SLAC-PUB-2934
June 1982
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SOFT GLUON EFFECTS IN PERTURBATIVE QUANTUM CHROMODYNAMICS*

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(Submitted to Nuclear Physics B)

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*Work supported in part by the Department of Energy under contract DE-AC03-76SF00515.

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ABSTRACT

The problem of the inclusion of sub-leading logarithms to all orders of the perturbative expansion is considered and solved in the case of multiple soft gluon emission. The evaluation of the entire set of the sub-dominant contributions is performed in a consistent way at two loop level in the impact parameter space for the effective quark form factor. The effect of the inclusion of the single logarithms is found to be large also at one loop level as compared with the dominant double logarithms and negligible corrections are seen arising from the second order terms. We stress the relevance of the single logarithmic effects by computing the cross section for production of hadrons at large acollinearity angles in electron-positron annihilation.

1. INTRODUCTION

The question about the validity of the Quantum Chromodynamics as the theory describing the physics of the strong interactions has represented a main topic in the experimental and theoretical work in the past few years.[1] The appealing possibility that a perturbative treatment of the theory can give some insight about the dynamics of the processes involving hadrons has shown both virtues and limits of such approaches. In fact, even it seems clear that a general qualitative agreeement can be obtained in the comparison with the experimental data, the efforts toward a more precise, quantitative, analysis have been, so far, unsuccessful. The reason is that even if the short distance behaviour of the theory is quantitatively described by a perturbative treatment, the large distance non-perturbative effects still elude a systematic analysis.[2]

A particularly interesting class of processes in perturbative QCD is represented by the semi-inclusive semi-hard processes which are characterized by the presence of two large but different mass scales. Examples are the cross section electron+positron → A+B+X with A and B hadrons at a relative transverse momentum Q_T^2 much smaller than Q^2 ($Q_T^2 \ll Q^2$) the total center-of-mass energy of the initial lepton pair and the Drell-Yan cross section h+h → $\mu^+\mu^\mp X$ with the lepton pair transverse momentum Q_T^2 much smaller than the pair mass M^2 ($Q_T^2 \ll M^2$). For these processes a simple perturbative expansion is not sufficient; large logarithms of the ratio of the two scales $\ell n\, Q^2/Q_T^2$ appear in the perturbative series and must be resummed to all orders in the running coupling constant to obtain a meaningful answer. The presence of double logarithmic corrections in the form of a Sudakov-type[3] form factor is a common characteristic of these reactions. When inclusive cross sections are considered the double logarithms disappear due to the compensation of the real and virtual contributions and standard resummation techniques can be applied. This is not the case in the particular regions of the phase space considered above in the semi-inclusive processes. Due to the enhanced importance of the soft bremsstrahlung the compensation of real and virtual contributions is incomplete and double logarithms appear. Furthermore to deal with double logarithmic corrections the standard resummation techniques are no longer applicable and particular procedures must be developed. Such an extension of the range of applicability of the standard perturbative analysis makes the study of these processes relevant to investigate the perturbative phase of the theory.

The use of this improved perturbative analysis is furthermore the correct theoretical framework to deal with the dynamics of multiple soft bremsstrahlung and has a more general range of interests than only the examples mentioned before: it can give a link between hard and soft physics. As one example in the electron positron annihilation the largest part of the cross section is given by the two jet configurations. The kinematical region $Q_T^2 \simeq Q^2$ is only a small fraction of the total phase space. The study of the intermediate transverse momentum region can give a link between the hard physics (3 jet events) and the soft physics (2 jet events).

The first step toward the understanding of the physics involved in the semi-inclusive processes has been made by Dokshitzer, Dyakonov and

Troyan.[4] In an important paper Parisi and Petronzio[5] have summed the leading double logarithmic contributions to the Sudakov-type form factor. A first attempt to include single logarithmic terms has also been performed. More recently various authors[6-8] have stressed the importance of the inclusion of non-dominant terms in the perturbative expansion. It is the purpose of this paper to make a systematic analysis of the leading double-logarithmic and sub-leading single-logarithmic contributions in the semi-inclusive processes or equivalently in the effective quark form factor. The analysis is carried in the impact parameter space. We stress the importance of a correct treatment of the kinematics and of the inclusion of the two loop corrections to treat the sub-leading corrections in a consistent way. By including the corrections at order α^2 we resum leading and sub-leading logarithms to all orders in the perturbative expansion. The sub-leading corrections are found to be large having a value comparble to the leading ones at present energies even if the coefficient of the terms arising from the two-loop amplitudes is small. We compare our results with previous analyses and we show the appearance of a new set of terms which have not been included before. In our derivation we use the formalism of the jet calculus of Konishi, Ukawa and Veneziano[9] and the Altarelli-Parisi equation in its generalized form proposed by Bassetto, Ciafaloni and Marchesini.[10]

The outline of the paper is the following. In Section 2 we describe the formalism and consider the soft limit in which we evaluate the parton distributions. After solving the evolution equations we compare the result with the previous ones and analyze the differences. The inclusion of the two loop corrections is made in Section 3 where also the effects of the soft approximation are estimated by computing the physical cross section $e^+e^- \rightarrow A+B+X$. In Section 4 we comment on the result and draw some conclusions. A short note on this work has appeared elsewhere.[11]

2. THE SOFT LIMIT IN THE EVOLUTION EQUATIONS

Any process which can be described by using the Quantum Chromodynamics theory in its perturbative phase is characterized by the presence of a parton (quark or gluon) with a large virtual mass Q^2. The parton may emit quarks and gluons by bremsstrahlung. Two comletely different situations can be distinguished: One case is when a single "hard" parton (b) emitted at large transverse momentum $p^2 \simeq Q^2$ with respect to the parent parton (a) (Fig. 1a.). This configuration is exemplified by the process $e^+e^- \rightarrow q\bar{q}g$ with the gluon at large angles with respect to both quark and antiquark (the 3-jet cross section). A simple perturbative analysis at a finite order in the coupling constant α_s is sufficient to describe such cross sections. A second case is when the initial parent parton give rise to a series of branching processes in which many-parton states are generated made by both soft and collinear quarks and gluons at small transverse momenta $p^2 \ll Q^2$ with respect to the initial parton (Fig. 1b). Such configurations can be also described perturbatively in the limit in which the virtualness of the partons are strictly ordered $Q^2 \gg k_1{}^2 \gg k_2{}^2 \gg \ldots \gg k_n{}^2$ in the physical axial gauges by tree like graphs in the leading logarithmic approximation by using an improved perturbation theory. An algorithm, the Jet Calculus, has been proposed by Konishi, Ukawa and Veneziano[9] to deal with such "semi-hard" configurations. In the Jet Calculus the many parton states are described by parton distribution functions $D(Q^2,\{x_i\})$ in terms of the initial energy and longitudinal momenta of the final partons. A further extension of the Jet Calculus formalism to include also explicitly the transverse momenta degrees of freedom in the parton distributions has been developed by Bassetto, Ciafaloni and Marchesini[10]. In their approach the parton distributions $D(Q^2,\{x_i\},\{p_{Ti}\})$ are functions also of the transverse momenta of the emitted partons. This extension is particularly useful for the problem we will study of the description of many soft partons configurations in the perturbation evolution. The generalized evolution equation they derived is in the case of one-particle inclusive distribution

$$D_a{}^b(Q^2,p_T,x) = \delta_{ab}\ \delta(1-x)\ \delta^{(2)}(p_T) + \sum_c \int_{Q_0{}^2}^{Q^2} \frac{dk^2}{k^2} \frac{\alpha_s(k^2)}{2\pi} \int_x^1 \frac{dz}{z} P_a{}^c(z)$$

$$\times \frac{d^2q_T}{\pi}\ \delta(z(1-z)k^2 - q_T{}^2)\ D_c{}^b\left(k^2, p_T - \frac{x}{z} q_T, \frac{x}{z}\right) \qquad (2.1)$$

where the a,b,c indices represent the type of initial, final and intermediate parton and the $\sum_c$ is over all the possible intermediate partons. x and p_T and z, q_T are the longitudinal and transverse momenta of the final (b) and intermediate (c) parton respectively with respect to the initial one (a).

Graphically Eq. (2.1) can be represented as in Fig. 2. The transverse momenta appear in the integrated part on distributions of Eq. (2.1) in the combination p_T - $x/z\ q_T$ which assures the correct invariance of the distributions under Lorentz transformations. The running coupling constant is defined to be $\alpha_s(k^2) = \beta_0{}^{-1}\ \ell n\ k^2/\Lambda^2$ with $\beta_0 = (11-2/3\ N_f)/4\pi$ for $SU(3)_{color}$ where N_f is the number of flavors. $P_a{}^c(z)$ are the standard Altarelli-Parisi probabilities. The scale $Q_0{}^2$ defines the lower limit of the range of applicability of the perturbative analysis and is such that $\alpha_s(Q_0{}^2)/2\pi \lesssim 1$.

In the particular channel a = c = quark the real part of the vertex function $\hat{P}_q{}^q(z)$ is given by

$$\hat{P}_q{}^q(z) = c_F \frac{1 + z^2}{1 - z + \epsilon(k_{\parallel})} \qquad (2.2)$$

where

$$\epsilon(k_{\parallel}) = \frac{k^2}{2k_{\parallel}{}^2} = \frac{k^2\eta^2}{4(k\cdot\eta)^2}$$

with η the gauge vector. Throughout this paper we shall use the light-like gauge $\eta^\mu A_\mu = 0$ with $\eta = (1,\vec{0},-1)$, $\eta^2 = 0$ which gives $\epsilon(k_{\parallel}) = 0$.

We will be interested in the configurations in the parton distributions dominated by the emission of soft partons. To specialize the evolution cascade to the case when soft emissions take place let us consider the iteration structure of Eq. (2.1). The soft emission range can be reproduced if at each vertex (black blob in Fig. 2) the parton (c) carries most part of the longitudinal momentum and has a limited transverse momentum with respect to the parent. Such conditions can be satisfied by taking the limits $z \to 1$ and $q_T{}^2 \ll Q^2$ in Eq. (2.1). By looking at Eqs. (2.1) and (2.2) we can see that the limit $z \to 1$ directly enters in the vertex functions and furthermore the real part of the quark-quark channel (2.2) is large in this limit. The large contribution comes form the infrared singular behavior and accounts for the high propability of emitting soft gluons by a quark like with $z \to 1$. When also the virtual contributions are included in the total vertex function the singularity itself disappears. The qq vertex gives among the possible $P_i{}^j(z)$ probabilities the largest contribution in the soft

$z \to 1$ limit. In order to pick up the dominant contribution to the multiple soft emission amplitude in Eq. (2.1) it is sufficient to iterate the quark channel, i.e.

$$\sum_c \equiv \sum_q$$

in Eq. (2.1). The matrix equation (2.1) can be reduced to the equation containing only the non-singlet channel (NS) which gives the largest contribution to the amplitude for producing a final quark from an initial quark by emitting only soft gluons. As we will see later this is also true when two loop vertex functions will be introduced. By iterating Eq. (2.1) in the non-singlet channel in the limit $z \to 1$ the contributions to the $q \to q$ amplitude are given by graphs as in Fig. 3 with emission of soft gluons, the quark lines carrying a large fraction of the longitudinal momentum.

As has been pointed out by many authors[12,13] a common feature of the processes in which the soft bremsstrahlung has a dominant role is given by the correct evaluation of the argument of the coupling constant α_s. The probability of emitting a soft parton is proportional to the running coupling constant evaluted at the transverse momentum of the parent and not at its total virtualness. It has been shown[13] that this behavior can be incorporated in the evolution equations by simply rescaling the argument of the coupling constant from k^2 to $k^2(1-z)$ in Eq. (2.1) i.e. $\alpha_s(k^2) \to \alpha_s(k^2(1-z))$. When soft emission effects have a relevant role, in fact, as it is the case for example when structure or fragmentation functions are evaluated in the $x \to 1$ limit, the balance between the real and virtual contributions is broken. The large fraction of momentum carried by one parton limits the available phase space for the real emissions which are then unable to compensate the large, opposite in sign effects of the virtual contributions. As a consequence large corrections appear in the perturbative expansion of the form $\alpha\ \ell n\ 1-x$ which must be summed. It has been proven[14] that the use of the rescaled coupling constant in the evolution equations (2.1) resums these large corrections at all orders in the perturbative series in the leading logarithmic approximation in the inclusive processes. Also an explicit two loop calculation[15] confirm this feature of the rescaled coupling constant in the case of inclusive processes. Furthermore, intuitive arguments and an overall consistency support the generality of this choice also in dealing with semi-inclusive processes. As first observed in Refs. 4,5 in fact in the Drell-Yan cross section for small transverse momenta of the lepton pair the argument of the running coupling constant in the quark Sudakov form factor is in fact the transverse momentum of the annihilating partons. By using the rescaled coupling constant and taking the logrithmic derivative with respect to the scale Q^2 in Eq. (2.1) one has in the NS channel

$$Q^2 \frac{\partial}{\partial Q^2} D(Q^2,x,p_T) = \int_x^1 \frac{dz}{z} \left[\frac{\alpha_s(Q^2(1-z))}{2\pi} P(z)\right]_+ \frac{d^2q_T}{\pi}\, \delta(z(1-z)Q^2 - q_T{}^2) \times D\left(Q^2, p_T - \frac{x}{z} q_T, \frac{x}{z}\right) \tag{2.3}$$

$P(z)$ represent the q-q vertex. Here the + sign represent the usual regularization procedure, i.e.

$$\left[\frac{\alpha_s(Q^2(1-z))}{2\pi} P(z)\right]_+ = \frac{\alpha_s(Q^2(1-z))}{2\pi} P(z) - \delta(1-z) \int_0^1 \left[\frac{\alpha_s(Q^2(1-z))}{2\pi} P(z)\right] dz \tag{2.4}$$

with $P(z) = 1+z^2/1-z$ in the light-like gauge. To solve Eq. (2.3) it is convenient to take the Fourier transform from the transverse momentum p_T to the impact parameter b_T[5]. Defining the transformed distributions

$$D(Q^2,b_T,x) = \int d^2p_T \exp\left(-i \frac{b_T \cdot p_T}{x}\right) D(Q^2,p_T,x) \tag{2.5}$$

Equation (2.3) becomes

$$\begin{aligned} Q^2 \frac{\partial}{\partial Q^2} D(Q^2,b,x) &= \int_x^1 \frac{dz}{z} C_F \left[\frac{\alpha_s(Q^2(1-z))}{2\pi} \frac{1+z^2}{1-z}\right] \\ &\times \frac{d^2q_T}{\pi} \exp\left(-i \frac{b_T \cdot p_T}{z}\right) \delta(z(1-z)Q^2 - q_T^2)\, D\left(Q^2,b,\frac{x}{z}\right) \\ &= \frac{C_F}{2\pi} \int_x^1 \frac{dz}{z} \int dq^2 \left[\alpha_s(Q^2(1-z)) \frac{1+z^2}{1-z}\right] \\ &\times \delta(z(1-z)Q^2 - q^2)\, J_0\left(\frac{bq}{z}\right) D\left(Q^2,b,\frac{x}{z}\right) \end{aligned} \tag{2.6}$$

With $b = |b_T|$, $q = |q_T|$ and J_0 the Bessel function of the first kind. The last equality in (2.6) follows from the integration over the angular variable in the two dimensional space d^2q . The use of the Fourier transformed equation (2.6) guarantees the conservation of the transverse momentum in the evolution. By taking the limit defined above $z \to 1$ and setting $z = 1$ in each slowly varying function of z, one has the solution:

$$D(Q^2,b,x) = D(Q_1{}^2,b,x) \exp(T_1(Q^2,Q_1{}^2,b)) \tag{2.7}$$

with

$$T_1(Q^2,Q_1{}^2,b) = -\frac{C_F}{\pi}\int_{Q_1{}^2}^{Q^2}\frac{dk^2}{k^2}\left[\int_0^{k^2}\frac{dq^2}{q^2}\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)\right.$$

$$\left.-\int_0^{ck^2}\frac{dq^2}{q^2}\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)J_0(bq)\right]$$

where c is a finite arbitrary constant ($c < 1$) explained later. Equation (2.7) shows that in the case of soft gluon approximation the x and b evolution factorize. The two terms in the square brackets reflect the regularization procedure in the + sign in Eq. (2.6) (see (2.4)) to separate virtual and real contributions. The appearance of two different upper limits in the integration over dq^2 also is related to the real and virtual contributions. A simple way to see the emergence of two different scales is looking at the derivation of Eq. (2.7) from Eq. (2.6). In fact by using the regularization procedures in Eq. (2.4) one can write

$$Q^2\frac{\partial}{\partial Q^2}D(Q^2,b,x) = \frac{C_F}{2\pi}\int_x^1\frac{dz}{z}\left[\alpha_s(Q^2(1-z))\frac{1+z^2}{1-z} - \delta(1-z)\int_0^1\frac{dy}{y}\alpha_s(Q^2(1-y))\frac{1+y^2}{1-y}\right]$$

$$\times J_0\left(\frac{bq}{z}\right) dq^2\,\delta(Q^2z(1-z)-q^2)\,D\left(q^2,b,\frac{x}{z}\right) \qquad (2.8)$$

By setting $z = 1$ in all the slowly varying functions of z and using the delta function one can write in the soft limit.

$$Q^2 \frac{\partial}{\partial Q^2} D(Q^2,b,x) \simeq \frac{C_F}{2\pi} \int_0^{cQ^2} \left[\frac{\alpha_s(q^2)}{q^2} \left(2 - \frac{2q^2}{Q^2} + \frac{q^4}{Q^4} \right) - \delta\left(\frac{q^2}{Q^2}\right) \int_0^1 \frac{1+y^2}{Q^2(1-y)} \alpha_s(Q^2(1-y))dy \right] J_0(bq)\, dq^2\, D(Q^2,b,x)$$

$$= \frac{C_F}{2\pi} \int_0^{cQ^2} \left[\frac{\alpha_s(q^2)}{q^2} \left(2 - \frac{2q^2}{Q^2} + \frac{q^4}{Q^4} \right) - \delta\left(\frac{q^2}{Q^2}\right) \int_0^{Q^2} \frac{1+y^2}{k'^2} \alpha_s(k'^2)dk'^2 \right] dq^2\, J_0(bq)\, D(Q^2,b,x)$$

$$= \frac{C_F}{2\pi} \left[\int_0^{cQ^2} \frac{dq^2\, \alpha_s(q^2)}{q^2} \left(2 - \frac{2q^2}{Q^2} + \frac{q^4}{Q^4} \right) J_0(bq) - \int_0^{Q^2} \frac{dk'^2\, \alpha_s(k'^2)}{k'^2} \left(2 - \frac{2k'^2}{Q^2} + \frac{k'^4}{Q^4} \right) \right] D(Q^2,b,x) \tag{2.9}$$

The upper limit cQ^2 in the first terms come from the maximum value of the transverse momentum under which the soft gluon appproximation can be applied to real emission. Then if one approximates the real emission part of evolution equation all over the available phase space, $c = 1/4$. Now it should be stressed that this restriction, however, does not apply to the virtual contributions where the limit is the total virtuality. The scale Q_1^2 in Eq. (2.7) even if arbitrary must be chosen to limit the integration over k^2 to be within the perturbative range i.e. $\alpha_s(Q_1^2)/2\pi \lesssim 1$. For this purpose it is then sufficient to consider a value for Q_1^2 much bigger than the fundamental parameter Λ^2 in the running coupling constant $Q_1^2 \gg \Lambda^2$. The impact parameter b can vary from 0 to ∞. Since the evaluation of the exponent in Eq. (2.7) gives terms containing logarithms of the products Q^2/Λ^2 and Q^2b^2, $Q_1^2b^2$, to absorb the possible large contributions of the type $\ell n Q_1^2 b^2$ we choose

Q_1^2 to be $Q_1^2 = 1/b^2$ which together with the condition $Q_1^2 \gg \Lambda^2$ fixes the upper limit on b^2 to be $b^2 \ll 1/\Lambda^2$ which reflects the relevance of non-perturbative effects at large values of b, $b \simeq O(1/\Lambda)$.

Let us now consider the exponent in Eq. (2.7). By putting Q_1^2 $1/b^2$ and dividing the integration region $(0,k^2)$ to $(0,ck^2)$, (ck^2,k^2) in the first term the formula (2.7) becomes

$$T_1(Q^2,b^2) = -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dk^2}{k^2}\left[\int_0^{ck^2}\frac{dq^2}{q^2}\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)[1-J_0(bq)]\right.$$
$$\left.+\int_{ck^2}^{k^2}\frac{dq^2}{q^2}\alpha_s(Q^2)\left(1-\frac{q^2}{k^2}+\frac{q^2}{2k^4}\right)\right] \quad . \tag{2.10}$$

The explicit evaluation of Eq. (2.10) is performed in the Appendix giving

$$T_1(Q^2,b^2) = -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dq^2}{q^2}\left\{\alpha_s(q^2)\left[\ln\frac{Q^2}{q^2}-\frac{3}{4}\right]+2\ln\frac{e^\gamma}{2}\alpha_s(b^{-2})\right\}$$
$$\simeq \frac{C_F}{\pi\beta_0}\left[L\ln\left(1-\frac{B}{L}\right)+B\right]+\frac{2c_F}{\pi\beta_0}(\ln 2-\gamma)\frac{B}{L-B}$$
$$-\frac{3C_F}{4\pi\beta_0}\ln\left(1-\frac{B}{L}\right)+\theta\left(\frac{1}{L}\left(\frac{B}{L}\right)\right)+\text{(power corrections)} \tag{2.11}$$

with $L = \ln Q^2/\Lambda^2$ and $B = \ln Q^2b^2$, and γ the Euler constant. In the last equality terms of the type $(1/L)^\gamma (B/L)^n$, $\gamma \geq 1$, $n \geq 0$ have been neglected together with power type contributions $(Q^2b^2)^{-n}$. In Eq. (2.11) the first term in the last equality is made by contributions of the type $B(B/L)^n$ $n \geq 1$ in the perturbative series. This term is the one obtain by Parisi and Petronzio[5] and corresponds to the double logarithmic approximation. The additional terms come from the sum to all orders of the perturbative expansion of the contributions $(B/L)^n$

(see the Appendix). In the limit $z \to 1$, $Q^2 \gg q^2$ also these last terms are large and give contributions comparable to the $B(B/L)^n$ ones since the region $Q^2 \gg q^2$ corresponds in impact parameter space to $Q^2b^2 \gg 1$ and the sum is over the large logarithms $\ln Q^2b^2$.* The classification we make picks up the dominant contributions provided that $L \to \infty$ and the (B/L) fixed limits are taken. The kinematical constraint $Q^2b^2 \gg 1$ results in a lower bound on the integration in b^2:$b^2 \gg 1/Q^2$. The second and third term in Eq. (2.11) then cannot be neglected with respect to the leading double logarithms $B(B/L)^n$ and their effect must be taken into account at present energies even if when the energy increases the terms $B(B/L)$ tend to become more important.

Among the sub-leading contributions $(B/L)^n$ contained in the second and third term in Eq. (2.11) a distinction can be made. The contributions coming from the term proportional to $B/(L-B)$ have been considered by Rakow and Webber[7]. In their formulation this term arises from the inclusion of configurations in which the emission of two or more gluons with large transverse momenta add vectorially to give a smaller total transverse momentum. Due to the exact treatment of the kinematics in our formulation these contributions arise naturally in the solution Eq. (2.11). As correctly noticed by the authors of Ref. 7 the effect of this term is to give a slight modification of the double logarithmic first terms of Parisi-Petronzio thus supporting the validity of the result of Ref 5. The reason can be easily understood by oberving that effectively the $B/L-B$ term can be incorporated in the first term by changing by a constant the definition of B or equivalently by changing

*Further discussions about this point will be made in Sec. 3.

the value of Λ in L. In fact one can write by substituting

$B \to B' = B + 2\, \ln e^{\gamma}/2 = B + a.$

$$L\, \ln\left(1 - \frac{B}{L}\right) + B \to L\, \ln\left(1 - \frac{B+a}{L}\right) + B + a$$

$$= L\, \ln\left(1 - \frac{B}{L}\right) + B - a\,\frac{B}{L-B} + 0\left(\frac{1}{L}\left(\frac{B}{L}\right)^n\right)$$

The inclusion of the second term in Eq. (2.11) can be then obtained in 1/L and B by a change of the definition of the terms in the perturbative series by a constant.

Of a different nature is instead the last term in Eq. (2.11). The appearance of such terms not included in Ref. 7 arise from the correct treatment of the non-sigular terms in z in the vertex functions of the evolution equation (2.3) and as can be seen in the derivation in the Appendix physically is related to the single logarithms due to the mass singularities in the quark evolution: mass singularity coming from the virtual correction. The appearance of such contributions has been performed in Refs. 4,6.*

In Fig. 4 we show the relative effects of the various terms in Eq. (2.11). The last term in Eq. (2.11) appears to give the significant correction to the leading double logarithmic form and changes significantly the shape of the effective quark form factor $\exp(T(Q^2,b))$. This fact shows the importance of the inclusion of the single logarithmic terms to obtain a sensible approximation in the soft emission range. It is unlikely that the neglected corrections

*Due to the use of a different gauge in our case the coefficient is 3/4 instead of 3/2 as in Refs. 4,6. When physical gauge independent quantities are considered, however, result coincides with these ones (see Eq. (3.13)).

$(l/L)^{\gamma}(B/L)^{n}$ or the power corrections $1/(Q^2b^2)$ [Ref. 6] can significantly change the above result as far as Q^2 is large. To this purpose it is however necessary to consider the inclusion of two loop corrections and in general the effect of the result from more loop corrections. To include all the sub-leading contributions of the type $(B/L)^{n}$ in a consistent way it is in fact important that to all orders of the perturbative expansion the entire class of terms with the correct coefficient is evaluated and that there are no other similar terms left out. This will be the subject of the next section.

3. SECOND ORDER CORRECTIONS

In the previous section we have discussed and solved the evolution equations (2.3) in the soft limit. The appearance of terms of the type $(B/L)^n$ in Eq. (2.11) shows that also at one loop level single logarithms are present provided that the correct kinematics is taken into account and that non-singular terms in the vertex functions P(z) are kept. In this section we will investigate the problem of how the result in Eq. (2.11) will be changed by the inclusion of the two loop corrections and the effect of higher order loop corrections.

Let us first consider the modification of the evolution Eq. (2.3) which arise from the insertion of second order corrections. This problem has been investigated in the inclusive processes in Ref. 15 by analyzing two loop graphs in the light-like gauge in the dimensional regularization scheme. By including the two loop corrections in the evolution equation (2.5) the resulting equation is in impact parameter space*

$$Q^2 \frac{\partial}{\partial Q^2} D(Q^2,b,x) = \int_x^1 \frac{dz}{z} \int dq^2 \left[\frac{C_F}{2\pi} \alpha_s \frac{1+z^2}{1-z} + \left(\frac{\alpha_s}{2\pi}\right)^2 P_2(z)\right]_+ \times \delta(z(1-z)Q^2 - q^2)\, J_0\left(\frac{bq}{z}\right) D\left(Q^2,b,\frac{x}{z}\right) \quad (3.1)$$

where $P_2(z)$ is the second order kernel. As for the one loop equation (2.3) the dominant contribution to the integral in the soft limit $z \to 1$ region is given by the part of the kernel containing terms proportional only to $(1-z)^{-1}$. By isolating such terms in $P_2(z)$ we have that the

*The configuration in which the virtualities of adjacent quarks are the same order does not produce logarithmic contributions since in this case there are no infrared singularities which lead to logarithmic effects.

$P_2(z)$ is

$$P_2(z) = \left[C_F C_G\left(\frac{67}{9} - \frac{\pi^2}{3}\right) + 2C_F N_F T_F \left(-\frac{10}{9}\right)\right] \frac{1}{1-z} \equiv K \frac{1}{1-z} \tag{3.2}$$

with $C_G = N = 3$ for $SU(3)_{color}$ and $T_F = 1/2$. The less dominant terms of the type $\ln z/1-z$, $1/1-z \ln z \ln(1-z)$ and $1/1-z \ln^2 z$ in the limit $z \to 1$ in P_2 give only the non-dominant contribution $[(1/L)^{\gamma}(B/L)^n]$ in the b space which we neglect in our approximation. Also at two loop level is the non-singlet discussed which gives the dominant contribution in the soft limit.

The coupling constant which appears in (3.1) is the rescaled one. The introduction of such coupling constant also at two loop level requires that this choice of scale resums contributions at all orders in the perturbative expansion which have not been already included in the first order rescaled one $\alpha_s(k^2(1-z))$. This statement is non-trivial and a rigorous proof has not yet been given. However, we assume that it is the case supported by the physical argument in Sec. 2. If this is the case the coefficient must be that given by Eq. (3.2). No double counting is involved. In fact the different form of the terms taken into account by $\alpha_s{}^2(k^2(1-z))$ can be seen by expanding the coupling constant at order $\alpha_s{}^2$. One has

$$\int_{1/b^2}^{Q^2} \frac{dk^2}{k^2} \int_x^1 \frac{dz}{z} \frac{\alpha_s{}^2(k^2(1-z))}{1-z} K = \int_{1/b^2}^{Q^2} \frac{dk^2}{k^2} \int_x^1 \frac{dz}{z} \times K\left[\frac{\alpha_s{}^2(k^2)}{1-z} - 2\,\beta_0 \frac{\alpha_s{}^3(k^2)}{1-z} \times \ln(1-z) + \ldots\right] \tag{3.3}$$

Equation (3.3) shows that the contributions at each order of $\alpha_s(k^2)$ are clearly different from that included in the first order rescaling coupling. The term in Eq. (3.3) proportional to β_0 implies that the dominant term in the three loop kernel is proportional to K ℓn 1-z/1-z. The appearance of such terms satisfies the observation made in Ref. 14 about an upper bound on the contributions to the anomalous dimension at any order in the perturbative series. An explicit three loop calculation should show the above structure giving further support to the correctness of the use of the rescaled coupling constant also at two loop level. Furthermore, a detailed analysis[16] of the Sudakov form factor show that the anomalous dimension have only a single logarithm. This fact supports also the above assumption.*

By following the same procedure used to solve Eq. (2.6) in the soft limit one gets from Eq. (3.1) the solution

$$D(Q^2,b,x) = D(b^{-2},b,x) \exp[T_1(Q^2,b^2) + T_2(Q^2,b^2)] \tag{3.4}$$

with $T_2(Q^2,b)$ is up to the correction we neglect

$$T_2(Q^2,b^2) = -\frac{C_F}{\pi}\frac{K}{2\pi}\int_{1/b^2}^{Q^2}\frac{dq^2}{q^2}\,\ell n\left(\frac{Q^2}{q^2}\right)\alpha_s^2(q^2)$$

Up to now we have analyzed and solved the evolution equations for the parton distributions in the impact parameter space in the soft limit. Let us consider now the use of the previous results in the evaluation of a physical quantity. We will consider the case of the cross section for the process electron-positron annihilation into the hadrons A,B $e^+e^- \to A+B+X$ in the particular configuration in which the detected particles are at large relative angles (acollinarity angle $\theta \simeq 180^0$).

*Of course strictly speaking the Sudakov form factor is not exactly equal to the quantity we are calculating.

In the transverse momentum space this corresponds to configurations in which the two hadrons have a relative transverse momentum ($Q_T^2 = Q^2\sin^2\theta/2$) $Q_T^2 \ll Q^2$ where Q^2 is the total energy of the electron positron pair. In this particular kinematical configuration the cross section is dominated by the emission of soft and collinear partons. This in fact must be the case if the detected hadrons are observed in the acollinear configuration. The emission of soft partons slightly changes the direction of the parent initial quark and antiquark leading to acollinear configurations of the final particles. In this picture the inclusion of non-dominant contributions physically can be related to the less probabale case of more "semi-soft" emissions at larger transverse momenta which compensate each other vectorially in the evolution to build up a small total transverse momentum.

For the purpose of including such type of configurations the analysis in impact parameter is essential. The impact parameter, in fact, guarantees the exact conservation of the transverse momenta during the evolution. This correct treatment is not easily attainable in the transverse momentum space due to the complexity of the kinematics[6]. Due to the properties of the transform from p_T to b space a direct comparison of the various logarithmic contributions between momentum and impact prameter spaces is only possible at leading double logarithmic level.[17] The structure of the sub-leading terms being completely different in the two spaces. For this reason to compute quantities in the transverse momentum space first a consistent approximation must be obtained in the b space and the inverse Fourier transform can be evaluated only at the end. It is important to notice that, by carrying such type of analysis, also the value of the cross section at zero total transverse momentum $Q_T = 0$ can be computed by first pinciples. We should add that it is this very fact that gives for this cross section a better chance of isolating the non-perturbative soft hadronization mechanism in the comparison with the experimental data. Graphically the cross section $e^+e^- \rightarrow A+B+X$ can be represented as in Fig. 5. Due to the factorization of the soft bremsstrahlung the cross section can be written:

$$\frac{1}{\sigma_{TOT}} \frac{d\sigma}{dx_A dx_B d^2Q_T} = \frac{1}{\sigma_{TOT}} \frac{1}{2} \sum_{q\bar{q}} \int d^2p_T{}^A \, d^2p_T{}^B \, d^2p_T{}^S \, \delta^{(2)}\left(Q_T - \frac{p_T{}^A}{x_A} - \frac{p_T{}^B}{x_B} - p_T{}^S\right)$$
$$\times D_q{}^A(Q^2,p_T{}^A,x_A) \, D_{\bar{q}}{}^B(Q^2,p_T{}^B,x_B) \, S(Q^2,p_T{}^S) \qquad (3.5)$$

graphically represented in Fig. 6. The sums extend over all types of quark flavors. $p_T{}^A$ and $p_T{}^B$ are the transverse momenta of the hadrons with respect to the initial q and $\bar{q}$ respectively and $p_T{}^S$ is the relative transverse momentum of $q\bar{q}$ pair in the blob $S(Q^2,p_T{}^S)$. x_A, x_B are the fraction of longitudinal momenta of the hadrons (see Fig. 6). $D_q{}^A$ and $D_{\bar{q}}{}^B$ are the q and $\bar{q}$ densities and $S(Q^2,p_T{}^S)$ represents the set of the two particle irreducible (2PI) graphs with the external photon vertex included. Due to the factorization of the soft bremsstrahlung[18] in Fig. 6, there are no lines conecting the central blob S with the D blobs and there is no soft line connecting the two D blobs.

Let's examine the various contributions to the cross section in Eq. (3.5). First of all we are using the light-like gauge. We choose the direction of the gauge vector η along the quark line in Fig. 6. Such choice makes the structure of the graphs contributing to the cross section in Fig. 6 more simple. In fact due to the gauge vector the soft gluon coupling to the quark line[19] is suppressed. As a result at the double logarithmic level (collinear and soft) is sufficient to consider the evolution of the antiquark only. In our analysis which includes also the single logarithms it is necessary to consider the evolution of the quark and include the single logarithms contributions also in the central blob S. It should be noted that the single logarithms arise from both soft and collinear type singularities. The graphical structure of the terms contributing to the cross section is given by Fig. 7 where in the D′ blobs only collinear singularities are present. The evolution of the quark leg can be described by an equation similar to Eq. (2.1).

$$Q^2 \frac{\partial}{\partial Q^2} D_q'(Q^2,p_T,x) = \int_x^1 \frac{dz}{z} \left[\frac{\alpha_s(Q^2(1-z))}{(1-z)}\right] \times \frac{d^2q_T}{\pi} \delta(Q^2(1-z)z - p_T^2) \times D'\left(Q^2,p_T,\frac{x}{z}q_T,\frac{x}{t}\right) \tag{3.6}$$

where the different $(1-z)_+$ kernel is due to the presence in this case of only mass singularities. By taking the Fourier transform and solving the evolution equation (3.6) in the b space one obtains the solution

$$D_q'(Q^2,b,x) = D_q\left(\frac{1}{b^2},b,x\right) \exp[T_q(Q^2,b^2)] \tag{3.7}$$

where

$$T_q(Q^2,b^2) = -\frac{C_F}{4\pi}\int_{1/b^2}^{Q^2} \frac{dq^2}{q^2}\alpha_s(q^2) + (\text{correction})$$

We have not included in Eq. (3.6) the α_s^2 correction. Since, due to the counting rules for the contribution of the mass singularities and to the decoupling of the soft singularities, the effect of the two-loop kernel is to give only contributions of the type $\alpha^2 \ln$ in the exponent of Eq. (3.7) which can be neglected in our approximation. The right hand blob in Fig. 7 corresponds to the evolution of the antiquark line, at the two loop level it is given by Eq. (3.1). The solution, as discussed previously, is given then by Eq. (3.4).

Let us analyze the 2PI central blob in Fig. 7. The evaluation of $S(Q^2,b)$ at order α_s is given by the contribution of the first four graphs in Fig. 8. By calculating in the light-like gauge with the gauge vector aligned along the quark line we have that the cross section is in $\overline{\text{MS}}$ subtraction scheme

$$\frac{1}{\sigma}\frac{d\sigma}{dx_1 dx_2} = \delta(1-x_1)\,\delta(1-x_2) - \frac{C_F\alpha_s}{\pi}\frac{1}{(1-x_2)_+} \tag{3.8}$$

where x_1 and x_2 are the longitudinal momenta of q and $\bar{q}$ respectively. The dependence on x_2 only of the second term is related to our choice of the gauge vector. By taking the Fourier transform with respect to $p_T{}^S$, the relative transverse momentum of the $q\bar{q}$ pair, S can be written in the soft gluon approximation

$$S(Q^2,b^2) = \int dx_1 dx_2 \frac{d^2p_T{}^S}{\pi} \exp(-ip_T{}^S\cdot b)\,\frac{1}{\sigma}\frac{d\sigma}{dx_1 dx_2} \times \delta(p_T{}^{S2} - (1-x_1)(1-x_2)Q^2) \tag{3.9}$$

by substituting Eq. (3.8) into (3.9) and integrating, one has

$$S(Q^2,b^2) = 1 + C_F \frac{\alpha_s(Q^2)}{\pi} \ln Q^2b^2 + (\text{const. term}) + \mathcal{O}(\alpha_s^2) \tag{3.10}$$

The appearance of the large logarithm $\ln Q^2b^2$, due to the choice of Q^2 as the renormalization point and also to our formulation, can spoil the perturbative expansion of S. The expression (3.10) must be then summed. This summation to all orders may be performed by changing the renormalization point from Q^2 to $1/b^2$. This procedure suggests

$$S(Q^2,b^2) = \left[1 + \mathcal{O}\left(\alpha_s\left(\frac{1}{b^2}\right)\right)\right] \exp\left(\frac{C_F}{\pi}\int_{1/b^2}^{Q^2} \frac{dq^2}{q^2}\,\alpha_s(q^2)\right) \tag{3.11}$$

Although the above resummation is assumed, we shall make some comments concerning our two assumptions later. Now considering Eq. (3.5) and taking the Fourier transform on the right hand side of the various factors in the integrals then substituting Eq. (3.4), (3.7) and (3.11) and Fourier transforming back we get

$$\frac{1}{\sigma_{TOT}}\,\frac{d\sigma}{dx_A dx_B d^2Q_T} = \frac{1}{2}\int bdb\; J_0(bQ_T)\left[D_q{}^A\left(\frac{1}{b^2},b,x_A\right)\, D_{\bar{q}}{}^B\left(\frac{1}{b^2},b,x_B\right) + (q \leftrightarrow \bar{q})\right] \cdot \exp[T(Q^2,b^2)] \tag{3.12}$$

where

$$\begin{aligned} T(Q^2,b^2) &= T_1(Q^2,b^2) + T_2(Q^2,b^2) + T_q(Q^2,b^2) + \frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dq^2}{q^2}\,\alpha_s(q^2) \\ &= -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dq^2}{q^2}\left\{\ln\frac{Q^2}{q^2}\left[\alpha_s(q^2) + \frac{K}{2\pi}\,\alpha_s{}^2(q^2)\right]\right. \\ &\quad \left. + 2\ln\frac{e^{\gamma}}{2}\,\alpha_s\left(\frac{1}{b^2}\right) - \frac{3}{2}\,\alpha_s(q^2)\right\} \end{aligned} \tag{3.13}$$

In Eq. (3.13) the running coupling constant must be expanded at two loop level

$$\alpha_s(q^2) = \frac{1}{\beta_0 \ln(q^2/\lambda^2)} - \frac{\beta_1 \ln[\ln(q^2/\Lambda^2)]}{\beta_o{}^3 \ln^2(q^2/\lambda^2)}$$

with

$$\beta_0 = \frac{33 - 2N_F}{12\pi} \quad \text{and} \quad \beta_1 \frac{153 - 19N_F}{24\pi^2}$$

Using the above form of α_s the explicit form of T is

$$T = \frac{C_F}{\pi\beta_0}\left[L\ \ell n\left(1 - \frac{B}{L}\right) + B\right] + \frac{2C_F}{\pi\beta_0} K(1)\frac{B}{L-B} - \frac{C_F K}{2\pi^2\beta_0{}^2}\left[\ell n\left(1 - \frac{B}{L}\right) + \frac{B}{L-B}\right]$$

$$- \frac{3C_F}{2\pi\beta_0}\ell n\left(1 - \frac{B}{L}\right) + \frac{C_F\beta_1}{\pi\beta_0{}^3}\left\{\frac{B}{L-B} + \frac{L}{L-B}\ell n\left(1 - \frac{B}{L}\right) + \frac{B}{L-B}\ell n\ L\right.$$

$$\left. + \frac{1}{2}\ell n^2\left(1 - \frac{B}{L}\right) + \ell n\left(1 - \frac{B}{L}\right)\ell n\ L\right\} \tag{3.14}$$

with $k(1) \overline{=} \ell n\ 2-\gamma$. Let us make some observations about the results obtained. The inclusion of the various contributions to take into account terms of type (α log) has been made in the evolution of the $\bar{q}$ and q lines and in the central blob. The two loop terms enter explicitly only in the kernel P_2 for the evolution in the antiquark line due to choices of the alignment of the gauge vector. The resulting large logarithms $\ell n\ Q^2b^2$ are exponentiated also in the central blob S as given by Eq. (3.11). By looking at Eq. (3.11) and thinking of the graphs included by such summation there could be still open the possibility that other graphs different in structure from the iteration of the first order set of graphs in Fig. 9 as the ones represented for example in Fig. 9 can give contributions of the type ($\alpha\ \ell n\ Q^2b^2$) not included in Eq. (3.11). This possibility can be ruled out by a complete non-trivial evaluation of all the graphs contributing to the cross section (3.5) at order α_s^2. An explicit calculation of the Sudakov form factor at two loop level[20] shows in fact that the coefficient of the double pole in dimensional regularization exactly coincides with the coefficient of $\alpha^2 \ln^2 Q^2b^2$ in the expansion of the exponential e^T with T given by Eq. (3.14). This result supports the validity of our conjecture about the exponentiation of the central blob and the correctness of our result. Moreover the result for the exponent in Eq. (3.13) coincides with the one obtained in Ref. 8 by Collins and Soper by using a completely different formalism if in our Eq. (3.13) the contributions from the two loop graphs are not included.

In principle following our formalism and procedure the inclusion of the neglected corrections $(1/L)^\gamma$ $(B/L)^n$ in the exponent Eq. (3.14) would be possible provided a two loop calculation of the central blob and a three loop evaluation of the kernel of the equation (3.1) is performed. From Eq. (3.12), it is easy to obtain the energy-energy correlation cross section. In fact in Eqs. (2.10), (3.4) we have chosen the starting scale of the perturbative evolution to be $1/b^2$ to eliminate logarithmic corrections other than $\ln Q^2b^2$. We restrict our analysis in fact to the perturbative region of b such that $b^2 \ll 1/\Lambda^2$ $\alpha_s(1/b^2)/2\pi \ll 1$. When $1/b^2$ is small one should use another scale $M_0{}^2(\alpha_s(M_0{}^2)/2\pi \ll 1)$ as the starting value of the evolution. Within this region the density $D(1/b^2,b,x)$ can be expanded in terms of $\alpha_s(1/b^2)$: $D(1/b^2,b,x) = D(1/b^2,x) + O[\alpha_s(1/b^2)]$. $D(1/b^2,x)$ is the usual decay function which satisfies the sum rule

$$\sum_A \int dx_A x_A \, D^A\left(\frac{1}{b^2}, x_A\right) = 1 \quad .$$

By summing over the final hadrons and integrating over $x_{A,B}$ using the previous sum rule with the substitution $Q_T^2 = Q^2 \sin^2\theta/2$ the cross section in Eq. (3.12) becomes

$$\frac{1}{\sigma_{TOT}} \frac{d\Sigma}{d\cos\theta} = \frac{Q^2}{4} \int bdb\ J_0(bQ_T)\ \exp(T(Q^2,b^2)) \qquad (3.15)$$

which is the energy-energy correlation cross section[21] and θ is the acollinearity angle. Equation (3.15) is valid, due to our approximation, in the region of large acollinearity angles $\theta \lesssim 180^0$. The total effective form factor exp $T(Q^2,b^2)$ is plotted in Fig. 4 (solid line). The perturbative calculation we have performed is valid for $Q^2 \gg \Lambda^2$ if b is sufficiently small, i.e. $1/b^2 \gg \Lambda^2$. By looking at the effective form factor $\exp(T(Q^2b^2))$ in Fig. 4 we see the Sudakov-suppression of large values of b (i.e. small values of p_T). This suppression makes the region of small values of b relevant in the integration over b and supports the validity of the perturbative evaluation $\alpha(1/b^2)/2\pi \ll 1$. This very fact shows that the double-logarithmic approximation is clearly not enough and one must take into account also single-logarithmic corrections since the important integration region in b is outside the region in which the double-logarithmic approximation is valid. However the curve in Fig. 4 shows that the $\exp[T(b)]$ has still a rather long tail in the large b region at present energies. As can be easily undertood when the suppression is not strong enough uncertainties can come from perturbative contributions and from non-perturbative effects which are also important at large b. In a previous work[22] we have carried the analysis of the experimental data. The result is represented in Fig. 10. The use of the formula

(3.15) without the inclusion of the hadronization effects compared with the experimental data shows disagreement with the data when the single logarithmic corrections are included. The reason may be understood by looking at the weaker suppression in Fig. 4 in the large b region. The single logarithmic terms leave more room to non-perturbative effects in this region. Here the role of the sub-leading contributions emerges. The leading double logarithmic form factor in fact, strongly suppressing the non-perturbative hadronization effects when substituted in Eq. (3.15), can give a good agreement with the data[7] also if non-perturbative hadronization is neglected and the simple parton formula is used. Such agreement is however artifical and is only due to the flexibility of the leading formula. By varying the value of the parameter $\bar{\Lambda}$ one can in fact change the slope and the intercept of the cross section (3.15) to fit the data. The inclusion of the hadronization effects does not affect such behavior. The inclusion of a limited intrinsic transverse momentum in fact affect the region of large b but here the strong suppression in the leading formula is sufficient to make the inclusion of the hadronization effects indistinguishible from the pure parton formula. A different situation happens however when the expression (3.14) for the effective form factor is included in the cross section (3.15). The role of the hadronization is crucial to obtain an agreement with the data.[23] The reason is that the inclusion of single logarithmic corrections, including additional terms in the perturbative expansion, reduces the flexibility of the parton level formula and, giving a weaker suppression at large values of b, enhances the role played by non-perturbative hadronization effects. At present energies the $\exp(T(Q^2,b^2))$ (see Fig. 4) has still a long tail in the large b region. This fact suggests that the large b region is still important. When the energy increases however the suppression of such region becomes stronger and a pure perturbative treatment without the inclusion of hadronization may be possible.[22] This observation is in agreement with the belief that non-perturbative effects become less important when the energy increases and seems to be verified also by this calculation. We discuss this and related questions in a separate paper.[23]

4. IMPACT PARAMETER SPACE AND REMARKS

In the previous section we have calculated the energy-energy correlation cross section Eq. (3.15). By looking at the b-space effective form factor exp $T(Q^2,b^2)$ we have found that the leading (double logarithmic contributions) and subleading (single logarithmic contributions) terms are of equal importance at present energies. Furthermore this fact stresses the importance of non-perturbative effects in the comparison with the data. In this section we clarify and discuss the approximation scheme in b space and make some final remarks.

The effective form factor exp $T(Q^2,b^2)$ has the following formal structure as far as the logarithmic dependence on Q^2, b^2 is kept

$$T(Q^2,b^2) = \sum_{\ell=1}^{\infty} \sum_{j=0}^{\ell+1} C(\ell,j)\, B^j\, L^{-\ell} \tag{4.1}$$

$$= \sum_{\ell=1}^{\infty} [C_\ell^0\, B(B/L)^\ell + C_\ell^1 (B/L)^\ell + C_\ell^2 (1/L)(B/L)^\ell + \ldots]$$

where $L = \ell n\, Q^2/\Lambda^2$ and $B = \ell n\, Q^2 b^2$. The leading (double logarithmic) contribution corresponds to the sum of the first term in Eq. (4.1). The sub-leading (single logarithmic) contribution corresponds to the second term. This classification can be achieved by taking the limit $L \to \infty$ with B/L fixed. The calculation of T in this paper is equivalent to the evaluation of C_ℓ^0 and C_ℓ^1 in Eq. (4.1). Furthermore the exponentiation of the soft gluon effects performed in the previous sections produces one constraint on b, i.e., $\alpha(1/b^2) \ll 1$. This constraint in fact is automatically satisfied by the "Sudakov-suppression" of the large b region when Q^2 is large. Equation (3.14) can be considered to be a good approximation to the total effective form factor in the region $1/Q \ll b \ll 1/\Lambda$.

In order to obtain physical quantities like Eq. (3.15) the inverse transform back from b space must be performed. The integration region of b is in principle over all the positive values although practically $1/Q \lesssim b \lesssim 1/\Lambda$. The effective form factor obtained Eq. (3.14) is, however, reliable only when $1/Q \ll b \ll 1/\Lambda$. Therefore at an arbitrary energy Q^2 various corrections must be taken into account to Eq. (3.15). The case in which b is large has been already discussed in detail in the previous sections. Let us make a comment about the case when b is very small. The very small b region corresponds to the emission of hard gluons with large p_T. Therefore some corrections are expected in the above processes. Such processes can, however, be treated purely perturbatively (3,4,...,N jet process) and such contributions to the cross section in the particular kinematical configuration considered in this paper are clearly suppressed. There is another corrections coming from kinematics. The effective form factor Eq. (3.14) was obtained after two approximations. One is the soft gluon approximation specified in the previous sections. Another is that we only summed up to subleading contributions with $Q^2b^2 \gg 1$. Explicitly we neglected the power corrections $(1/Q^2b^2)^n$ and $(1/L)^m(B/L)^n$ type contributions. Such corrections come from for example: Energy conservation and/or the emissions of the non-soft gluons. It is a difficult problem whether such corrections can also be resummed. However such corrections can produce only a non-dominant contributions when transformed back to momentum space. This important fact has been proven by Ellis et al.[6]

and related problems have also been discussed in detail by the same authors. Therefore in principle some possible corrections in the very small b region can be safely neglected.

In this paper we chose $L = \ln Q^2/\Lambda^2$ and $B = \ln Q^2b^2$ as expansion parameters in the calculations of the effective form factor. However, one can always change these expansion parameters by finite amount; $L \to L'$ and $B \to B'$ as explicitly explained in Section 2. This ambiguity may be related to the so-called scheme dependence. Therefore there is the possibility depending on the scheme of drawing other curves in Figs. 4 and 10.[24] It is unlikely however that such changes can lead to different shapes of the b-space effective form factor in the important region of b (i.e., damping part*) and therefore to different conclusion from the ones in Section 3.

ACKNOWLEDGEMENTS

We are grateful to S. Drell for the warm hospitality we have received at the SLAC Theory Group. One of us (L.T.) would like to thank also L. Susskind for having made his stay at Stanford possible.

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*The extremely small b region is unimportant also due to the kinematical factor b in Eq. (3.15).

APPENDIX

In this Appendix we integrate explicitly the evolution Eq. (2.10) to obtain Eq. (2.11). We consider the limit of $L \equiv \ln Q^2/\Lambda^2 \to \infty$ with B/L fixed ($B \equiv \ln Q^2 b^2$) and keep the only terms which do not vanish in this limit. Therefore the contributions of order $(1/L)(B/L)^n$ and also powers $(1/Q^2 b^2)^n$ are neglected. As it is discussed in the text, such neglected terms do not produce significant contributions to the cross section when transformed back to the momentum space when Q^2 is sufficiently large.

We start with eq. (2.10):

$$T_1(Q^2,b^2) = -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dk^2}{k^2}\left\{\int_0^{ck^2}\frac{dq^2}{q^2}\,\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)[1-J_0(bq)]\right.$$

$$\left.+\int_{ck^2}^{k^2}\frac{dq^2}{q^2}\,\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)\right\} . \qquad \text{(A.1)}$$

In order to pick up the leading contributions from each term in eq. (A.1), the function $[1 - J_0(bq)]$ is well approximated by the θ-function, $\theta(q - 1/b)$. Therefore let us rewrite eq. (A.1) as follows:

$$T_1(Q^2,b^2) = -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dk^2}{k^2}\left\{\int_0^{ck^2}\frac{dq^2}{q^2}\,\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)\theta(q-1/b)\right.$$

$$\left.+\int_{ck^2}^{k^2}\frac{dq^2}{q^2}\,\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)\right\}$$

$$-\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dk^2}{k^2}\int_0^{ck^2}\frac{dq^2}{q^2}\,\alpha_s(q^2)\left(1-\frac{q^2}{k^2}+\frac{q^4}{2k^4}\right)$$

$$\times\,[\theta(1-q/b) - J_0(bq)] \equiv -\frac{C_F}{\pi}\,[I_1 + I_2] \quad . \qquad \text{(A.2)}$$

Now I_1 is easily calculated to give

$$I_1 = \int_{1/b^2}^{Q^2} \frac{dk^2}{k^2} \int_{1/b^2}^{k^2} \frac{dq^2}{q^2}\, \alpha_s(q^2) \left[1 - \frac{q^2}{k^2} + \frac{q^4}{2k^4}\right]$$

$$= \int_{1/b^2}^{Q^2} \frac{dq^2}{q^2}\, \alpha_s(q^2) \left[\ell n \frac{Q^2}{q^2} - \frac{3}{4} + \frac{q^2}{Q^2} - \frac{q^4}{4Q^4}\right] \quad . \qquad \text{(A.3)}$$

The last equality in Eq. (A.3) is obtained by changing the order of integration of the variables k^2 and q^2. Now the integration of the $[\ell n\, Q^2/q^2 - 3/4]$ term gives the contribution of order $B(B/L)^n$ and $(B/L)^n$ respectively. On the other hand the contributions from the terms which contain inverse power of Q^2 give at most $\alpha(1/b^2)$, i.e., order $(1/L)(B/L)^n$, corrections since those contribution are estimated to be

$$\int_{1/b^2}^{Q^2} \frac{dq^2}{q^2}\, \alpha_s(q^2) \left[\frac{q^2}{Q^2} - \frac{q^4}{4Q^4}\right] \leq \alpha_s(1/b^2) \int_{1/b^2}^{Q^2} \frac{dq^2}{q^2} \left[\frac{q^2}{Q^2} + \frac{q^4}{4Q^4}\right]$$

$$= \alpha_s(1/b^2) \left[\frac{9}{8} - \frac{1}{b^2Q^2} - \frac{1}{8b^4Q^4}\right] \quad .$$

As far as $1/b^2 \gg \Lambda^2$, this term can be neglected as explained in the text. Then we have

$$I_1 \simeq \int_{1/b^2}^{Q^2} \frac{dq^2}{q^2}\, \alpha_s(q^2) \left[\ell n \frac{Q^2}{q^2} - \frac{3}{4}\right] \quad . \qquad \text{(A.4)}$$

The evaluation of I_2 is more complicated but straightforward. It is convenient to change the order of integration of the variables and look at the Q^2 and b^2 dependence

$$I_2 = \int_0^{cQ^2} \frac{dq^2}{q^2} \alpha_s(q^2)[\theta(1-q/b) - J_0(bq)] \int_{q^2/c}^{Q^2} \frac{dk^2}{k^2} \left(1 - \frac{q^2}{k^2} + \frac{q^4}{2k^4}\right)$$

$$- \int_0^{c/b^2} \frac{dq^2}{q^2} \alpha_s(q^2)[\theta(1-q/b) - J_0(bq)] \int_{q^2/c}^{1/b^2} \frac{dk^2}{k^2} \left(1 - \frac{q^2}{k^2} + \frac{q^4}{2k^4}\right)$$

$$\equiv J(Q^2,b^2) - J(1/b^2,b^2) \quad .$$

Now J becomes

$$J = \int_0^{cQ^2} \frac{dq^2}{q^2} \alpha_s(q^2) [\theta(1-q/b) - J_0(bq)]$$

$$\times \left(\ell n \frac{Q^2}{q^2} + \ell n\ c - c + \frac{c^2}{4} + \frac{q^2}{Q^2} - \frac{q^4}{4Q^4} \right) \quad .$$

In the integrand of J, the term which contains $\ell n\ Q^2/q^2$ is clearly dominant. Integrating such term we have

$$J(Q^2,b^2) \cong \int_0^{cQ^2} \frac{dq^2}{q^2} \alpha_s(q^2) [\theta(1-q/b) - J_0(bq)]\ \ell n\ Q^2/q^2$$

$$= \int_0^{c} \frac{dx^2}{x^2} \alpha_s(Q^2x^2) [\theta(1/Qb-x) - J_0(bQx)]\ \ell n\ 1/x^2 \quad .$$

Since the infrared singularity coming from the $x \ll 1$ region is regularized by the $[\theta - J_0]$ function and also we are considering the case of $b^2\Lambda^2 \ll 1$, we can expand the coupling constant

$$\alpha_s(Q^2x^2) = \frac{1}{\beta_0\ \ell n\ Q^2x^2/\Lambda^2} = \frac{1}{\beta_0} \sum_{n=0} \frac{1}{(\ell n\ Q^2/\Lambda^2)^{n+1}} (\ell n\ 1/x^2)^n \quad .$$

Then

$$J(Q^2,b^2) = \frac{1}{\beta_0} \sum_{n=0}^{\infty} \frac{1}{(\ln Q^2/\Lambda^2)^{n+1}} \int_0^c \frac{dx^2}{x^2} \left[\ln \frac{1}{x^2}\right]^{n+1} [\theta(1/Qb-x)-J_0(bQx)]$$

$$= \frac{1}{\beta_0} \sum_{n=1}^{\infty} \frac{(-)^n}{(\ln Q^2/\Lambda^2)^n} \frac{\partial^n}{\partial\epsilon^n}$$

$$\times \int_0^c dx^2 (x^2)^{\epsilon-1} [\theta(1/Qb-x) - J_0(bQx)] \Big|_{\epsilon=0}$$

$$\cong \frac{1}{\beta_0} \sum_{n=1}^{\infty} \frac{(-)^n}{(\ln Q^2/\Lambda^2)^n} \frac{\partial^n}{\partial\epsilon^n}$$

$$\times \left[\frac{1}{\epsilon} \left(\frac{1}{Q^2b^2}\right)^{\epsilon} - \frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{4}{Q^2b^2}\right)^{\epsilon} \right]_{\epsilon=0} .$$

Above the final equality comes after neglecting the terms which vanish when $Q^2b^2 \gg 1$ [6]. Using the identity

$$\frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{4}{Q^2b^2}\right)^{\epsilon} = \frac{1}{\epsilon} \exp[-\epsilon(\ln b^2Q^2 + 2\ln \gamma_E/2)]$$

$$- 2 \sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{2k+1} \epsilon^{2k+1}$$

where γ_E is the Euler constant and ζ the Riemann zeta-function, J is estimated to give

$$J(Q^2,b^2) = \frac{1}{\beta_0} \sum_{n=1}^{\infty} \frac{1}{(\ln Q^2/\Lambda^2)^n}$$

$$\times \left[2\ln \gamma_E/2 \ (\ln b^2Q^2)^n + \frac{n}{2} (2 \ln e^{\gamma_E}/2)^2 (\ln b^2Q^2)^{n-1} \right.$$

$$+ \frac{n(n-1)}{6} (2 \ln e^{\gamma_E}/2)^3 \ (\ln b^2Q^2)^{n-2}$$

$$\left. + \frac{2}{3} \zeta(3) \ n(n-1) \ (\ln b^2Q^2)^{n-2} + O(\ln b^2Q^2)^{n-3} \right]$$

$$\simeq \frac{1}{\beta_0} 2 \ln e^{\gamma_E}/2 \frac{\ln Q^2b^2}{\ln Q^2/\Lambda^2 - \ln Q^2b^2}$$

$$= 2 \ln e^{\gamma_E}/2 \int_{1/b^2}^{Q^2} \frac{dq^2}{q^2} \alpha_s(1/b^2) \quad . \qquad (A.5)$$

The constant term and also those which are inverse powers of Q^2 in J and the contribution from the coupling constant at two-loop can be easily evaluated to give negligible contributions [order $(1/L)(B/L)^n$] in a similar way.

A similar calculation can be performed to show that $J(1/b^2,b^2)$ is at most of order $(1/L)(B/L)^n$ [or $\alpha_s(1/b^2)^n$] as can be easily conjectured from the fact that the relevant scale is in this case only b^2 $(1/b^2 >> \Lambda^2)$.

Finally, I_2 becomes in the approximation considered in this paper

$$I_2 = \int_{1/b^2}^{Q^2} \frac{dq^2}{q^2} (2\ln e^{\gamma_E}/2) \ \alpha_s(1/b^2) \quad . \qquad (A.6)$$

The final form of T_1 is given from Eqs. (A.2), (A.4) and (A.6)

$$T_1(Q^2,b^2) = -\frac{C_F}{\pi}\int_{1/b^2}^{Q^2}\frac{dq^2}{q^2}\left[\alpha_s(q^2)\left(\ell n\frac{Q^2}{q^2}-\frac{3}{4}\right)+(2\ell n\, e^{\gamma_E}/2)\,\alpha_s(1/b^2)\right]$$

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FIGURE CAPTIONS

1. a) A typical "hard" process. b) A "semihard" branching process. p_T is the transverse momentum of the "observed" b parton with respect to the parent parton a.
2. Graphical representation of the evolution Equation (2.1).
3. The contribution to the $q \to q$ amplitude by emission of gluons only (NS channel) given by the iteration of Eq. (2.1) with $p_q{}^q(z)$ as kernel.
4. The contributions of the various terms in the last equality of Eq. (2.11) for the effective form factor $\exp(T_1(Q^2,b^2)$. The first double logarithmic (dashes); first plus second (dot-dashes); first second and third (dots). The solid line represents the total final contribution (Eqs. (3.13) and (3.14)) with (solid) and without (double dots-dash) the two loop $\alpha_s{}^2$ term. $Q = 30$ GeV, $\Lambda = 0.1$ GeV, $N_F = 5$.
5. The cross section $e^+e^- \to A + B + x$.
6. The cross section $e^+e^- \to A + B + x$ in its factorized form Eq. (3.5) with kinematics.
7. The factorized structure of the cross section in the light-like gauge. The D′ blob contains only collinear singularities.
8. The graphs contributing at order α_s to the central blob $S(Q^2,p_T{}^s)$.
9. Some of the graphs contributing to the cross section at order $\alpha_s{}^2$.
10. Comparison of the cross section Eq. (3.15) with the data of Ref. 24.

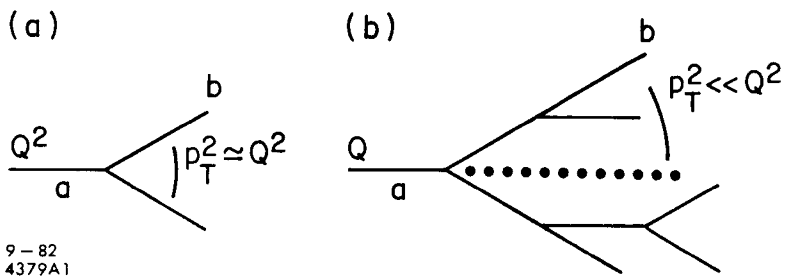


Fig. 1

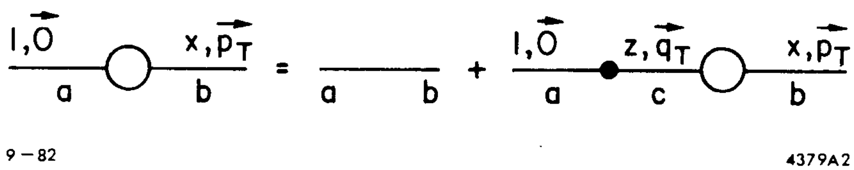


Fig. 2

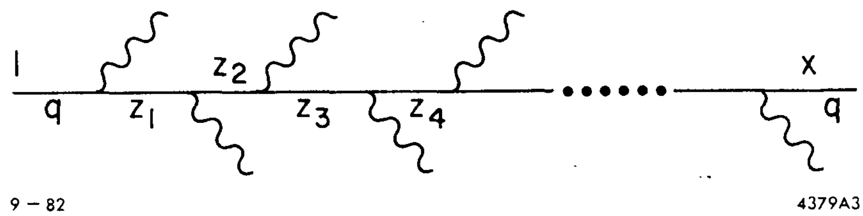


Fig. 3

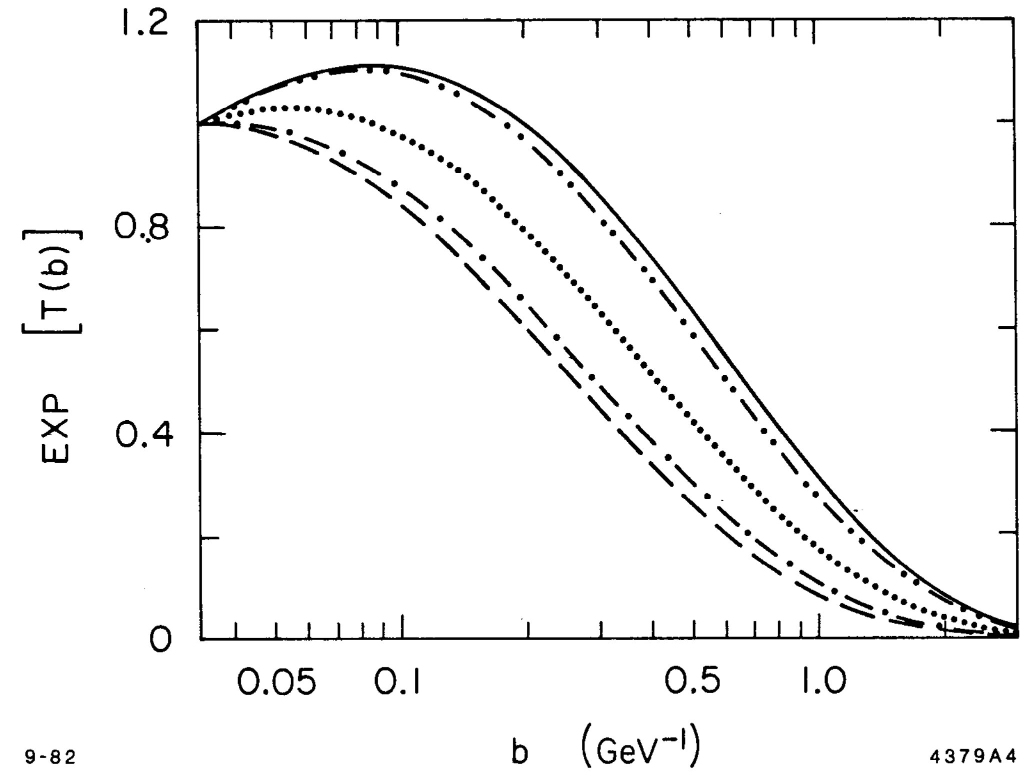


Fig. 4

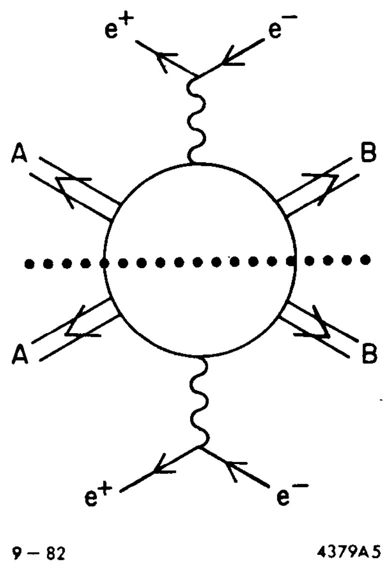
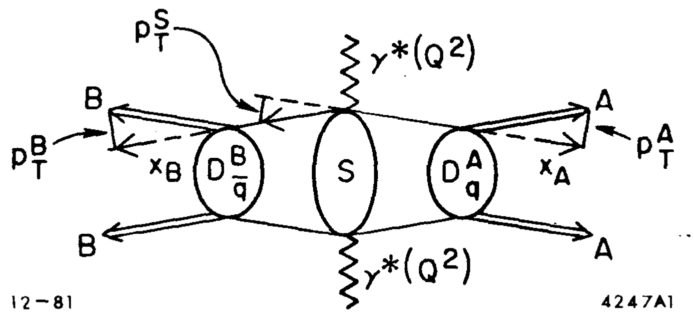


Fig. 5



p_T^S
$\gamma^*(Q^2)$
B
A
p_T^B
x_B
$D_{\bar q}^B$
S
D_q^A
x_A
p_T^A
B
A
$\gamma^*(Q^2)$
12-81
4247A1

Fig. 6

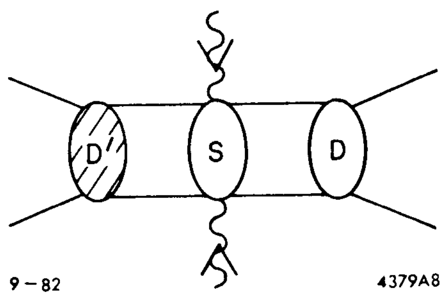


Fig. 7

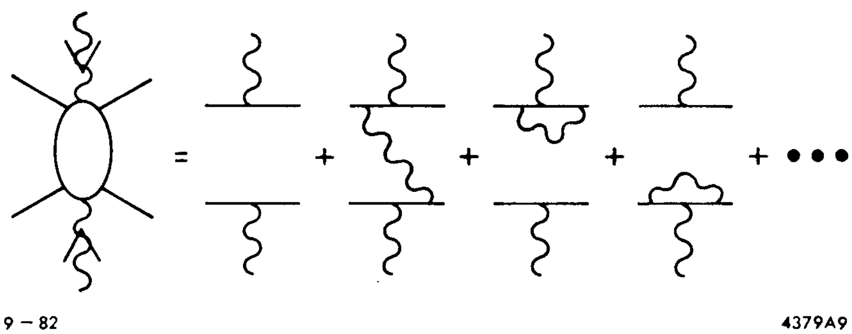


Fig. 8

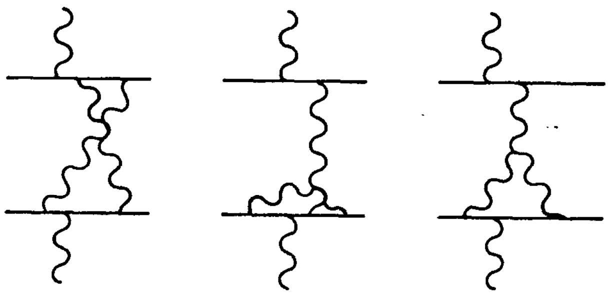
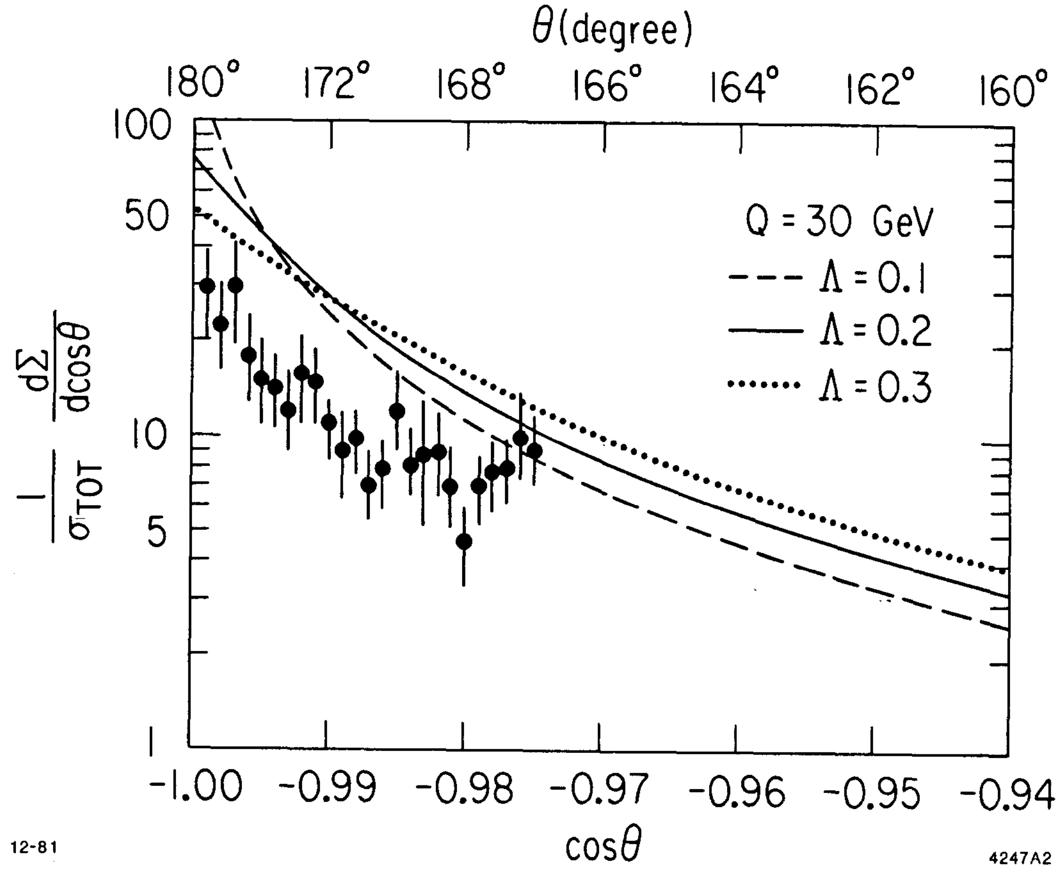


Fig. 9



θ(degree)
180°
172°
168°
166°
164°
162°
160°
100
50
10
5
1
Q = 30 GeV
Λ = 0.1
Λ = 0.2
Λ = 0.3
1/σTOT dΣ/dcosθ
-1.00
-0.99
-0.98
-0.97
-0.96
-0.95
-0.94
cosθ
12-81
4247A2

Fig. 10