate Z_3 with the view to understanding the origin of anti-screening.[26] We shall return to this infrared aspect of the problem in Section 9.

Now, as explained by Frenkel and Taylor[7], Eq. (51) is at first sight paradoxical since the positivity of ρ_1 should again lead to the conclusion that QCD, like QED, screens the charge and this is in contradiction with asymptotic freedom. However, as is manifest in Eq. (33), the positivity of ρ_1 assumes that $q_L{}^2$ is positive and this is violated by the prescription for dealing with the gauge singularity. In fact, what we actually proved was that

$$\rho_1(q^2,q_L) \geq 0 \qquad \text{when } q_L \neq 0$$

but that

$$\rho_1(q^2,0) \geq 0 \qquad \text{when } q_L = 0$$

Frenkel and Taylor pointed out that in perturbation theory it is this negative piece which solves the paradox and gives rise to asymptotic freedom. We can generalize their argument in a way which naturally leads into the renormalization group.

On dimensional grounds it is natural to introduce the dimensionless function $d(q^2/q^2{}_L) \equiv q^2\rho(q^2,q_L)$ in terms of which (51) reads

$$Z_3 = 1 - \int \frac{dq^2}{q^2}\, d\left(\frac{q^2}{q^2{}_L}\right) \tag{52}$$

Recall that the sign of the β-function is governed by the (gauge invariant) logarithmically divergent piece of (52)[1,4]. Now it is obvious that the large q^2 behaviour of the integrand corresponds to the limit $q^2{}_L \to 0$ <u>and this is precisely where the sign of d changes from positive to negative.</u> Thus we prove that

$$(Z_3)_{div} \geq 1 \tag{53}$$

corresponding to anti-screening and asymptotic freedom. Notice that this result goes beyond perturbation theory in the same sense that the original Lehmann proof of screening in QED $[(Z_3)_{div} \leq 1]$ does. Although a straightforward perturbative calculation of $d(\infty)$ naturally reproduces the standard result, it is worth emphasizing that this argument proves that QCD must be asymptotically free without the necessity of performing any calculation! In this regard we should mention that the whole analysis can be directly performed in the gauge $n \cdot q = 0$ which simplifies things considerably. The point is that with this choice, $\partial_L A^a_0 = 0$ so A^a_0 decouples from the dynamics and can be treated as a c-number. The evaluation of the commutator relevant to the sum rule is now trivial because $\partial_0 A^a_i$ can effectively be replaced by the canonical coordinate E^a_i. One can then derive a sum rule of the form (50), as in the covariant case, and prove that the corresponding spectral function is in fact negative definite leading directly to (53).

A couple of further comments are worth making before turning to a discussion of the renormalization group. The structure of the "paradox" implicit in (51), namely the apparent inconsistency of asymptotic freedom with the absence of ghosts, is strikingly similar to the work of Oehme and Zimmerman.[2] They work in a positive definite sector of an indefinite matric Hilbert space thereby maintaining covariance and positivity and likewise find an inconsistency. Our experience with the axial gauge suggests, though by no means proves, that a similar resolution must occur in their work. Thus, one expects that a hidden "kinematical" singularity analogous to the q_L^{-2} in axial gauge violates the positivity constraint in such a way as to produce anti-screening.

Our final comment of this section concerns the sum rule for ρ_2. Recall that, although ρ_2 has no definite sign, it is bounded from above by ρ_1. Thus we deduce that $L(q_L) \lesssim 1$. Since $L(q_L) \propto g^2\langle\tilde{A}\cdot\tilde{A}\rangle$ one might hope to use this to extract interesting bounds on the coupling constant. However we have not been able to do so and view this second sum rule as of limited value.

8. THE RENORMALIZATION GROUP EQUATIONS

In the argument at the end of the last section where we proved that $(Z_3)_{div} \geq 1$ we were somewhat cavalier in treating the hidden cut-off parameter essential for renormalizing the theory. In this section we shall concentrate on this aspect of the problem by considering the renormalization group equations for the propagator in axial gauge. The presence of the extra kinematical variable q_L (the "gauge parameter") introduces some minor complications, though, as we shall see, it also supplies a potentially powerful tool for probing the infrared structure of the theory. We shall work with the dimensionless function

$$d\left[\frac{q^2}{\mu^2}, \frac{q^2{}_L}{\mu^2}, g(\mu^2)\right] \equiv q^2 D(q^2, q_L) \tag{54}$$

For simplicity we have suppressed all indices. As usual μ represents the arbitrary mass scale at which we have chosen to normalize the theory. If this scale is changed (e.g. $\mu^2 \to \lambda\mu^2$) then the renormalizability of the theory requires that this be equivalent to a rescaling of d by a factor Z: explicitly

$$d\left[\frac{q^2}{\mu^2}, \frac{q^2{}_L}{\mu^2}, g(\mu^2)\right] = Z\left[\frac{q^2{}_L}{\mu^2}, g(\mu^2), \lambda\right] d\left[\frac{q^2}{\lambda\mu^2}, \frac{q^2{}_L}{\lambda\mu^2}, g(\lambda\mu^2)\right] \tag{55}$$

Differentiating this with respect to λ and setting $\lambda = 1$ leads to the renormalization group equation[1]

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial T} + \beta(g) + \gamma(T,g)\right] d(t,T,g) = 0 \tag{56}$$

Here we have introduced $t \equiv -\ell n\, q^2/q_0{}^2$, $T \equiv -\ell n\, q_L{}^2/q_0{}^2$, $\beta \equiv \partial g/\partial\lambda|_{\lambda=1}$ and $\gamma(T,g) \equiv \partial/\partial\lambda \;\; \ell n Z(T,g,\lambda)|_{\lambda=1}$. The solution to (56) can be expressed in various forms; two of the more interesting and useful are

$$d(t,T,g_0) = d[0,T-t,g(-t)]e^{-\int_0^t dt' \lambda[T-t',g(-t')]} \tag{57}$$

and

$$d(t,T,g_0) = d[t-T,0,g(-T)]e^{-\int_0^T dt' \lambda[T-t',g(-t')]} \tag{58}$$

where $g(-t)$ is the usual running coupling constant derived from $\beta(g)$. In QCD $\beta(g) \xrightarrow{g\to 0} -bg^3$ with $b > 0$ leading to asymptotic freedom.[1] This is reflected in the large q^2 behaviour of g:

$$g^2(-t) \xrightarrow{t\to-\infty} \frac{g_0{}^2}{1 - 2bg_0{}^2t} \tag{59}$$

Equation (57) implies that the large q^2 behaviour of d is related to its free field behaviour at fixed q^2 but with $q^2{}_L \to 0$ (since $T-t \to \infty$). This feature of the axial gauge was exploited in a less arcane fashion in the previous section when we proved that $(Z_3)_{div} \geq 1$. Recall, incidentally, that the exponential contributions in (57) and (58) typically induce powers of $\log q^2$ in the usual way.

A curious and potentially powerful feature of this gauge is the duality exhibited between q^2 and $q^2{}_L$ as shown explicitly in Eqs. (57) and (58). Just as the large q^2 limit can be approached by making $q_L{}^2$ small, so the small q^2 limit can, in principal, be approached by making $q_L{}^2$ large! For example, take $q^2{}_L \to \infty$ (i.e. $T \to -\infty$) in Eq. (58) then on the right-hand-side we need the value of d for small coupling $[g(-T) \to 0]$ and for "t" ($= t-T$) $\to \infty$ (i.e. $q^2 \to 0$). Although this is a delicate limit it illustrates how the behaviour in the infrared (at fixed coupling) can, in principle, be related to a similar behaviour but with an asymptotically free coupling constant. Such a connection between the ultra-violet and infrared is, of course, inherent in the renormalization group equations. Whether it can actually be exploited is not clear since it inevitably involves invoking boundary conditions in regions where we have little knowledge. In order to explore this possibility and to see whether one can exploit the presence of the extra variable q_L in the axial gauge we present a possible scenario in the following section.

9. THE INFRARED BEHAVIOUR OF THE PROPAGATOR, INSTANTONS AND CONFINEMENT

Naively, the infrared (IR) behaviour of the propagator is expected to tell us something about confinement; a highly singular behaviour for the transverse part is suggestive of a long-range confining force. On the other hand, the propagator is gauge dependent so it is difficult to know what meaning, if any, one can ascribe to its IR behaviour. The fact that it is gauge dependent does not, necessarily, mean that it does not contain physics; rather, it is that the physics is obscure and therefore difficult to extract. In QED the problem is easily circumvented since $\rho_1(q^2)$ is gauge invariant; indeed one can show that its IR behaviour leads to $1/q^2$ for the propagator corresponding to the classical $1/r$ Coulomb potential. As already shown earlier, no such simple solution emerges in QCD.

Because of this problem attention has generally been focused on the gauge invariant Wilson loop[10] (ω) rather than the propagator:

$$\omega \equiv \langle 0 | \mathrm{Tr}\, P\, e^{ig\oint A^a{}_\mu \lambda_a dx^\mu} | 0 \rangle \qquad (60)$$

Here P is the path ordering symbol which orders the λ_a matrices around the loop integral. For a rectangular loop lying parallel to the time axis $\omega \sim e^{-V(R)t}$ for large t; V(R) can be interpreted as the potential energy of a static quark-antiquark pair separated by a distance R. Thus it is the Wilson loop rather than the propagator that determines the (gauge-invariant) static long-range force. For a non-confining theory one expects $\omega \sim e^{-t}$ whereas for a confining one [where $V(R) \sim R$, for example] one expects $\omega \sim e^{-A}$, A being the area of the loop. The area law can therefore be taken as a signal for confinement.

In a recent paper[9] we showed how to relate the physics buried in the propagator to the Wilson loop. This is done via the following inequality:

$$\omega \leq e^{-1/2\, g^2 \oint dx_\mu \oint dy_\nu\, \tilde{D}^{ab}{}_{\mu\nu}(x-y)\, \delta_{ab}} \tag{61}$$

Notice that this involves bounding a gauge invariant physical object, ω, by a gauge dependent unphysical object. Thus, if in some gauge, D is highly singular in the IR, then (61) implies that $\omega \leq e^{-A}$, indicative of confinement. If, on the other hand, it behaves mildly, as in QED, then $\omega \leq e^{-t}$ and no conclusions can be drawn concerning confinement. Thus, the inequality (61) circumvents the gauge-dependence problem inherent in using the behaviour of the propagator as a signal for confinement: explicitly, (61) implies that a highly singular IR behaviour of the gluon propagator in any one gauge is sufficient to prove confinement. Physically it corresponds to the statement that the actual long-range static potential is at least as strong as the naive potential derived from the gluon propagator. We should exmphasize that the proof of (61) involves only that piece of $D_{\mu\nu}$ proportional to $g_{\mu\nu}$ (effectively only the transverse piece). In QED this piece is unaffected by a gauge transformation; in QCD, on the other hand, it remains gauge dependent and it is this property that presumably allows a highly singular behaviour to develop.

Most of the work on the IR behaviour of the gluon propagator in QCD has been within the context of the Schwinger-Dyson equations. Typically these are truncated and self-consistent solutions sought for the ensuing equations. There are several claims in the literature that this leads to a $1/q^4$ behaviour[11,13] which, if valid, would superficially, at least, be tantamount to a proof of confinement.

In this section we would like to explore two aspects of the problem within the axial gauge: (a) the constraints imposed on the IR behaviour due to analyticity and the positivity of the spectral functions; (b) the possibility that the IR limit can be obtained by taking $q_L^2 \to \infty$ using the renormalization group and that a highly singular behaviour is correlated with the existence of instantons.

a. Constraints due to Analyticity and Positivity

In Section 3 we showed that in a space-like gauge the $D_i(q^2,q\ )$ satisfy a Lehmann representation, Eq. (18a):

$$D_i(q^2,q_L) = \int_0^\infty \frac{\rho_i(q'^2,q_L)}{q'^2 - q^2}\, dq'^2 \tag{18a}$$

This expresses the analyticity of $D_i(q^2,q_L)$ as a function of q^2 at fixed q_L. If $\rho_i \geq 0$ then there is no way it can produce a singularity in D_i stronger than the simple pole already manifest in the integrand. Loosely speaking, a singularity can only be made stronger by cancelling it against a similar singularity and this is not possible if $\rho_i \geq 0$. A simple example is provided by the suggested $1/q^4$ behaviour: this corresponds to $\rho \sim \delta'(q^2)$ which is not positive definite. This can be generalized; consider the complex variable $z = re^{i(\pi-\theta)}$ where r is its modulus and θ its phase. The real axis is approached from above by taking the limit $\theta \xrightarrow{\epsilon\to 0^+} \epsilon/r$. Suppose $D(z)$ were of the form $z^{-\nu}$ (ν arbitrary) so that the complex plane were cut along the positive real axis; the discontinuity across this cut is

$$2i\rho = \text{disc } D(z) \overset{\epsilon\to 0}{=} r^{-\nu} \sin\nu(\pi - \theta)$$

One can immediately see that this is positive only when $\nu \leq 1$; incidentally when $\nu = 1$ we can easily check that this gives the standard δ-function, as it should. This argument can be generalized to other classes of singularities and is, in fact, a general consequence of the positivity. Thus we conclude that for $\rho_i(q^2,q_L) \geq 0$,

$$\lim_{q^2\to 0} D_i(q^2,q_L) \leq \frac{1}{q^2} \qquad (62)$$

i.e. <u>the positivity of the spectral function restricts the propagator from being any more singular in the IR than it is in QED!</u> Now, for $n^2 < 0$ we have proven that $\rho_1 \geq 0$ provided $q_L \neq 0$ as is the case here since we are not interested in large q^2. Thus $D_1(q^2,q_L)$ cannot be more singular than $1/q^2$. However, as discussed above it is the combination

$$D_t(q^2,q_L) \equiv D_1(q^2,q_L) - D_2(q^2,q_L) \qquad (63)$$

which is relevant for confinement since this is the coefficient of $g_{\mu\nu}$. However, its spectral function, $(\rho_1-\rho_2)$, was <u>also</u> proven to be positive (independent of q_L, incidentally) and so D_t also cannot be more singular than $1/q^2$. <u>We therefore conclude that in space-like axial gauges one can learn nothing about confinement from studying the gluon propagator!</u>

The situation in time-like gauges ($n^2 > 0$) is considerably more subtle since there is no longer any reason to believe that D_i are analytic in q^2. Unfortunately, however, the argument leading to the interpretation of (61) strictly speaking breaks down. Essential in its proof[9] was the requirement that n^μ be perpendicular to the loop, i.e. $n^\mu \oint dx_\mu = 0$. In order to make the connection to the static potential and therefore confinement it was necessary to line the loop up parallel to the time axis and this necessitates $n^\mu \oint dx_\mu \neq 0$. Of course, one might argue that in Euclidean space such a restriction is unnecessary since, there, space and time are on a more equal footing. Thus orienting the loop perpendicular to the time axis can still lead to a valid criterion for confinement and one can justify using (61) for $n^2 > 0$.

Given that this is valid one might then argue that the absence of a Lehmann representation now allows $D_1(q^2, q_L)$ to develop a highly singular behaviour, especially since there is no longer the restriction that $\rho_1 \geq 0$. However, some care must be taken; as emphasized in Section 3, though D_1 cannot be proven to be analytic in q^2 it is still expected to be analytic in $q_0{}^2$ at fixed $\bar{q}^2$, as expressed in Eq. (18b). In a time-like gauge we can, for simplicity, identify $q_0 = q_L$. As already mentioned there is no positivity constraint on ρ_1 so D_1 is permitted to have a highly singular behaviour in $q_0{}^2$ and therefore presumably in q^2. However we showed in Section 4 that even when $n^2 > 0$ $\rho_1 - \rho_2 \geq 0$ (regardless of the size of q). Thus D_t, the invariant relevant to confinement, cannot have a more singular behaviour than a simple pole in q^2. Note that this does not contradict the work of Baker, Ball and Zachariasen[11] who claim to have found a self-consistent behaviour of $1/q^4$ for the IR structure of $D_1(q^2, q\ )$. A crucial input into their calculation was to set $D_2 = 0$ and look only at D_1. However it is the combination $D_1 - D_2$ that is important for confinement and according to our arguments this is always guaranteed to be no more singular than it is in QED. We therefore extend our conclusions to $n^2 > 0$ and claim it is not possible to unambiguously learn about confinement from the IR behaviour of the gluon propagator in any axial gauge. We should stress that it is certainly not inconceivable, though we believe unlikely, that the IR behaviour of D_1 can by itself be related to a criterion for confinement.

b. The IR Limit from $q^2{}_L \to \infty$ and its Connection to Instantons

We showed at the beginning of this section that the small q^2 limit can be derived by taking $q^2{}_L \to \infty$. On the other hand we have just seen that it does not seem possible to use the axial gauge gluon propagator to learn about confinement and this considerably weakens the potency of using the large $q^2{}_L$ ruse. Nevertheless this limit still seems worth exploring since there is evidence[11] that D_1 acquires a highly singular behaviour which is presumably non-perturbative in origin. Indeed it is expected that this is connected to the presence of non-trivial topological excitations (instantons) which permeate the theory.[8] Although the work of Ref. 11 was outside of the domain of perturbation theory it made no explicit reference to instantons. We shall now explore how the $q_L{}^2 \to \infty$ limit can be used to show that such non-perturbative pieces, lead to a change in the IR power behaviour. The argument presented is rather schematic and is given simply to illustrate how the connection might evolve.

In perturbation theory the dimensionless propagator can be expanded as follows:

$$d(t,T,g) = \sum_{n=0}^{\infty} a_n(t,T)g^{2n} \tag{65}$$

In QCD there are well-known non-perturbative contributions and there can be expected to add a term of the form[1,8]

$$d(t,T,g) = e^{-c/g^2} f(t,T) \tag{66}$$

This may well be modified by powers of g; however, for the present purposes we wish only to concentrate on the effects due to the essential singularity in (66). Consider the limit $q^2{}_L \to \infty$ (i.e. $T \to -\infty$); then in the right-hand-side of Eq. (58) we only need the asymptoticallyfree value of g. In that case the presence of the anomalous dimension term γ will, as usual, only produce logarithmic corrections; these are, in any case, independent of t and so we shall ignore them since we are looking for a power behaviour. Substituting (66) into (58) requires

$$e^{-c/g^2_0} f(t,T) \approx e^{-c/g^2(-T)} f(t-T,0) \tag{67}$$

where logarithmic corrections (as well as ordinary perturbative effects arising from (65) have been temporarily neglected). When $T \to -\infty$ we can use (59) to obtain the equation

$$f(t,T) = e^{2bct} f(t-T,0) \tag{68}$$

Notice that $g_0{}^2$ has dropped out of this equation. We can now take the IR limit explicitly, i.e. $q^2 \to 0$ or $T \to -\infty$, to obtain $f \sim e^{2bct} = (q^2{}_0/q^2)^{bc}$. It should be emphasized that the perturbative contribution (65) only leads to powers of logarithms as does the anomalous dimension term. It is the crucial presence of the essential singularity term e^{-c/g^2} that leads to a power law behaviour. This, therefore, strongly suggests that the presence of instantons will induce, in general, a power law behaviour of the form

$$D_1(q^2,q_L) \xrightarrow{q^2\to 0} (q^2{}_0/q^2)^{1+bc} \tag{69}$$

Note, incidentally, that if this argument is valid the scale of the power is the same as that which governs the logarithmic scale of asymptotic freedom! Obviously if there is some constraint that forbids such a power law behaviour in certain amplitudes, as discussed above, then c must effectively be zero. We therefore conclude on general grounds that if non-perturbative pieces of the form (66) are present and the theory is renormalizable then pieces of the propagtor will have a highly singular IR behaviour.

SUMMARY

This paper has been devoted to a general study of the gluon propagator in axial gauge. We have emphasized throughout the essential difference between space-like and time-like gauges; roughly speaking, quantization in time-like gauges is considerably more subtle since it tends to interfere with the notion of equal time commutation relations and causality.[27] Consequently, properties such as the Lehmann representation can only be proven in space-like gauges. In such gauges, however, there is an apparent contradiction between the positivity of the Hilbert space and the idea of anti-screening (or asymptotic freedom). We spent some time elaborating and generalizing an argument of Frenkel and Taylor on a solution to this paradox which revolves around the treatment of the so-called gauge singularity. We pointed out that their argument survives any permissible generalization of the usual principal value prescription and can in fact be put into non-perturbative terms.

Although the IR behaviour is considerably more dynamical than the UV we pointed out tha positivity of certain spectral functions constrains the corresponding invariant to be no more singular in the IR than it is in QCD. Indeed we showed that the very piece that determines an extremum of the Wilson loop falls into this class regardless of whether the gauge is time-like or space-like. Thus we claim that all we can show from a dynamical IR study of the propagator in this gauge is that the force law between static quarks must be at least Coulombic; a not terribly useful result. Nevertheless other pieces of the propagator that do not contribute to this may well be highly singular. We point

out that this can be investigated by taking the limit where the parameter $q_L^2 \to \infty$. This is made explicit in the appropriate renormalization-group equations where we find the interesting result that this limit does indeed drive the theory into the IR ($q^2 \to 0$) but with an asymptotically free coupling constant. This is then used to demonstrate that the change in power law behaviour in the IR can be directly correlated with the presence of terms e^{-c/g^2} presumably arising from instantons. This suggests that the non-perturbative power behaviour $1/q^4$ found in Ref. 11 can, in fact, be associated with instantons, though probably not with confinement. This exercise clearly suggests that in covariant gauges where positivity can be relaxed there may well be a similar technique based on the renormalization group for correlating a highly singular IR behaviour with the effects of instantons. In that case one could conclude that this is tantamount to a proof of confinement.

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12. By this we mean the piece that survives when the non-Abelian

coupling is turned off (i.e., when the group is U(1)).

13. Thus, none of our arguments apply to the work of S. Mandelstam, Phys. Rev. D20, 3223 (1979); working in a covariant gauge he finds a $1/q^4$ singularity for the propagator.

14. E. Abers and B. W. Lee, Phys. Rep. 9C, 1 (1973).

15. We shall typically suppress the explicit dependence of the ρ_i on n^2.

16. Thus, from both points of view, the breakdown of q^2-analyticity for time-like n^2 is related to the impossibility of fully exploiting the causal nature of the commutator.

17. Throughout this discussion we have ignored the possibility of subtractions (or their equivalent) in writing the respresentations. Roughly speaking, since asymptotic freedom determines the large q^2 behaviour this is an irrelevant issue (see Ref. 2).

18. Typically, throughout this section Latin indices will refer to transverse (T) components; in some cases we shall make this explicit in order to emphasize the limitation.

19. Indeed (29b) can be viewed as a consistency condition on the definition of ρ_2.

20. We should reemphasize at this stage that the complete set $|N\rangle$ are not necessarily the asymptotic states of the theory $|N_p\rangle$; nevertheless the absence of ghosts in the axial gauge means that squares of matrix elements are positive definite.

21. Thus neglecting ρ_2 as in the truncation scheme of Ref. 11 does no violence to positivity constraints.

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24. Presumably care must be taken that setting K = 0 [i.e. $\Lambda(\infty,\bar{x}_T)$ $= -\Lambda(-\infty,\bar{x}_T)$] does not contradict boundary conditions being imposed on A^a_μ; if it does then, of course, the principal value prescription could lead to erroneous results.

25. These formulae are written in terms of "unrenormalized" fields; a rescaling of the gluon field by a factor $Z_3^{1/2}$ produces a sum rule for Z_3^{-1} in terms of the "renormalized" spectral function; see Refs. 5 and 6. Our results do not depend on this rescaling; the ability to do this is, of course, the basis of the renormalizaton group equations discussed in Section 9; see also Refs. 5 and 6.

26. Since Z_3 in this formalism is the coefficient of the $\delta(q^2)$ contribution to $\rho(q^2)$ a physical interpretation can only be made within the context of perturbation theory. For, in general, the singularity at $q^2 = 0$ could be quite different. Indeed, even in QED the use of the electron pole in the electron propagator to isolate Z_2 is similarly artificial since its singularity is typically a gauge dependent cut. This has been used by the author as a starting point for a confinement criterion for dynamical quarks in the full theory of QCD; see G. B. West. Phys. Rev. Lett. 46, 1365 (1981).

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