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CORRECTIONS TO THE MUONIUM HYPERFINE SPLITTING OF RELATIVE ORDER $\alpha^2(m_e/m_\mu)$*

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ABSTRACT

We present the results of an analytic calculation of the corrections of relative order $\alpha^2(m_e/m_\mu)$ to the muonium ground state hyperfine splitting due to exchanged photons. We compare theory and experiment, taking into account these corrections and some radiative-recoil contributions described in an accompanying paper.

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The comparison of the measured hyperfine splitting (hfs)[1] in the muonium ground state with the quantum electrodynamics (QED) prediction[2] currently provides the most stringent test of relativistic two-body bound state theory. This paper describes the calculation of the recoil corrections (excluding radiative corrections) of relative order $\alpha^2(m_e/m_\mu)$. (Here α is the fine structure constant and m_e and m_μ are the electron and muon masses.) The $\ell n\ \alpha^{-1}$ terms in order $\alpha^2(m_e/m_\mu)$ have been calculated previously[3,4] and the $\ell n(m_\mu/m_e)$ terms have been shown to vanish.[5] We present here the results of an analytic calculation of the remaining non-logarithmic contributions. The so-called radiative-recoil corrections are described in an accompanying paper.[6]

In the Bethe-Salpeter approach, the energy levels of a two-particle bound state are given by the positions of the poles in the four point function. Since the Coulomb potential dominates the QED bound state problem, we start with an analysis of the Coulomb ladder. Each loop of the ladder is separated into two pieces. One piece, which we call the "nonrelativistic loop", is treated exactly and leads to a three-dimensional wave equation whose solution yields the lowest order wave functions and energy levels. The other piece, which we call the "remainder loop," is treated perturbatively along with the various two-particle irreducible parts of the four-point graphs. There are infinitely many ways to partition the Coulomb ladder loops into a lowest order part and a perturbative part and, indeed, a variety of approaches have been presented in the literature.[7] In general, one must include at least the Coulomb-Schrödinger part of each loop in lowest order, since this part is essentially non-perturbative in nature.

For muonium, it is convenient to take advantage of the small mass ratio ($m_e/m_\mu \ll 1$) to incorporate more physics into the lowest order problem. By requiring the muon to be on its positive energy mass shell, and setting aside some terms involving m_μ^{-1}, one arrives at the Gross equation[8] for the bound state wave functions

$$(H_e + \tilde{V})\psi_n = E'_n \psi_n \quad , \tag{1}$$

where $H_e = \vec{\alpha}_e \cdot \vec{p} + \beta_e m_e$ is the Dirac Hamiltonian, $\tilde{V}$ is an effective potential,[9] ψ_n includes a factor of $\frac{1}{2}(1+\beta_\mu)$, and E' is related to the total energy E through $E' = (E^2 - m_\mu^2 + m_e^2)/2E$. Obviously, Eq. (1) incorporates the relativistic properties of the electron, while most of the dynamics of the muon have been suppressed except for effects associated with the reduced mass.

The shifts in the energy levels can be calculated from standard perturbation theory, which through second order in the perturbation yields (see, e.g., Ref. 4)

$$\Delta E' = \langle 0|\tilde{K}|0\rangle \left\{1 + \left\langle 0\left|\left[\frac{\partial \tilde{V}}{\partial E'} + \frac{\partial \tilde{K}}{\partial E'}\right]\right|0\right\rangle\right\} + \sum_{n\neq 0} \frac{\langle 0|\tilde{K}|n\rangle\langle n|\tilde{K}|0\rangle}{E'_0 - E'_n} \ . \tag{2}$$

As a consequence of the reduction to the three-dimensional wave equation, the four-dimensional kernels $\tilde{K}$ are terminated by Coulomb interactions at each end. In carrying out the calculation, one finds that the precise decomposition of contributions between first and second order in $\tilde{K}$ is somewhat artificial, since a given $\tilde{K}$ does not correspond to a definite power of α. In fact, it is sometimes convenient to remove certain pieces of the first order terms and combine them with second order terms to obtain an expression that is simpler to evaluate. Here we shall describe the main lines of the calculation without attempting to detail these refinements.

First we discuss the first order in $\tilde{K}$. Any given kernel contains many orders in α and m_e/m_μ; but in general, as the number of loops excluding "nonrelativistic loops" increases, the least order in α increases. For the present work, it turns out that we need up to two such loops. We can use the propagator decomposition described earlier and the Gross equation to rearrange the kernels. In this way we organize the kernels so that there are no more than two loops and no "remainder loops". The final set of kernels is shown in Fig. 1. Note that the subtracted one-loop kernels have the effect of removing lower-order contributions that are over-counted in the two-loop kernels. For example, each of the kernels in the first line of Fig. 1 contains leading order (Fermi splitting) contributions; altogether, the leading order appears exactly once.

It is interesting that one arrives at the set of kernels shown in Fig. 1 regardless of the choice of wave equation.[10] Different choices of wave function manifest themselves as different decompositions into lowest and second order contributions in Eq. (2). That is, any feature not incorporated in the wave function is restored in the second order terms. The use of the Gross equation means that the relativistic properties of the electron and certain recoil corrections are in the wave function rather than in the sum over states.

In carrying out the evaluation of the first-order perturbation theory matrix elements, it is useful to have covariant denominators for all of the photon propagators, rather than the awkward noncovariant denominators contained in the Coulomb photon exchanges C and the transverse photon exchanges T. In order to accomplish this, we carry out a transformation <u>within</u> the kernels to the covariant Feynman gauge. It is still convenient to distinguish photons with spatial indices (denoted by V) from those with

temporal indices (denoted by $\mathcal{O}$). The net effect of the gauge transformation on the kernels is that $T \to V$ and $C \to \mathcal{O}$ for the set shown in Fig. 1. There are, in addition, residual gauge terms associated with the external fermion lines. Those associated with the muon line vanish because the muon is on mass shell in the Gross equation. Those associated with the electron line give contributions that tend to be smaller than the order of interest, partly because of cancellations between the two-loop and one-loop kernels. Gauge terms arising from the graphs labeled (TCT) do contribute in relative order $\alpha^2(m_e/m_\mu)$. However, these contributions are precisely cancelled by terms that arise from the second order perturbation expressions.

At this stage we wish to expand the first order energy shifts in powers of m_e/m_μ. Thus, it is tempting simply to expand the muon factor of each graph in powers of $1/m_\mu$. However, the presence of previously calculated terms of relative order $\alpha(m_e/m_\mu)\,\ell n\,(m_\mu/m_e)$ shows that this is not possible.[11] We organize the calculation so that these terms as well as the leading order contributions, can be identified and extracted at the start. We also note that individual graphs contain spurious non-recoil contributions. These are eliminated before integration by combining kernels generated by permuting photon connections on the muon leg (denoted by a brace in Fig. 1). This procedure also eliminates spurious $\alpha^2(m_e/m_\mu)\,\ell n\,(m_\mu/m_e)$ terms.[12] Thus, we are finally able to make a direct m_μ^{-1} expansion before integration.

Having carried out these procedures, we find that in the two-loop contributions the muon factor always contains at least one δ-function of the time component of momentum. For most of the contributions we can, in the order of interest, neglect the dependence of the kernels on the

momentum variables of the wave function. Then we are left with certain seven-dimensional integrals to evaluate. We treat these by combining denominators using Feynman parameters and carrying out the momentum integrations. We are then able to construct six-dimensional integrals involving only non-relativistic propagators that lead to identical parameter integrals. These six-dimensional integrals can be evaluated easily by Fourier transforming to coordinate space. There are some cases in which we must retain the wave function momentum dependence in the kernels. However, it is then legitimate to make non-relativistic approximations, which eliminate the time-components of the momentum integrations. The nine-dimensional integrals that arise in this way are relatively straightforward to evaluate using complex integration. The twelve-dimensional integral associated with the kernel VOV can be computed by using a variant of the method of Dalgarno and Lewis.[13]

Finally, let us discuss the calculation of the second-order energy shifts. In second order in $\tilde{K}$ at least one of the kernels must involve the hyperfine interaction, but one may be spin independent. A spin independent contribution does arise from R between Coulomb potentials and from the convection part of a transverse photon interaction. Because of the structure of $\tilde{K}$, the contribution where the hyperfine interaction is taken twice in the sum over states has two or more Coulomb interactions between hyperfine interactions. It can be worked out easily by using the Dalgarno-Lewis method. The result, including contributions from the $\partial\tilde{K}/\partial E'$ term in Eq. (2), agrees with a calculation of Caswell and Lepage.[14] The contributions involving a spin independent factor combine naturally with certain terms from the first order contribution; this procedure will

not be detailed here. The resulting sum over states is also evaluated by using the Dalgarno-Lewis method.[15]

The contributions from the kernels of Fig. 1 are

$$\Delta\nu = E_F\left\{1 + \frac{3}{2}\alpha^2 - 3\frac{\alpha}{\pi}\frac{m_R \ln(m_\mu/m_e)}{m_\mu - m_e} + \alpha^2\frac{m_R}{m_e+m_\mu}\left[2\ln\alpha^{-1} - \underline{8\ln 2 + 3\frac{11}{18}}\right.\right.$$

$$\left.\left. + \mathcal{O}(\alpha, (m_e/m_\mu))\right]\right\} \qquad (3)$$

where $m_R = m_e m_\mu/(m_e+m_\mu)$ and $E_F = (8/3)\alpha^4 m_R^3/m_e m_\mu = (16/3)\alpha^2 c R_\infty (m_R^3/m_e^2 m_\mu)$. This expression includes all terms previously calculated that do not involve radiative corrections, and a new contribution, which is underlined. Because we have expanded in powers of m_μ^{-1}, the new term is not applicable to positronium. Its contribution to muonium is -2.2 kHz. Other new contributions arising from radiative corrections are given in an accompanying paper. They have a net value +1.1 kHz (uncertainty guessed to be ~0.5 kHz). With this plus other known contributions,[2] the theoretical value of the muonium hfs is

$$\Delta\nu(\text{theory}) = 4\ 463\ 303.3(1.7)(3.0)\ \text{kHz} \quad . \qquad (4a)$$

Here we have use the ac Josephson value for α. The 1.7 kHz uncertainty comes from combining a 1.4 kHz uncertainty in m_μ (the muon magnetic moment)[1] with a 1.0 kHz uncertainty in the ac Josephson value for α. The bulk of the 3.0 kHz theoretical uncertainty (reduced from 5.0 kHz by the work reported here) is due to a refined binding correction to the electron anomalous moment.[16] This value is in good agreement with the experimental result

$$\Delta\nu(\text{experiment}) = 4\ 463\ 302.88(.16)\ \text{kHz} \quad . \qquad (4b)$$

Since the theory prediction uses as input the value of the fine structure constant α, one can regard the measurement of the muonium hfs as a means for determining α. Recent refinements in the theory of the electron anomalous magnetic moment have resulted in greater precision in the determination of α from pure elementary particle physics.[17] Comparison of these results could, when the theoretical uncertainties in the muonium hfs are reduced further, give bounds on the scales of internal electron structure purely from particle physics measurements.[17] Comparison of α from the muonium hfs with condensed matter determinations provides an important test of the internal consistency of QED, as well as of our understanding of the theory of the condensed matter measurements. We list below the values of α determined from the muonium hfs, the electron anomalous moment,[17] the ac Josephson effect,[18] the quantum Hall effect,[19] and a combination of the quantum Hall and ac Josephson measurements:[19]

$$\begin{aligned}
\alpha^{-1}\ \text{(muonium hfs)} &= 137.035969(21)(46) \\
\alpha^{-1}\ \text{(anomalous moment)} &= 137.035993(5)(9) \\
\alpha^{-1}\ \text{(ac Josephson)} &= 137.035963(15)(?) \qquad (5) \\
\alpha^{-1}\ \text{(quantum Hall)} &= 137.035968(23)(?) \\
\alpha^{-1}\ \text{(ac-J. and q.H.)} &= 137.035965(12)(?)\ .
\end{aligned}$$

The first errors listed are experimental and the second theoretical. The question mark in the condensed matter determinations indicates that the theoretical uncertainties are unknown; they are possibly very small in comparison to the experimental ones. The agreement is satisfactory, but further reductions in the errors and a better theoretical understanding of the condensed matter measurements are clearly desirable.

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7. See for example R. Barbieri and E. Remiddi, Nucl. Phys. B141, 413 (1978), and Refs. 3, 4 and 14.
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9. We have used both $\tilde{V} = (m_\mu/E)V_c$, where V_c is the Coulomb potential, and $\tilde{V}_c = \left(1 - \beta_e \frac{m_e}{E}\right)V_c$. Of course, they give the same results.
10. The wave function must be at least as accurate as $\left[1+\left(\vec{\alpha}_e\cdot\vec{p}/2m_e\right)\right]\left[1-\left(\vec{\alpha}_\mu\cdot\vec{p}/2m_\mu\right)\right]\psi_S$, where ψ_S is the Schrödinger wave function. Otherwise, one must consider kernels involving more than two loops.
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piece from the two-loop kernels with the second order perturbation expression.

15. Alternatively, one may use the second option mentioned in footnote 9. Then the sum over states does not appear since the wave function incorporates that part of the physics.

16. This correction was estimated by S. J. Brodsky and G. W. Erickson, Phys. Rev. 148, 26 (1966). It is analogous to the self-energy correction $G_{SE}(Z\alpha)$ in the Lamb shift. The errors given are estimated upper limits on the theoretical uncertainty rather than standard deviations.

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FIGURE CAPTION

Fig. 1. Kernels contributing to the muonium hfs to the order of interest. Dashed lines represent Coulomb interactions (C) and wavy lines represent transverse photons (T). The brace indicates that photon lines are to be inserted in the muon leg in all possible ways. The labeling in parentheses indicates the order of attachment of the photon lines to the electron leg.

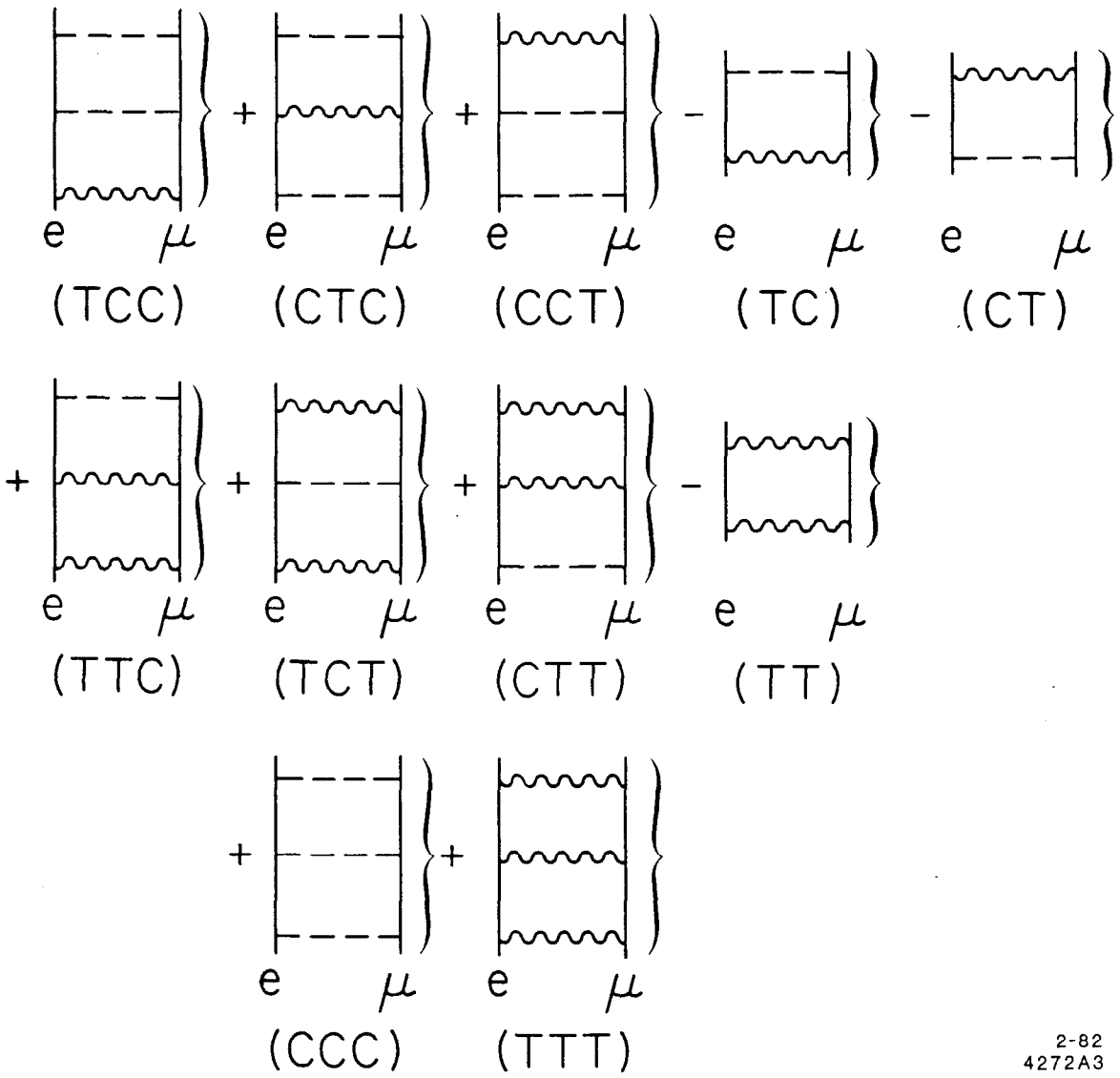

Fig. 1