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ZERO RANGE FOUR PARTICLE EQUATIONS*

H. Pierre Noyes
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

We prove that the zero range four particle equations are one variable equations of the same form as the three particle equations with the two particle amplitudes replaced by the appropriate analytic continuation of the on shell three particle amplitude. Thanks to the Faddeev-Yakubovsky combinatorics we believe that the N particle zero range equations can be written in terms of N-1 particle amplitudes in the same way.

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We have shown[1] that by applying the conventional two particle zero range boundary condition $k\text{ctn}\delta = \lambda(k^2)$ for each pair to the asymptotic form of the three particle wave function we can derive three particle equations of the Faddeev form[2,3] with $t(q,\bar{q};z-\tilde{p}^2)$ replaced by $\tau(z-\tilde{p}^2)$ where the on shell two particle amplitude is

$$\tau^{\pm}(\tilde{k}^2) = -\left[\pi\mu\left(k\text{ctn}\delta - \sqrt{-k^2 - i0^+}\right)\right]^{-1} \equiv -\frac{N^2}{\tilde{k}^2 + \varepsilon} + \hat{\tau}^{\pm}(\tilde{k}^2) \tag{1}$$

$$\tilde{k}^2 = k^2/2\mu \quad ; \quad \mu = m_1 m_2/(m_1 + m_2)$$

and ε is the two particle binding energy. For consistency with time reversal invariance and on shell three particle unitarity $\lambda(k^2)$ must be chosen so that τ has no singularities for negative k^2 other than bound state poles, as will be true for example of a class of Castillejo-Dalitz-Dyson[4] solutions of the Low equation. By following the same procedure we derive in this communication the corresponding four particle equations, and indicate the generalization to N particle equations using the Faddeev-Yakubovsky[5] combinatorics.

Karlsson and Zeiger[6] have shown that if the conventional three particle theory is formulated using interacting two particle states rather than plane waves as a basis that the equations for any finite number of partial waves contain only the half on shell amplitudes $t_\ell(k;\tilde{q}^2 \pm i0^+) = \tau_\ell^{\pm}(\tilde{q}^2)[(k/q)^\ell + (\tilde{k}^2 - \tilde{q}^2)\, f_{q^2}(k)]$. Here $f_{q^2}(k)$ is a real function[7] measuring the departure of the wave function from the asymptotic form at short distance and the factored form holds for any short range interaction for any finite angular momentum ℓ. In what follows we will confine ourselves to $\ell = 0$ since the generalization to any finite number of partial waves is immediate. KZ factor out the Jost function rather than τ_ℓ. If we look at the Faddeev amplitudes $M_{ab}(p_a,q_a;p_b,q_b;z)$ on shell, that is with $\tilde{q}^2 = z - \tilde{p}^2$, it follows immediately that for any three particle system generated by two particle short range interactions, this amplitude will always have the form

$$M_{ab}(p_a,p_b;z) = \tau_a(z-\tilde{p}_a^2)\delta_{ab}\delta(p_a-p_b)/p_a p_b + \tau_a(z-\tilde{p}_a^2)Z_{ab}(p_a,p_b;z)\tau_b(z-\tilde{p}_b^2) \tag{2}$$

If we take the zero range limit of the KZ equations,[8] we find a one variable integral equation for Z_{ab} depending only on two particle observables, and by invoking a dispersion-theoretic representation for $\tau(z-\tilde{p}^2)$ can transform it[9] into the once iterated Faddeev form

$$\begin{aligned} Z_{ab}(p_a,p_b;z) &= -\bar{R}_{ab}(p_a,p_b;z) - \sum_{c=a\pm}\int_0^\infty p_c^2\,dp_c\,\bar{R}_{ac}(p_a,p_c;z)\,Z_{cb}(p_c,p_b;z) \\ &= -\bar{R}_{ab}(p_a,p_b;z) - \sum_{c=a\pm}\int_0^\infty p_c^2\,dp_c\,Z_{ac}(p_a,p_c;z)\,\bar{R}_{cb}(p_c,p_b;z) \end{aligned} \tag{3}$$

with

$$\bar{R}_{ab} = \frac{1}{2}\,\bar{\delta}_{ab}\int_{-1}^{1}\left[p_a^2/2\mu_b + p_b^2/2\mu_a + p_a p_b \xi/m_c - z\right]^{-1} d\xi \;;\; \bar{\delta}_{ab} = 1 - \delta_{ab} \tag{4}$$

Here we have dropped the factors τ_a and τ_b with which the equation must be clothed to restore the once iterated Faddeev form. Since we will derive a similar equation below, we do not provide here the direct derivation of this equation from the zero range boundary conditions in the three particle space. Thanks to Eqs. (1) and (2) by comparison with OB Eqs. (IV.7) and (IV.8) we find that the physical elastic scattering and rearrangement amplitudes $\mathscr{K}_{ab} = N_a Z_{ab} N_b$ and that the physical breakup amplitude is

$$\mathscr{B}_{0b}\left(p_a, q_a; p_b^{(0)}; z\right) = \sum_a \hat{\tau}_a(z-\tilde{p}_a^2) Z_{ab}\left(p_a, p_b^{(0)}; z\right) N_b - \frac{N_a^2 Z_{ab}\left(p_a, p_b^{(0)}; z\right) N_b}{\tilde{p}_a^2 - \varepsilon_a - z} \tag{5}$$

Hence the cross sections for the physical processes can immediately be obtained from OB Eqs. (III.8), (III.15), (IV.37) and (IV.39).

In order to extend our treatment to the four particle case we define the (3,1) configurations with $r = 1,2,3,4$ and the (2,2) configurations with $r = 5,6,7$ geometrically in Fig. 1 and algebraically in Table I. We see that, analogous to the treatment by Yakubovsky, we must consider 18 initial and 18 final configurations and construct our theory in terms of the amplitudes F_{ab}^{rt} where the symbols are only defined when $a \subset r$ and $b \subset t$. Starting from a state of four free particles, we project out the state in which all angular momenta are zero and obtain the radial wave function

$$\begin{aligned} U\left(x_a^r, y_a^r, z_r\right) = {} & \frac{\sin p_a^{r(0)} x_a^r}{p_a^{r(0)}} \frac{\sin q_a^{r(0)} y_a^r}{q_a^{r(0)}} \frac{\sin s_r^{(0)} z_r}{s_r^{(0)}} \\ & - \sum_{r'=1}^{7} \sum_{a' \subset r'} \int_0^\infty p_{a'}^{r'2} dp_{a'}^{r'} \int_0^\infty q_{a'}^{r'2} dq_{a'}^{r'} \int_0^\infty s_{r'}^2 ds_{r'} \\ & \times \frac{F_{a'b}^{r't}\left(p_{a'}^{r'}, q_{a'}^{r'}, s_{a'}^{r'}; E\right)}{\tilde{p}_{a'}^{r'2} + \tilde{q}_{a'}^{r'2} + \tilde{s}_{r'}^2 - E - i0^+} \\ & \times \int d\Omega \, \frac{\sin p_{a'a}^{r'r}(\Omega) x_a^r}{p_{a'a}^{r'r}(\Omega)} \frac{\sin q_{a'a}^{r'r}(\Omega) y_a^r}{q_{a'a}^{r'r}(\Omega)} \frac{\sin s_{r'r} z_r}{s_{r'r}(\Omega)} \end{aligned} \tag{6}$$

In order to apply our zero range boundary condition to this wave function, we must first reduce the spacial dependence to the coordinate y_a^r of the distinguished pair, which can be done by Fourier transformation yielding

$$U_{p_a^r s_r}(y_a^r) = \frac{\sin q_a^{r(0)}(y)}{q_a^{r(0)}} \frac{\delta\left(p_a^r - p_a^{r(0)}\right)}{p_a^r p_a^{r(0)}} \frac{\delta\left(s_r - s_r^{(0)}\right)}{s_r s^{(0)}} \delta_{ab}\delta_{rt}$$

$$- \pi\mu_r^a F_{ab}^{rt}(p_r^a, k_r^a, s_r; E)\, e^{ik_r^a y_r^a}$$

$$- \sum_{r'=1}^{7} \sum_{a' \subset r'} \bar{\delta}_{aa'} \int p_{a'}^{r'2} dp_{a'}^{r'} \int q_{a'}^{r'2} dq_{a'}^{r'} \int s_{r'}^2 ds_{r'} \frac{F_{a'b}^{r't}}{\tilde{p}^2 + \tilde{q}^2 + \tilde{s}^2 - E}$$

$$\times \int d\Omega \frac{\sin q_{a'a}^{r'r}(\Omega) y_a^r}{q_{a'a}^{r'r}(\Omega)} \frac{\delta\left(p_a^r - p_{a'r}^{r'r}(\Omega)\right)}{p_a^r p_{a'a}^{r'r}(\Omega)} \frac{\delta\left(s_r - s_{r'r}(\Omega)\right)}{s_r s_{r'r}(\Omega)} \quad ;$$

$$\bar{\delta}_{aa'} = 1 - \delta_{aa'} \tag{7}$$

where we have kept only the asymptotic form of the amplitude corresponding to the distinguished pair, consistent with our zero range assumption, and used the on shell value for q_a^r defined in Table I. Applying our zero range boundary condition $U'/U = k_a^r \text{ctn}\delta_a$ in the limit $y_a^r \to 0^+$ we find by invoking Eq. (1) that

$$F_{ab}^{rt} = \tau_a(E - \tilde{p}_a^{r2} - \tilde{s}_r^2)\left[\delta\delta - \sum_{r'=1}^{7} \sum_{a' \subset r'} \bar{\delta}_{aa'} \int R_{aa'}^{rr'} F_{a'b}^{r't}\right] \tag{8}$$

Since this equation still contains disconnected scattering processes when $r = r'$, we move these to the left hand side of the equation and obtain

$$\sum_{c}\left[\delta_{ac}\delta+\bar{\delta}_{ac}\tau_{a}\int R^{r}_{ac}\right]F^{rt}_{cb}=\tau_{a}\left[\delta_{rt}\delta_{ab}\delta\delta-\sum_{r'}\bar{\delta}_{rr'}\sum_{a'\subset r'}\bar{\delta}_{aa'}\int R^{rr'}_{aa'}F^{r't}_{a'b}\right] \tag{9}$$

By examination of this equation on the left for the (3,1) configurations, we find that this is simply the zero range Faddeev equation $M = t(1-\int\bar{R}M)$ clothed with the momentum conserving $\delta(s_r - s_r^{(0)})/s_r s_r^{(0)}$ of the four particle spectator and with the energy W replaced by $E-\tilde{s}_r^2$. But, as noted above, this equation also holds in the time reversed form $M = (1-\int M\bar{R})t$ obtained by applying the boundary condition to the first scattering rather than the last. Hence $(1-\int M\bar{R})(1+\int\bar{R})M = M$ providing an <u>algebraic</u> inversion of the operator on the left in Eq. (9) which when applied makes the driving term in the equation for F^{rt}_{ab} into $M^{r}_{ab}(E-\tilde{s}_r^2)\delta(s_r - s_r^{(0)})/s_r s_r^{(0)}$.

For the (2,2) configurations, the only terms which couple are $F^{r}_{a\bar{a}}$ and $F^{r}_{\bar{a}a}$, establishing immediately that $F^{r}_{aa} = t_a(E-\tilde{p}_a^{r2}-\tilde{s}_r^2)\delta(p^r_a - p^{r(0)}_a)\times\delta(s_r - s_r^{(0)})/p^r_a p^{r(0)}_a s_r s_r^{(0)}$. The coupled terms appear to give a problem since neither component of either pair scatters from the other, the spectator momentum factors out, and we anticipate a factored form. The factored solution is immediate in the Schroedinger equation in configuration space, but in the integral equation we get contributions in the iterations to any finite order in the multiple scattering series. Blankenbecler[10] has pointed out to the author that the same problem occurs in the conventional theory; it is mentioned by Mitra, Gillespie, Sugar and Panchapakesan.[11] However, if we iterate the two pair equation once we find that

$$M^r_{ab}\left(p'^r_a, p^{r(0)}_a; E - s^2_r\right) = \delta_{ab}\ t_a\left(E - \tilde{p}^{r2}_a - \tilde{s}^2_r\right)\ \delta\left(p^r_a - p^{r(0)}_a\right) / p^r_a p^{r(0)}_a$$

$$- \frac{t_a\left(E - \tilde{p}^{r2}_a - \tilde{s}^{(0)2}_r\right)\ t_b\left(E - \tilde{p}^{r2}_b - \tilde{s}^{(0)2}_r\right)}{\tilde{p}^{r2}_a + \tilde{p}^{r(0)2}_b + \tilde{s}^{(0)2}_r - E} - \int \bar{R}M \quad ;$$

$$r \subset 5,6,7 \tag{10}$$

Here we have used the fact that in this configuration $p^r_b = q^r_a$ as can be seen immediately from Fig. 1. But $E = \tilde{p}^{r(0)2}_a + \tilde{p}^{r(0)2}_b + \tilde{s}^{(0)2}_r$ showing that there is an on shell singularity in the first iterate. Hence we can multiply Eq. (10) through by this singularity and remove the unwanted multiple scattering term $\int \bar{R}M$. In configuration space this singularity does lead to the factored form $t_a t_b e^{ik^r_a y^r_a}\, e^{ik^r_b y^r_b}$ as expected. Further, we see that for these configurations we also have Eq. (2) with $Z_{ab} = -\delta_{\bar{a}b}(\tilde{p}^{r2}_a - \tilde{p}^{r(0)2}_a - i0^+)^{-1}$. Thus we have the Faddeev form for the equations and the algebraic inversion proceeds just as in the (3,1) case.

In the three particle equation we can see explicitly from Eq. (4) that the factorization of t allows the reduction of the equation to one variable with a geometrical kernel involving an integration over the angle $\cos^{-1}\xi$ between $\hat{p}_a$ and $\hat{q}_a$. All that happens for higher angular momentum states is that we acquire additional rotation matrices as functions of this angle and additional indices which are given explicitly in BO. The reduction occurs because of the δ-function for the spectator which puts the two body scatterings in the three particle space. In the four body case we have an extra integration in momentum, but also an extra δ-function, so the same reduction occurs. Hence we obtain by inverting the left hand side of Eq. (9) (as discussed above) the one variable equations for the zero range four particle problem

$$ {}^{(4)}M^{rt}_{ab}(p^r_a,p^t_b;E) = {}^{(3)}M^{r}_{ab}(p^r_a,p^t_b;E)\left[\delta_{rt}\delta\left(s^r_a - s^{r(0)}_a\right)\Big/ s_r s^{(0)}_r \right. $$

$$ \left. - \sum_{r'}\sum_{a'\subset r'}\int_0 dp^{r'}_{a'}\ {}^{(4)}\bar{R}^{rr'}_{aa'}(p^r_a,p^{r'}_a;E)\ {}^{(4)}M^{r't}_{a'b}(p^{r'}_{a'},p^t_b;E)\right] \tag{11} $$

where

$$ {}^{(4)}\bar{R}^{rr'}_{aa'}(p^r_a,p^{r'}_{a'};E) = \bar{\delta}_{rr'}\bar{\delta}_{aa'}\int d\Omega\int_0^\infty dq^{r'}_{a'}\int_0^\infty ds_{r'}\left(\tilde{p}^{r'2}_{a'} + \tilde{q}^{r'2}_{a'} + \tilde{s}^2_{r'} - E - i0^+\right)^{-1} $$

$$ \times\ p^{r'2}_{a'}\, q^{r'2}_{a'}\, s^2_{r'}\,\delta\left(p^r_a - p^{r'r}_{a'a}(\Omega)\right)\delta\left(s_r - s_{r'r}(\Omega)\right)\Big/ p^r_a p^{r'r}_{a'a}(\Omega)\, s_r s_{r'r}(\Omega) \tag{12} $$

and we have replaced the F^{rt}_{ab} which refer explicitly to the four particle case by ${}^{(4)}M^{rt}_{ab}$ with an eye to generalization to the N particle case. Just as in the three particle case, we could obtain an alternative equation by applying our boundary condition to the first scattering rather than the last; that is, we also have the equation ${}^{(4)}M = \left(1 - \int {}^{(4)}M\ {}^{(4)}\bar{R}\right){}^{(3)}M$. We also have the generalization of Eq. (2), namely

$$ {}^{(4)}M^{rt}_{ab} = {}^{(3)}M^{r}_{ab}\,\delta_{rt}\,\delta_{ab} + {}^{(3)}M^{r}_{ab}\ {}^{(4)}Z^{rt}_{ab}\ {}^{(3)}M^{t}_{ab} \tag{13} $$

Hence by one iteration of Eq. (11) we can obtain an integral equation for the smooth function ${}^{(4)}Z$ in which the primary singularities have been factored out. Thus knowing Z we can immediately recover all the physical four particle cross sections in a manner strictly analogous to the three particle case discussed in OB. We are grateful to V. Vanzani for showing[12] that the form of our four particle equations is identical to the form of one set of such equations he has developed in the conventional theory,[13] except that the off shell behavior in his equations requires a convolution over ${}^{(3)}M$ which prevents the factorization we have found in the zero range theory.

Since our theory does not rest on a Hamiltonian model for the interactions; we are required[14] to prove that the resulting equations are unitary. In the three particle case the unitarity condition $M_{ab} - M^*_{ab} = -\sum_{cd} M_{ac}(R_0 - R^*_0)M^*_{db}$ follows immediately from the form of the Faddeev equations and the two particle on shell unitarity condition $t_a - t^*_a = -t_a(R_0 - R^*_0)t^*_a$, as was pointed out by Friedman, Lovelace and Namyslowski[15] and discovered independently by Kowalski.[16] Using a matrix notation, the proof is simple:

$$
\begin{aligned}
-\sum_{cd} M_{ac}(R_0 - R^*_0)M^*_{db} &= -\sum_{c}\Big(\delta_{ac} - \sum_{c'} M_{ac'}R_0\bar{\delta}_{c'c}\Big)\big(t_c - t^*_c\big)\Big(\delta_{ab} - \sum_{c''}\bar{\delta}_{cc''}R^*_0 M^*_{c''b}\Big) \\
&\quad - \sum_{cd}\bar{\delta}_{cb}\, M_{ac}(R_0 - R^*_0)M^*_{db} \\
&= M_{ab} - M^*_{ab} - \sum_{c'} M_{ac'}R_0\bar{\delta}_{c'c}t^*_c\Big(\delta_{cb} - \sum_{c''}\bar{\delta}_{cc''}R^*_0 M_{c''b}\Big) \\
&\quad + \sum_{c}\Big(\delta_{ac} - \sum_{c'} M_{ac'}R_0\bar{\delta}_{cc'}\Big)t_c \sum_{c''}\bar{\delta}_{cc''}R^*_0 M^*_{c''b} \\
&\quad - \sum_{cd}\bar{\delta}_{cd}\, M_{ac}(R_0 - R^*_0)M^*_{db} \qquad (14)
\end{aligned}
$$

where the first line diagonal term is obtained by using the Faddeev equations in appropriate order and the two particle unitarity condition (which is on shell thanks to the δ-function implied by $R_0 - R^*_0$); the unwanted terms in the last equation vanish by a second application of the Faddeev equations. In order to convince those who find this somewhat symbolic proof inadequate, the proof has been carried through using the explicit integral expressions and including the bound state poles[1] and checked by Erwin Alt.[17] But, since the four particle equations depend now only on factored three particle input, which has just been shown to satisfy three particle on shell unitarity, the same steps immediately establish the unitarity of our four particle equations. Just as in the three particle case, the two forms of the equation lead to time reversal invariance.

We claim that the generalization to the N-particle case is now transparent. We write our N-particle equation in configuration space using the full Faddeev-Yakubovsky combinatorial decomposition and reduce this to a one variable equation in the distinguished coordinate. Applying our zero range boundary condition as before, the two particle amplitude factors out. Transferring the appropriate configurations to the left hand side we obtain spectator problems in reduced spaces which can be inverted in the same way that we demonstrated explicitly for the (3,1) and (2,2) configurations above. The driving term on the right now has N-2 δ-functions rather than 2 before the inversion, and N-3 δ-functions after the inversion. Hence, just as before, we can obtain one variable equations driven by the appropriate analytic continuation of the N-1 particle amplitudes. The integral equations that provide these continuations have no singularities other than bound state poles provided only the two particle amplitudes themselves have no such singularities, as already required. Time reversal invariance follows from two forms of the equations as before. Unitarity is immediate from an obvious generalization of the FLN proof. The reduction of the kernel to one variable follows from standard applications of angular momentum techniques, which of course become increasingly tedious as the number of particles increases, but which have to be faced in any exact N-particle theory. We therefore claim to have proved that the N-particle zero range equations are always one variable equations of the form

$$
{}^{(N)}M^{C(N)C'(N)}_{C(N-1)\ldots} = {}^{(N-1)}M^{C(N)}_{C(N-1)\ldots} \times \left[\delta_{C(N)C'(N)\ldots} - \sum_{C(N'')} \bar{\delta}_{CC''} \sum_{C''(N-1)\subset C''\ldots} {}^{(N)}\bar{R}^{CC''}_{C''(N-1)\ldots} \; {}^{(N)}M^{C''C'}_{C''(N-1)\ldots}\right] \tag{15}
$$

and in reverse order. Finally, the essential singularities can always be factored out by an obvious generalization of Eq. (13).

The physics lying behind the remarkably simple result we have obtained is simply that by sticking to two particle on shell scatterings of the pairs as the driving mechanism and making the angular momentum reduction, the only variable content left on which these amplitudes can depend, thanks to momentum conservation, is the appropriate analytic continuation to negative energies required by the uncertainty principle. The factorization is quite general for short range interactions as was proved long ago.[7] The simplification was conjectured a decade ago,[14] but could not be proved because of the reluctance of this author to abandon "left hand cuts" in the two particle input, which turns out to be the key to success. In the relativistic generalization of this approach, which we claim to be immediate and which has been shown to work in the three particle case,[18,19] this assumption turns out to be analogous to the "locality" assumption of quantum field theory. Our theory differs in that it can be kept consistently to sectors in which only a finite number of particles enter by using particle functions rather than field functions as the basis. The basic trick in the relativistic generalization is simply to assume that "particle" and "quantum" bind to make a state with the same mass and quantum numbers as the "particle". As in the non-relativistic theory presented in this communication, unitarity and time reversal invariance are immediate. "Crossing" and relativistic spin are under investigation.

As already noted, confidence in the four particle approach was gained thanks to detailed study an earlier version of the equations by V. Vanzani. Conversations with W. Sandhas, H. Haberzettl, and E. Alt were also helpful. Recent discussions with R. Blankenbecler, L. Biedenharn, and M. Orlowski were instrumental in bringing this work to completion. The author is grateful to the Humboldt Stiftung for an award and to E. Schmid and the University of Tübingen for hospitality during the first pass at the four particle zero range equations.

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TABLE I

Four Particle Coordinates

$$M = \sum_{i=1}^{4} m_i = \sum_{i=1}^{4} m_{r_i}$$

$r \subset 1,2,3,4$	$r \subset 5,6,7$
$M_r = M - m_{r_4}$; $M\nu_r = M_r m_{r_4}$	$M\nu_r = (m_{r_1} + m_{r_2})(m_{r_3} + m_{r_4})$
$r_i = r - 1 + {}_4 i$	$r_i = r - 5 + {}_3 i$

$$s_r^2 = 2\nu_r \tilde{s}_r^2$$

$$a,b,c \subset r \quad ; \quad \mu_a^r = m_{r_1} m_{r_2} / (m_{r_1} + m_{r_2})$$

$$(q_a^r)^2 = 2\mu_a^r (\tilde{q}_a^r)^2$$

$$a = (r_1, r_2) \quad ; \quad b = (r_2, r_3) \quad ; \quad c = (r_3, r_1)$$

$M_r n_a^r = m_{r_3}(m_{r_1} + m_{r_2})$; $(\tilde{p}_a^r)^2 = 2n_a^r (p_a^r)^2$ $\quad$ $\bar{a} = (r_3, r_4)$; $\bar{b} = (r_1, r_4)$; $\bar{c} = (r_2, r_4)$

$M_r n_b^r = m_{r_2}(m_{r_3} + m_{r_1})$; $(\tilde{p}_b^r)^2 = 2n_b^r (p_b^r)^2$ $\quad$ $\bar{\mu}_a^r = m_{r_3} m_{r_4} / (m_{r_3} + m_{r_4})$, etc.

$M_r n_c^r = m_{r_1}(m_{r_2} + m_{r_3})$; $(\tilde{p}_c^r)^2 = 2n_c^r (p_c^r)^2$ $\quad$ $(\tilde{p}_{\bar{a}}^r)^2 = 2\bar{\mu}_a^r (p_{\bar{a}}^r)^2$, etc.

If the four particle c.m. energy normalized to zero at four particle breakup threshold is called E, then the on shell condition $E = \tilde{p}^2 + \tilde{q}^2 + \tilde{s}^2$ is configuration and channel invariant. The on shell momentum for the distinguished pair is defined as $k_a^r = [2\mu_a^r(E - \tilde{p}_a^{r2} - \tilde{s}_r^2)]^{\frac{1}{2}}$. The spacial coordinates corresponding to $\underline{p}_a^r$, $\underline{q}_a^r$ and $\underline{s}_r$ are $\underline{x}_a^r$, $\underline{y}_a^r$ and $\underline{z}_r$ respectively. In order to express the four particle wave function in terms of a single set of coordinates we will need to know the geometrical connections

$\underline{p}_a^r(\underline{p}_{a'}^{r'},\underline{q}_{a'}^{r'},\underline{s}_{r'}) \equiv \underline{p}_{aa'}^{rr'}$, $\underline{x}_a^r(\underline{x}_{a'}^{r'},\underline{y}_{a'}^{r'},\underline{z}_{r'}) \equiv \underline{x}_{aa'}^{rr'}$, etc. which are readily obtained from Fig. 1. If we take out the dependence on the orientation of the configurations in space, which can be done by an appropriate application of rotation matrices analogous to that done with care for the three particle case in OB, these transformations will depend on the three direction cosines $(\hat{p}_a^r\cdot\hat{q}_a^r)$, $(\hat{p}_a^r\cdot\hat{s}_r)$, $(\hat{q}_a^r\cdot\hat{s}_r)$ which we symbolize collectively by Ω. The reduction of the plane wave basis $\exp i(\underline{p}\cdot\underline{x}+\underline{q}\cdot\underline{y}+\underline{s}\cdot\underline{z})$ to the scalar form used in the text is greatly facilitated by the identity

$$\underline{p}_a^r\cdot\underline{x}_{aa'}^{rr'} + \underline{q}_a^r\cdot\underline{y}_{aa'}^{rr'} + \underline{s}_r\cdot\underline{z}_{rr'} = \underline{p}_{a'a}^{r'r}\cdot\underline{x}_a^r + \underline{q}_{a'a}^{r'r}\cdot\underline{y}_a^r + \underline{s}_{r'r}\cdot\underline{z}_r \quad .$$

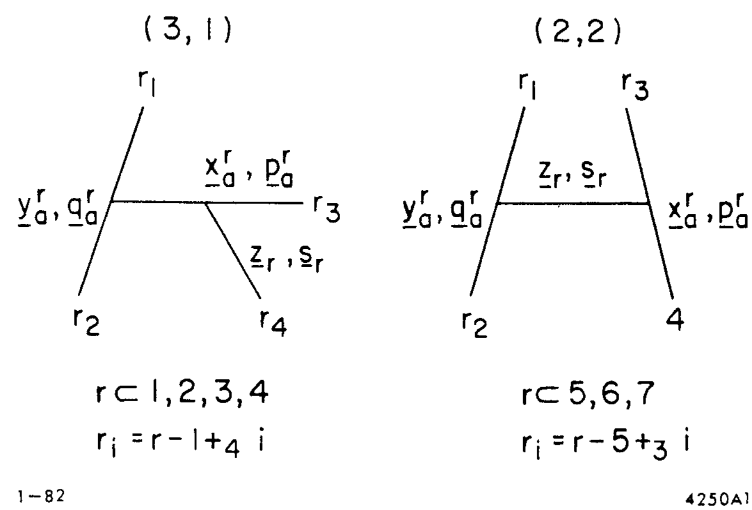

Fig. 1. Geometrical definition of the four particle coordinates used in this study for the (3,1) and (2,2) configurations. Algebraic details are given in Table I.