SLAC-PUB-2904
April 1982
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A NOTE ON CYLINDRICAL WAVES WHICH PROPAGATE AT THE VELOCITY OF LIGHT*

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INTRODUCTION

The continuous linear acceleration of ultra relativistic particles in free space by an electromagnetic field requires the presence of a cylindrical wave component with phase velocity that differs negligibly from c and with non-vanishing electric field component in the direction of propagation. Lawson and Woodward have pointed out the fact that certain geometries proposed for laser driven acceleration fail to satisfy these requirements.[1] On the other hand, complex wave number plane wave fields which do satisfy these requirements have been constructed by Palmer, who also points out that any cylindrical wave with the required properties can be formed from superposition of plane waves of the form which he has obtained.[2]

The situation is analogous to that which occurs in standard waveguide theory. There it is also true that any waveguide mode can be constructed by superposition of plane waves. Nevertheless, the study of the general properties of cylindrical waves has proved to be a very powerful tool for the analysis of waveguides and similar structures. Because this may also prove to be the case for fields with propagation velocity c we present a brief study of their properties below.

A BRIEF REVIEW OF CYLINDRICAL WAVES

By definition, cylindrical waves with sinusoidal time dependence are solutions of Maxwell's equations in which the fields have the general form

$$\vec{E} = \vec{E}(x,y)\, \exp i(kz - \omega t)$$
$$\vec{B} = \vec{B}(x,y)\, \exp i(kz - \omega t) \quad . \qquad (1)$$

* Work supported in part by the Department of Energy, contracts DE-AC03-76SF00515 (SLAC) and DE-AT03-81ER40029 (UCSD).
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(Presented at the Laser Acceleration of Particles Workshop, Los Alamos, New Mexico, February 18-23, 1982.)

Since each field satisfies the vector wave equation we have

$$\nabla^2\vec{E} + \frac{\omega^2}{c^2}\vec{E} = \nabla^2\vec{B} + \frac{\omega^2}{c^2}\vec{B} = 0$$

and hence the transverse variation of each field component satisfies the two dimensional Helmholtz equation

$$\nabla^2 f(x,y) + K^2 f(x,y) = 0 \qquad (2)$$

where

$$K^2 = \frac{\omega^2}{c^2} - k^2 \qquad (3)$$

writing

$$\vec{E}(x,y) = \vec{E}_T + \hat{z}E_z \quad , \quad \hat{z}\cdot\vec{E}_T = 0$$

$$\vec{B}(x,y) = \vec{B}_T + \hat{z}B_z \quad , \quad \hat{z}\cdot\vec{B}_T = 0 \qquad (4)$$

and substituting in Maxwell's equations yields[3] (Gaussian units)

$$ik\hat{z}\times\vec{E}_T - \hat{z}\times\vec{\nabla}E_z = i\frac{\omega}{c}\vec{B}_T \qquad (5a)$$

$$ik\hat{z}\times\vec{B}_T - \hat{z}\times\vec{\nabla}B_z = -i\frac{\omega}{c}\vec{E}_T \qquad (5b)$$

$$\vec{\nabla}\times\vec{E}_T = i\frac{\omega}{c}B_z\,\hat{z} \qquad (6a)$$

$$\vec{\nabla}\times\vec{B}_T = -i\frac{\omega}{c}E_z\,\hat{z} \ . \qquad (6b)$$

From Eq. (3) one obtains

$$\vec{E}_T = \frac{ik}{K^2}\vec{\nabla}E_z - \frac{i\omega/c}{K^2}\hat{z}\times\vec{\nabla}B_z \qquad (7a)$$

$$\vec{B}_T = \frac{ik}{K^2}\vec{\nabla}B_z + \frac{i\omega/c}{K^2}\hat{z}\times\vec{\nabla}E_z \ . \qquad (7b)$$

The standard TE (TM) modes are obtained by setting E_z (B_z) equal to zero and choosing a suitable set of solutions of (2) for B_z (E_z). The transverse fields are then obtained from (7).

THE SPECIAL CASE $k = \omega/c$

For waves which propagate at velocity c, the quantity K^2, which appears in the denominator of Eq. (7) vanishes, so that the previously described procedure fails. Setting $E_z = B_z = 0$ provides an obvious way out of the difficulty with Eq. (7) and leads to the usually discussed TEM modes. There are additional possibilities, however, which we discuss below.

Setting $K^2 = 0$ in (2) we obtain:

$$\nabla^2 f = 0 \quad . \tag{8}$$

Setting $k = \omega/c$ in (5) and carrying out a little vector algebra yields

$$\vec{\nabla} B_z = - \hat{z} \times \vec{\nabla} E_z \quad . \tag{9}$$

Equation (8) tells us that B_z and E_z satisfy Laplaces equation, and (9) tells us that they are related by the Cauchy-Riemann equations. Thus if we write

$$W(x+iy) = B_z(x,y) + iE_z(x,y) \quad , \tag{10}$$

Equations (8) and (9) are satisfied wherever W is analytic. We note that in the general case the modes are neither TE nor TM as Eq. (9) implies that non-vanishing $\vec{\nabla} E_z$ implies non-vanishing B_z.

An additional set of useful restrictions on the transverse fields is obtained by taking the divergence of (5), which when combined with (6) yields

$$\vec{\nabla} \cdot \vec{E}_T = - ikE_z \tag{11a}$$

$$\vec{\nabla} \cdot \vec{B}_T = - ikB_z \tag{11b}$$

Equations (5), (6), (10), and (11) are a set of necessary relations which the fields must satisfy. The relations are not, however, independent since satisfaction of a suitable subset implies the others. It is sufficient, for example, to satisfy (6a), (11a), (5a) and either (10) or the two dimensional <u>vector</u> Laplace equation for $\vec{E}_T$. The simplest procedure for generating a solution is to begin by specifying any solution of the two dimensional vector Laplace equation. If one identifies this solution with $\vec{E}_T$, then (6a), (11a), and (5a), provide explicit formulas for B_z, E_z, and $\vec{B}_T$, respectively. Alternatively, if one identifies this solution with $\vec{B}_T$, then (6b), (11b), and (5b) provide explicit formulas for E_z, B_z, and $\vec{E}_T$. Thus each solution of the vector Laplace equation generates two independent solutions.[4]

SOME SIMPLE APPLICATIONS

A solution of the type applied by Palmer to the grating problem[2] is obtained by setting

$$E_x = E \cos q y \, e^{-qx}$$

$$E_y = 0 \quad .$$

Then from (6a)

$$B_z = \frac{i}{k}\frac{\partial E_x}{\partial y} = -\frac{iq}{k} E \sin q y \, e^{-qx}$$

from (11a)

$$E_z = \frac{i}{k}\frac{\partial E_x}{\partial x} = -\frac{iq}{k} E \cos q y \, e^{-qx}$$

and from (5a)

$$\vec{B}_T = \hat{z} \times \left(\vec{E}_T + \frac{i}{k}\vec{\nabla}E_z\right)$$

$$= E\left[\left(1 - \frac{q^2}{k^2}\right)\hat{y}\cos qy + \frac{q^2}{k^2}\hat{x}\sin qy\right]e^{-qy}$$

The cylindrically symmetric solution suitable for acceleration is obtained by setting

$$E_z = \text{constant}$$

$$H_z = 0$$

Then from (6a) and (11a)

$$\vec{E}_T = -\frac{ik\vec{r}}{2} E_z$$

and from (5a)

$$\vec{B}_T = -\frac{ikr\hat{\theta}}{2} E_z \quad .$$

Finally, we recall that Lawson and Woodward[1] showed that any physical solution independent of y and valid in free space for all $x > 0$ must have vanishing E_z. This result is easily obtained in the present context from the fact that for y independent fields (11a) and (8) imply that E_z is constant and $E_x = ikE_z x + \text{constant}$.

REFERENCES

1. J. D. Lawson, Rutherford Laboratory report RL-75-043 (1975); IEEE Transactions on Nuclear Science, NS-26, 4217 (1979); P. M. Woodward, Journal IEE93, Part IIIA, 1554 (1947).

2. R. B. Palmer, Particle Accelerators 11, 81-90 (1980).

3. For a more detailed discussion see, for example, J. D. Jackson, Classical Electrodynamics (Wiley, New York, 2nd Edition 1975), Ch. 8.

4. Related TEM modes are obtained by setting $\vec{E}_T = \vec{\nabla}(\vec{\nabla}\cdot\vec{F})$, $\vec{B}_T = \hat{z}\times\vec{E}_T$ (or $\vec{B}_T = \vec{\nabla}(\vec{\nabla}\cdot\vec{F})$, $\vec{E}_T = -\hat{z}\times\vec{B}_T$) where $\vec{F}$ is the solution to the vector Laplace equation referred to above. Note further that although $\hat{z}\times\vec{F}$ is also a solution of the vector Laplace equation, the solutions which it generates by the same procedure are linear combinations of those obtained with $\vec{F}$.