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SENSITIVITY OF A LASER DRIVEN GRATING LINAC TO GRATING ERRORS*

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ABSTRACT

The effect of grating errors on transverse beam stability is analyzed. We characterize grating errors by random groove displacements and find that transverse displacements due to such errors approach limiting values of the same order as the grating displacements themselves. It therefore appears that transverse stability requirements will not impose unusually stringent precision requirements on the grating structure.

INTRODUCTION

As described by Palmer,[1] the grating structure for a laser driven grating linac requires shaped groove spacings of the order of one half the laser wavelength and an overall length of several hundred meters. Random errors in the grooves are surely inevitable, and in view of the vast number of grooves, the effect of such errors upon beam stability must be assessed. We provide here an estimate of the relation between the magnitude of these errors and that of the mean deviation from the nominal orbit which these errors induce.

FORMULATION OF THE PROBLEM

We begin with a brief description of the strong focussing design discussed elsewhere in these proceedings.[1,2] The field components in synchronism with the electrons are written

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$$\vec{E} = E_o \cos py \, e^{-px} \left(\hat{z} \cos \phi - \hat{x} \frac{k}{p} \sin \phi\right) \tag{1}$$

$$\vec{B} = E_o \, e^{-px} \left[\hat{z} \sin py \cos \phi - \hat{x} \frac{p}{k} \sin py \sin \phi \right. \tag{2}$$

$$\left. + \hat{y} \left(\frac{p}{k} - \frac{k}{p}\right) \cos py \sin \phi \right] + B_o \, Sq(z) \, \hat{y} \quad .$$

Sq(z) is a square wave of amplitude ±1 and slowly increasing period L(z). The electron phase ϕ is given by

$$\phi = \phi_o - \frac{1}{2} \Delta \, Sq(z) \tag{3}$$

where here z is the coordinate of a particular electron. Over a grating section of length L/2 the fields vary as $\exp i(kz - \omega t)$ with $k \approx \omega/c$. The electron motion is ultra relativistic so that the variation of this quantity over a section is negligible. The value of $(kz - \omega t)$ for a particular electron is designated by ϕ. The phase shift, $\pm\Delta$, which occurs between sections is brought about by a shift of magnitude Δ/k of the grooves from section to section. The instantaneous shifts in the above formulas are, of course, an idealisation of shifts which occur in a distance small compared to L, and the modifications in the fields at these junctures which are required by Maxwell's equations are neglected in the analysis.

The transverse equations of motion of the electron are (taking $z \approx ct$ as the independent variable)

$$\frac{d}{dz} m\gamma \frac{dx}{dz} = -\frac{eE_o}{c^2} \left[\frac{p}{k} \cos py \, e^{-px} \sin \phi + \frac{B_o}{E_o} Sq(z)\right] \tag{4}$$

$$\frac{d}{dz} m\gamma \frac{dy}{dz} = -\frac{eE_o}{c^2} \frac{p}{k} \sin py \, e^{-px} \sin \phi \quad . \tag{5}$$

As described in Refs. 1 and 2, due to the operation of the strong focusing principle these equations have a stable straight line orbit with $\phi_o = 0$, $y = 0$, and x_o determined by

$$\frac{p}{k} e^{-px_o} \sin \frac{\Delta}{2} = \frac{B_o}{E_o} \quad . \tag{6}$$

There is also a family of more complicated stable orbits with slowly varying period L whose ϕ_o values lie in a relatively narrow band about zero. Here, however, we shall confine our attention to the simple $\phi_o = 0$ orbit and discuss the effect which grating errors have upon it alone. Setting $x_1 = x - x_o$ and assuming both x_1 and y small we have

$$\frac{1}{\gamma}\frac{d}{dz}\gamma\frac{dx_1}{dz} + K^2 \; Sq(z)\; x_1 = 0 \tag{7}$$

$$\frac{1}{\gamma}\frac{d}{dz}\gamma\frac{dy}{dz} - K^2 \; Sq(z)\; y = 0 \tag{8}$$

with

$$K^2 = \frac{|eE_o|}{\gamma m\, c^2}\, e^{-px_o} \sin\frac{\Delta}{2}\frac{p^2}{k} \tag{9}$$

$$= \frac{1}{\gamma}\frac{d\gamma}{dz}\tan\frac{\Delta}{2}\frac{p^2}{k}$$

where the second line of (9) follows from Eq. (1) and the obvious relation between E_z and the acceleration rate. Following Ref. 2 we choose the z variation of L so that $KL(z) \equiv \psi$ is constant.

It is apparent that the x_1 and y motion present identical problems so we confine our attention to the x_1 motion in the following. To take account of grating errors we introduce a driving force F(z) to the RHS of (7)

$$\frac{1}{\gamma}\frac{d}{dz}\gamma\frac{dx_1}{dz} + K^2 \; Sq\; x_1 = \frac{F(z)}{\gamma m\, c^2} \equiv f(z) \quad , \tag{10}$$

where F(z) represents corrections to the RHS of (4) which arise from these errors. The general form of the displacements induced by f is given by

$$x_1(z) = \int_0^z dz_1 \; g(z,z_1)\; f(z_1) \tag{11}$$

where $g(z,z_1)$ is the solution of Eq. (7) satisfying the boundary conditions

$$g(z,z_1) = 0$$
$$\left.\frac{\partial g}{\partial z}\right|_{z=z_1} = 1 \quad . \tag{12}$$

The quantity of interest is, of course, the expectation value of x_1^2 induced by the fluctuating force. It is given by

$$\langle x_1^2\rangle = \int_0^z dz_1 \int_0^z dz_2 \; g(z,z_1)\; g(z,z_2)\; \langle f(z_1)\; f(z_2)\rangle \tag{13}$$

EVALUATION OF $\langle x_1^2 \rangle$

In order to evaluate Eq. (13) we need an expression for the force correlation function $\langle f(z_1)\, f(z_2) \rangle$ and for the Green's function $g(z,z_i)$.

Let $\gamma m\, f_L$ represent the x component of force in Eq. (4) due to the laser, that is

$$\begin{aligned} f_L &= - \frac{eE_o}{\gamma m\, c^2} \frac{p}{k}\, e^{-px_o} \sin \frac{\Delta}{2} \\ &= - \frac{1}{\gamma} \frac{d\gamma}{dz} \frac{p}{k} \tan \frac{\Delta}{2} \quad . \end{aligned} \tag{14}$$

Taking account of the fact that the grating spacing s, the wavelength, and x_o are all of the same order of magnitude, we estimate that the value of $f(z_2)$ due to a displacement δs of a single groove at z_1 may be written

$$f(z_2) \approx \frac{\delta s}{s}\, f_L(z_1)\; C(z_1 - z_2) \tag{15}$$

where $C(0) = 1$. C may be expected to fall off on a scale of the order of s as z_2 recedes from z_1. Thus

$$f(z_1)\; f(z_2) \approx \left(\frac{\delta s}{s}\right)^2 f_L^2(z_1)\; C(z_1 - z_2) \quad . \tag{16}$$

We next estimate that the principal effect on $\langle f(z_1)\; f(z_2) \rangle$ may be attributed entirely to the displacement of the nearest groove, whence

$$\langle f(z_1)\; f(z_2) \rangle = \frac{\langle \delta s^2 \rangle}{s^2} f_L^2(z_1)\; C(z_1 - z_2) \quad . \tag{17}$$

Finally, as will be clear below, the scale over which f_L^2 and $g(z,z_i)$ vary is enormous compared to s. Hence for insertion into (13) we write $C(z_1 - z_2) \approx s\delta(z_1 - z_2)$ yielding finally

$$\langle f(z_1)\; f(z_2) \rangle = \frac{\langle \delta s^2 \rangle}{s}\, \delta(z_1 - z_2)\; f_L^2(z_1) \quad . \tag{18}$$

It seems clear that Eq. (18) has the correct dependence (for the purposes of Eq. (13)) on z_1, z_2, the laser field strength and $\langle \delta s^2 \rangle$. The effect of the crude approximations which we have made can only be to replace s by a quantity of the same order of magnitude. Therefore, we shall henceforth think of s as a quantity of the order of the grating spacing rather than as the grating spacing itself.

The determination of g by solving (7) subject to the boundary conditions (12) is simple in principle, especially if one takes advantage of the fact that γ, and hence K^2 and L vary very slowly .

with z. In the interest of obtaining our final result in a simple form, however, we prefer to proceed in a more approximate manner. In particular we wish to replace (7) by

$$\frac{1}{\gamma}\frac{d}{dz}\gamma\frac{dg}{dz} + Q^2(z)\ g = 0 \tag{19}$$

where

$$\cos QL = \cos\frac{\psi}{2}\ \cosh\frac{\psi}{2} \quad . \tag{20}$$

This has the effect of replacing the strongly focused betatron oscillations of wave number Q described by (7) by simple harmonic oscillations with the same wave number. As shown in the appendix, this procedure is well justified when $\psi/2$ is small. ($\psi/2 < \pi/4$ is small enough.) Continuing then with (19) and assuming γ slowly varying we obtain by inspection

$$g(z,z_1) = \left[\frac{\gamma(z_1)}{\gamma(z)\ Q(z)\ Q(z_1)}\right]^{1/2} \sin\int_{z_1}^{z} Q(z')dz' \quad . \tag{21}$$

Substitution of (21), (18), and (14) into (13) yields

$$\begin{aligned}\langle x_1^2\rangle = {} & \frac{K^2}{Q^2}\frac{\langle\delta s^2\rangle}{s}\frac{\tan^2\frac{\Delta}{2}}{\gamma(z)\ K(z)}\frac{p^2}{k^2} \\ & \cdot\left(\frac{d\gamma}{dz}\right)^2\int_0^z \frac{dz_1}{\gamma(z_1)\ K(z_1)}\sin^2\int_{z_1}^{z} Q(z')dz' \quad .\end{aligned} \tag{22}$$

In writing (22) we have made use of the fact that K/Q is z independent and have, in addition, assumed $d\gamma/dz$ to be constant. To complete the evaluation we replace the $\sin^2$ factor by its average value and make use of (8) to obtain

$$\begin{aligned}\langle x_1^2\rangle &= \frac{\langle\delta s^2\rangle}{2sk}\frac{K^2}{Q^2}\frac{1}{\sqrt{\gamma(z)}}\int_0^z \frac{1}{\sqrt{\gamma(z_1)}}\frac{d\gamma}{dz_1}dz_1 \\ &= \frac{\langle\delta s^2\rangle}{sk}\frac{K^2}{Q^2}\left[1 - \left(\frac{\gamma(0)}{\gamma(z)}\right)^{1/2}\right]\end{aligned} \tag{23}$$

Equation (23) is our main result. It shows that while the beam fluctuations grow initially, the effect quickly saturates on account of the "adiabatic damping" effect in (21). For $\psi < \pi/2$ Eq. (20) implies $Q^2/K^2 \approx \psi^2/48$.

As shown by the discussion of Ref. 2, the allowable ϕ_o range becomes small as ψ becomes small and $\psi < \pi/2$ is probably a smaller value than one would want to use. The specific design proposed in Ref. 2 has $\psi = \pi$, a value which is outside the established region of validity of the replacement of Eq. (7) by Eq. (18). The discussion in the appendix does, however, encourage one to believe that the qualitative behavior implied by (23) continues to hold, and for a preliminary estimate even to risk a quantitative application. Then we have $K/Q = 2$ and taking $s \approx 1/2\ \lambda$ we obtain

$$\langle \delta s^2 \rangle \approx \frac{\pi}{4} \langle x_1^2 \rangle \tag{24}$$

so that the groove displacement induces a particle displacement of the same order of magnitude.

Before concluding we recall that the above analysis has been confined to $\phi_o = 0$. Depending upon the sign, either the horizontal or vertical betatron wave number is reduced as ϕ_o is varied from zero, and approaches zero as the limit of stability is reached.[2] Since the betatron wave number Q^2 appears in the denominator of Eq. (23) it seems very likely that the $\langle x_1^2 \rangle$ induced by a specified $\langle \delta s^2 \rangle$ will increase and diverge as the stability limit of the ϕ_o range is approached. A further investigation to determine the effect which limitation on the attainable precision of grating ruling has upon the ϕ_o range would be desirable.

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APPENDIX

Following Bruck[3] we note that Eq. (10) can be formally transformed to (we neglect the z dependence of γ here, and hence of K^2 and L also)

$$\frac{1}{\gamma} \frac{d}{d\bar{z}} \gamma \frac{d}{d\bar{z}} \bar{x} + Q^2 \bar{x} = \bar{f}\big(z(\bar{z})\big) \tag{A1}$$

where

$$\bar{x} = \frac{1}{\sqrt{Q\beta}} x_1 \tag{A2}$$

$$\bar{z} = \int^z \frac{dz'}{Q\beta(z')} \tag{A3}$$

$$\bar{f} = (Q\beta)^{3/2} f \tag{A4}$$

and the Twiss matrix function β(z) is the periodic solution of

$$\frac{1}{2}\beta\frac{d^2\beta}{dz^2} - \frac{1}{4}\left(\frac{d\beta}{dz}\right)^2 + K^2 Sq(z)\ \beta^2 = 1 \quad . \tag{A5}$$

We find that

$$Q\beta(z) = \frac{Q}{K \sin QL}\left[\sin\frac{\psi}{2}\cosh\frac{\psi}{2} + \cos\left(\frac{\psi}{2} - 2Kz\right)\sinh\frac{\psi}{2}\right] \qquad 0 < z < \frac{L}{2}$$

$$= \frac{Q}{K \sin QL}\left[\sinh\frac{\psi}{2}\cos\frac{\psi}{2} + \cosh\left(\frac{3}{2}\psi - 2Kz\right)\sin\frac{\psi}{2}\right] \qquad \frac{L}{2} < z < L \tag{A6}$$

with $\beta(z+L) = \beta(z)$. One can readily verify that (A6) satisfies (A5) and continuity conditions on β and dβ/dz at z = 1/2 L and z = 0, L (periodicity condition). In the small ψ limit (A6) becomes

$$Q\beta(z) \approx 1 + \psi^2\left(\frac{1}{2} - \frac{z}{L}\right)\frac{z}{L} \quad ; \qquad 0 < z < \frac{L}{2}$$

$$\approx 1 - \psi^2\left(\frac{3}{2} - \frac{z}{L}\right)\frac{z}{L} \quad ; \qquad \frac{L}{2} < z < L \tag{A7}$$

so that Qβ oscillates about 1 with an amplitude $\psi^2/16$. The treatment given in the main text amounts to the neglect of the difference between the barred and unbarred quantities and in view of (A2), (A3), (A4), and (A7) appears to be well justified for $\psi < \pi/2$. For the $\psi = \pi$ case, inspection of (A6) indicates similar behavior for Qβ but with a maximum value of 2.41, minimum value of 0.5 and average value of 1.36. Hence the application of (23) to this case is not likely to be grossly in error.

REFERENCES

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