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GROUP THEORETIC APPROACHES TO NUCLEAR AND HADRONIC COLLECTIVE MOTION*

L. C. Biedenharn†
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

Three approaches to nuclear and hadronic collective motion are reviewed, compared and contrasted: the standard symmetry approach as typified by the Interacting Boson Model, the kinematic symmetry group approach of Gell-Mann and Tomonaga, and the recent direct construction by Buck.

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† Permanent address: Duke University, Durham, N.C. 27706.

I. INTRODUCTION AND SUMMARY

It is a pleasure to take part in this "Symposium on Group Theory and Its Applications in Physics" held in honor of Marcos Moshinsky on the occasion of his 60th birthday. The subject of the symposium is one to which Marcos, and many others of us here, have devoted much effort and interest, and a subject not as well appreciated, when we began many years ago, as it is today. This change is due not only to these combined research efforts, but also to the excellent organizational talents of the UNAM group under Marcos in arranging many successful colloquia over the years to proselytize and augment the group theoretic viewpoint.

I would like to discuss today *the applications of group theory* -- and hence symmetry techniques -- *to nuclear and hadronic collective motion*. Much of what I will discuss will be a review, but not all, for I hope to present a few new results and special aspects.

My introduction to symmetry techniques, like most of us here including Marcos, stemmed from the work of Wigner, beginning with angular momentum theory and leading, through Racah's work, into general Lie groups. The fact that the (orbital) harmonic oscillator shell model is SU(3) invariant became physically meaningful with the Elliott rotational model. One of the keys in elucidating this structure was the Bargmann-Moshinsky series[1,2,3] on the group theory of harmonic oscillators in the early 60's. These papers brilliantly exploited techniques that have been used repeatedly in the years following.

There are precisely *two* such techniques and they may be elegantly codified as (a) *The Jordan-Schwinger Map*,[4,5] and (b) *The Dirac Map*.[6]

ad(a): *The Jordan-Schwinger Map* takes $n \times n$ matrices and maps them into bilinear products over n boson operators. To be precise let $\{a_i, \bar{a}_i\}$ denote boson operators (in Dirac's original notation) obeying: $[\bar{a}_i, a_j] = \delta_{ij}$ with all other commutators zero. If $\{\underset{\sim}{A}^{(\alpha)}\}$ are a set of $n \times n$ matrices with *numerical* elements A^{α}_{ij}, define the mapping J by:

$$J: \quad \underset{\sim}{A}^{\alpha} \rightarrow \mathscr{A}^{(\alpha)} \equiv \sum_{ij} A^{\alpha}_{ij}\, a_i\, \bar{a}_j \quad . \tag{1.1}$$

Then one has the elementary (but extraordinarily useful!) result: *The map J preserves commutation relations*:

$$J([A,B]) = [J(A), J(B)] \quad . \tag{1.2}$$

Expressed in words, the *operators* $\{\mathscr{A}^{(\alpha)}\}$ obey the same commutation relations as the *numerical* matrices $\{A^{(\alpha)}\}$. Generalized to n^2 bosons (n independent copies of n bosons), this result is *definitive for all irreps of the unitary group U(n)* and can be extended easily to compact forms of the orthogonal group (as well as other Lie groups).

ad(b): To define the Dirac mapping requires that in addition to the n-boson operators $\{a_i, \bar{a}_i\}$ we construct the $1 \times n$ matrix $(a_1 \ldots a_n\ \bar{a}_1 \ldots \bar{a}_n) \equiv \Lambda$, its transpose $\tilde{\Lambda}$, and the matrix $\beta \equiv \begin{pmatrix} 0 & -\mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}$, where $\mathbb{1}_n$ is the $n \times n$ unit matrix.

The Dirac mapping of the $2n \times 2n$ numerical matrix A is then defined by:

$$D: \; A = (A_{ij}) \rightarrow \frac{1}{2} \sum_{i,j,k=1}^{2n} \Lambda_k\, \beta_{ki}\, A_{ij}\, \Lambda_j = \frac{1}{2} \tilde{\Lambda} \beta\, A \Lambda \quad . \tag{1.3}$$

If we restrict the matrices A,B,... to $2n \times 2n$ numerical matrices of the form βM, where M is *symmetric*, then one finds:

$$D([A,B]) = [D(A),D(B)] \quad . \tag{1.4}$$

For matrices of the restricted form, *the Dirac operator mapping preserves commutation relations*. The Dirac operator mapping thus has the same basic property as the Jordan-Schwinger boson operator mapping, but constitutes a generalization of the J-S map in that the matrices involved are larger ($2n \times 2n$ instead of $n \times n$). The price one pays for this generalization is that the admissible matrices must have a restricted form. (For matrices not of the restricted form, the Dirac map yields a c-number and not an operator.)

The Dirac map is especially adapted to the non-compact symplectic group $Sp(2n,\mathbb{R})$ and to its double covering, the metaplectic group. It may also be adapted to graded Lie algebras.[7]

These two maps underlie a huge amount of the current group theoretical applications in physics (and lately even in mathematics); both maps are now used routinely without much notice or comment. Despite this familiarity, I thought the present audience, especially, would enjoy seeing the structure codified in this elegant and easily comprehended way. (It is interesting to note that the maps have inverses.[8])

Let me turn now to my main theme, symmetry and collective motion. Following the literature, I will distinguish two ways of exploiting symmetry and discuss each separately in succeeding sections (Sections II and III). In the course of doing so, I will review the currently interesting interacting boson model and its relation to the earlier Bohr-Mottelson models (Section IV). The direct approach to collective motion will be discussed in Section V, and the recent criticisms of Louck, concerning this construction, will be discussed in this concluding section.

II. THE STANDARD SYMMETRY APPROACH TO NUCLEAR COLLECTIVE MOTION

By the standard symmetry approach, we mean the construction of a Hamiltonian which is invariant (or nearly so) under a group of symmetry transformations; group theory then allows one to construct basis states realizing the symmetry, and explicit matrix elements for physically interesting transition operators[3] (themselves classified by the symmetry). By "nearly invariant" we mean there may be *small perturbations* by non-invariant pieces of the Hamiltonian, which pieces are again classified and explicated by the symmetry. This is certainly completely standard.

a. *The Bohr Model*

One of the first such nuclear models (for N and Z, even) is the Bohr treatment[9,10] of the nucleus as a *liquid drop*. The radius of the drop is expanded as a Legendre series in Y_{LM} and the expansion truncated to $L = 0$ and 2 only. ($L = 1$ is eliminated by the center-of-mass constraint.) Thus:

$$r = a + \sum_{\mu} q_{\mu} \, Y_{2,\mu}(\theta_{\varphi}) \quad , \tag{2.1}$$

$$q_{\mu}^{*} = q_{-\mu} \quad , \quad \text{(reality condition)} \quad . \tag{2.2}$$

The model focusses on the five quadrupolar variables, $\{q_{\mu}\}$, and their conjugates, $\{\Pi_q\}$, and takes the Hamiltonian to be approximately that of a five-dimensional harmonic oscillator with a common frequency.

The spectrum, in lowest order, thus agrees with the frequently observed nearly harmonic spectrum, near closed shells, which typify the anharmonic five-dimensional vibrator.

This model is heuristic and taken to be a realization, approximately, of *incompressible quadrupolar flow* defined by quadrupolar surface vibrations.

Group theoretically the model is that of *broken SU(5) symmetry*, and, in accord with this view, the dynamics were greatly extended by Greiner,[11] among others,[12] who used as the model Hamiltonian all possible SO(3) invariant interactions constructible with four or fewer bosons.

The group theoretic classification of the states via the chain: $SU(5) \supset SO(5) \supset SO(3) \supset SO(2)$ was discussed by many, notable contributions[13,14] being made by Moshinsky and his group, especially to the transformation coefficients defined by this subgroup decomposition. (The explicit quantum numbers defined by this chain are given below in Section II-c.)

b. *The Bohr-Mottelson Unified Model*

The liquid drop model of collective nuclear motion is characterized by irrotational flow and (as discussed below) *small* moments of inertia (I_{liq}). At the opposite extreme for collective nuclear motion is *rigid body motion*, which in the Bohr-Mottelson approach[10] is modeled by a fixed nuclear wave function defined in the intrinsic frame of the rigid rotator. Thus one has a wave function of the form:

$$\psi = D^{J*}_{MK}\left(\{\theta_\lambda\}\right) \chi_{\text{intrinsic}} \quad , \tag{2.3}$$

which implies an adiabatic splitting of the internal motions ($\chi_{\text{intrinsic}}$) and the rotational rigid body motion (D^{J*}_{MK}) with a K-quantum number defined in the intrinsic frame. The adiabatic condition, on which the splitting is based, assumes that the rotational motion is very slow compared to the internal motions.

Because of the quantization of angular momentum an arbitrarily small rotational frequency cannot be assumed. It follows that the adiabatic splitting of the wave function cannot be a general property.[15] An alternative way to view the physics of this situation is to note that the adiabatic splitting implies that the relation between the body-fixed frame (the frame in which χ_{int} is defined) and the laboratory frame is well defined. But to define precisely the angular variables relating to the two (classical) frames implies by the uncertainty principle that *unlimitedly large angular momenta are involved.* That is to say, the intrinsic wave function is required to be essentially unchanged even for large rotational excitations. Using still other words, the adiabatic splitting necessarily implies that the rotational bands (effectively) do not terminate.

Unlimitedly large (rotational) bands are characteristic of noncompact groups (and, accordingly, infinite-dimensional unitary representations).

The group-theoretic structure that corresponds to the adiabatic Bohr-Mottelson unified model is the noncompact group $\mathbb{R}^5 \circledS SO(3)$ as found by Ui,[16] or more properly, the covering group $\mathbb{R}^5 \circledS SU(2)$ as found by Weaver et al.[17] This group has a rather elementary algebra consisting of a general angular momentum operator J (with $J \times J = iJ$) and a quadrupole operator Q with commuting components, $[Q_\mu, Q_{\mu'}] = 0$, which transforms as a quadrupole under J, i.e., $[J_m, Q_\mu] = iC^{212}_{\mu M \mu'}\, Q_{\mu'}$. The irreps of this group include all known examples (of *both integer* and of *half-integer*) quadrupolar rotational bands, and these irreps automatically obey the discrete symmetry structure (D_2) found earlier by Landau (molecular) and by Bohr (nuclei). *The K quantum number*, as one might expect, *is well-defined*.

A group-theoretic model for *terminating band structure* was found considerably earlier by Elliott:[18] this is the *compact* group generated by the *orbital* angular momentum L and a quadrupole operator Q, obeying the SU(3) algebra:

$$L \times L = i\,L \quad ; \quad [L_m, Q_\mu] = iC^{212}_{\mu m \mu'}\, Q_{\mu'}$$

and (2.4)

$$[Q_\mu, Q_{\mu'}] = iC^{221}_{\mu\mu' m}\, L_m \quad .$$

Let us remark that these group-theoretic state classification problems are by no means always straightforward! The labeling induced by SU(3) ⊃ SO(3) is a classic example of the difficulties that can occur. This problem was given a definitive discussion a few years ago by the combined efforts of the Montreal and Moshinsky UNAM groups in a paper[19] fittingly entitled "Everything You Always Wanted to Know About SU(3) ⊃ SO(3)."

Racah showed very early that an (orthonormal) labeling by a polynomial operator in the SU(3) generators was not possible. Nevertheless Elliott gave a (non-orthogonal and approximate labeling in terms of a "K-quantum number," equivalent to the heuristic basis:

$$[p\ q\ 0] \supset \begin{cases} 0 \le K \le \min(p,p-q)\ , \ K = \text{even integer} \\ K \le L \le \max(p,p-q)\ , \\ \text{If } K=0\ , \text{ then } L = \text{even integer only} \quad . \end{cases} \qquad (2.5)$$

The bands in the Elliott model must terminate, since the group is compact (and hence the unirreps are finite dimensional). Accordingly the values of the available angular momenta are limited in size; it follows that the K-quantum number is *necessarily ill-defined physically.*

This is the physical reason[20,21] behind the difficulty of the labeling problem for the subgroup chain: $SU(3) \supset SO(3)$.

c. *The Interacting Boson Model*[22]

The Interacting Boson Model‡ (IBM) for even-even nuclei has a completely different physical motivation. The underlying physical structure is the shell model, and attention is focussed on the (2N) nucleons outside closed shells. (For shells more than half-filled, one uses nucleon holes.) These nucleons are then assumed to *form pairs* (NN or PP pairs of unit isospin) having either $L = 0$ ("s-bosons") or $L = 2$ ("d-bosons"). These "bosons" -- N in number and of six types (1-s,5-d's) -- are assumed to interact as true bosons forgetting their origin as fermion pairs.

The Hamiltonian is taken to be rotationally invariant and to have all possible terms constructible from four or fewer bosons. There are nine possible terms and this general Hamiltonian reads:[24]

$$\begin{aligned} H = {} & \varepsilon_s \, s\cdot\bar{s} + \varepsilon_d \, \underset{\sim}{d}\cdot\underset{\sim}{\bar{d}} \\ & + \sum_{L=0,2,4} A_L\Big[(\underset{\sim}{d}\times\underset{\sim}{d})^L \times (\underset{\sim}{\bar{d}}\times\underset{\sim}{\bar{d}})^L\Big]^0 \\ & + B_0\Big[(\underset{\sim}{d}\times\underset{\sim}{d})^0\,(\bar{s}\;\bar{s}) + \text{h.c.}\Big]^0 \\ & + B_2\Big[(\underset{\sim}{d}\times\underset{\sim}{d}) \times (\underset{\sim}{\bar{d}}\;s) + \text{h.c.}\Big]^0 \\ & + C_0\Big[(\bar{s})^2\,(s)^2\Big] + C_2\Big[(ds)\times\bar{d}\;\bar{s}\Big]^0 \quad . \end{aligned} \tag{2.6}$$

‡ Actually there are two models: IBM-1 and IBM-2. In the former[23] no distinction is made between proton and neutron pairs whereas in IBM-2 the two types of pair are distinguished. Only IBM-1 enters in the sequel, but it is easy to extend the considerations to IBM-2.

Group theoretically the interacting boson model is a realization of *broken SU(6) symmetry*. Accordingly we must select a subgroup chain for the breaking pattern, and then use standard methods to label states and construct matrix elements.

SU(6) symmetry is unusually rich in subgroup chains and there are three chains of important physical interest:

(a) *Anharmonic vibrator:*

$$SU(6) \supset SU(5) \supset SO(5) \supset SO(3) \supset SO(2) \ ,$$

(b) *Axisymmetric rotor:*

$$SU(6) \supset SU(3) \supset SO(3) \supset SO(2)$$

(c) *γ-unstable rotor:*[25]

$$SU(6) \supset SU(4) \cong SO(6) \supset SO(5) \supset SO(3) \supset SO(2) \ .$$

For each of these chains, the general Hamiltonian of Eq. (2.6) may be specialized, such that a closed form results in terms of the linear and quadratic (Casimir) invariants of the subgroups in the chain.[24]

The labeling of the states in each chain may be given explicitly, but the most convenient labeling yields non-orthogonal states[13] such as encountered first in the Elliott model. (The group theoretic technique of "traceless bosons" is useful here.[26])

It is important to note that not only is a closed form available for the energies, and a complete labeling of the states, but analytic expressions can be given for matrix elements of transition operators.[27] This makes it possible to survey large quantities of data extensively for trends in the parameters of the Hamiltonian (which depend on N and Z).

The labeling of all states defined by the three subgroup chains above are given in the following diagram:

SU(6)

irrep: $[N\ \dot{0}]_6$

SU(5)

irrep: $[n_d\ \dot{0}]_5$

$n_d = N, N-1, \ldots, 0$

SU(4)

irrep: $[\sigma\ \sigma\ 0\ 0]$

$\sigma = N, N-2, \ldots, 0$ or 1

SU(3)

$[p\ q\ r] \cong [p-r\ \ q-r\ \ 0]$

p,q,r are all partitions of 2N into lexical triples of even integers.

SO(5)

irrep: $(v,0)$

$v = n_d, n_d-2, \ldots, 0$ or 1

$v = \sigma, \sigma-1, \ldots, 0$

SO(3)

irrep: (L)

<u>SO(5) Chain</u>

partition $v = 3\Delta + \lambda$
For each possible partition,
$L = 2\lambda, 2\lambda-2, 2\lambda-3, \ldots, \lambda$
($L = 2\lambda-1$ does not occur)

<u>SU(3) Chain</u>

$0 \le K \le \min(q, p-q)$
K = even integer.

$K \le L \le \max(q, p-q)$
If $K = 0$, only
L = even integer.

d. *How are the Bohr Model and the Interacting Boson Model Related?*

This question is both interesting and topical -- a session of the APS meeting in Baltimore (April 20-24, 1981) was devoted to answering it. (Iachello and Arima in noting the existence of the anharmonic vibrator chain took the essential equivalence of the models as a truism from the beginning.) The analyses at the APS meeting showed no clear consensus. Klein, Li and Vallierés, for example, assert[28] that the two models are "completely equivalent" whereas, by contrast, Gilmore and Feng insist[29] that "the two models are demonstrably not equivalent."

The existence of such contradictory positions reflects the fact that there is no common view as to precisely what constitutes each of the two models. Klein et al. (disavowing totally the liquid drop origins of the Bohr model) define their "Bohr model" to allow arbitrary non-polynomial boson interactions and, more importantly, restrict the Hilbert space to a *finite* basis (fixed for a given nucleus). The relation between this "Bohr model" and a "generalized IBM" (also allowing arbitrary interactions) is shown by an equivalence of bases, realized by means of a nonlinear (Holstein-Primakoff) boson mapping (see below).

By contrast the Gilmore-Feng assertion is based on the fact that the Bohr model has an unbounded spectrum, in clear contrast to that of the IBM. This view points to a valid physical distinction: the quadrupolar quanta of the original Bohr model are unlimited, whereas the SU(6) quanta of the IBM are fixed for a given nucleus by the number of valence pairs.

But mathematically, the two models show very close relationships. Gilmore and Feng point out that the ansatz, Eq. (2.1), does not in fact conserve volume, so that to conserve volume a *dynamical* $L = 0$ term must (in this view) be adjoined as a physical variable [equivalent to adjoining an s-boson; this then leads (non-linearly) to the IBM]. Technically this is correct [taking Eq. (2.1) as *exact*] but -- as we show in Section III -- is a misreading of the physics! Actually the ansatz given by Eq. (2.1) is simply a poor approximation for incompressible irrotational flow with *large* deformations; the s-boson in fact enters -- as we show later -- to allow the possibility of *volume-changing* monopole deformations!

If one simply adjoins an $L = 0$ dynamical variable, s, and imposes a conservation condition on the (total) number of bosons [this is a basis (or wave function) constraint] then we are led at once to a nonlinear realization of U(6) symmetry (and the IBM) involving explicitly only the five d-bosons:

$$\text{U(5) generators:} \quad \{d_\mu \, \bar{d}_\nu\} \quad ; \quad \mu,\nu = 1,\ldots,5 \tag{2.7a}$$

additional $U(\tilde{6})$ generators:

$$d_\mu \bar{s} \rightarrow d_\mu \left[N - \sum_1^5 d_\mu \, \bar{d}_\mu \right]^{1/2} \quad ; \quad \mu = 1,\ldots,5 \tag{2.7b}$$

$$s\bar{d}_\mu \rightarrow \left[N - \sum_1^5 d_\mu \, \bar{d}_\mu \right]^{1/2} \bar{d}_\mu \quad ; \quad \mu = 1,\ldots,5 \tag{2.7c}$$

$$s \, \bar{s} \rightarrow \left[N - \sum_1^5 d_\mu \, \bar{d}_\mu \right] \ \text{(constraint on basis)} \quad . \tag{2.7d}$$

One sees in this procedure the origins of (and the reasons for) the assumptions underlying the Klein et al. analysis. The existence of this nonlinear mapping relating the two models has been noted by many.[30-33]

It is not surprising that this extension of the Bohr model yields the IBM; in fact, Janssen, Jolos and Dönau[30] used precisely this path to develop the "IBM" in 1974, *prior* to the Iachello-Arima introduction of their model!

The relationship between these two models was analyzed by several authors[34-36] by the *method of coherent states* to yield the classical limit of the IBM in the form of a potential energy surface (involving the intrinsic shape variables); comparison with the Bohr model potential energy can then be made directly, including all symmetry limits of the IBM. The method of coherent states is an important group-theoretic technique[29] for analyzing symmetry structures but it would carry us too far afield to discuss it here.

In the final analysis the equivalence or not of the two models is a matter of definition (and personal taste). Suffice it to say, both models are useful and reflect quite different physical viewpoints.

III. THE KINEMATIC SYMMETRY APPROACH TO COLLECTIVE MOTION

The Bohr Model and the Interacting Boson Model have both been discussed as instances of the standard symmetry approach to collective motion in the preceding section. There is, however, an entirely different approach for exploiting symmetry. One begins by focussing attention on certain physically important operators (observables) which generate the symmetry. The Hamiltonian is assumed to be a function of these operators, but it need not be an invariant. This assumption alone suffices to ensure that any multiplet, characteristic of the symmetry, will at worst be *split by the Hamiltonian, but not mixed with other multiplets.*

One thus starts with a set of operators that obey (equal time) commutation relations characteristic of some algebra. These operators are identified with physical transition operators which, acting on a given state, use up most of their strength in transitions to a few nearby states. The algebra may be such that (because of dynamics) the stationary, or quasi-stationary, states fall into a few (unitary) irreducible representations of the group. This approach, which is largely attributed now to Gell-Mann had been partly developed earlier by Lipkin and Goshen,[37] and even earlier by Tomonaga.[38]

The focus upon transition operators as generators of the symmetry has one aspect that deserves emphasis: *commutation relations are kinematical statements, so that the algebraic structure is preserved independently of the dynamics of symmetry breaking.*

Following the model by which the weak and electromagnetic currents were exploited, Dothan, Gell-Mann, and Ne'eman considered the (symmetric) energy-momentum tensor, which couples to gravity, and showed that,[39] in the quark model, the time-derivative of the quadrupole moment of the zero-zero component of this tensor and the *orbital* angular momentum close on the algebra of $s\ell(3,\mathbb{R})$. This is the group of volume preserving deformations and rotations of three-space.

For nuclear and hadronic collective motion it is intuitively clear that the quadrupole moments are important operators, and a crucial point is how to treat properly these deformational degrees of freedom. The way to proceed has already been made clear in condensed matter physics where quantal treatments of collective phenomena such as sound waves, plasma oscillations, and the like are important. *One identifies the appropriate flow pattern of the desired collective motion and constructs the corresponding generator.*[40,41]

Consider for the moment two dimensions. Using the velocity potential $\phi = \frac{1}{2}(x^2 - y^2)$ one obtains a volume preserving irrotational flow. (The flow changes a region bounded by a circle into an elliptic boundary.) The generator for this flow is the operator: $\Pi \equiv [(\nabla\phi)\cdot\nabla + \text{h.c.}]$ which has the form: $\Pi = xp_x - yp_y$.

To apply this to nuclear collective motion (in three dimensions) we observe that ϕ is none other than a quadrupole operator, so we consider the total quadrupole moment generated by the nucleons (relative to the center-of-mass):

$$Q_\mu = \sum_{n=1}^{N} q_\mu^{(n)} \quad , \tag{3.1}$$

and construct the corresponding flow generator:

$$\Pi_\mu = \sum_{\substack{n=1 \\ i=1\ldots3}}^{N} \left[\left(\nabla_i^{(n)} q_\mu^{(n)}\right)\nabla_i^{(n)} + \text{h.c.}\right] \quad . \tag{3.2}$$

The five operators Π_μ generate, under commutation, the Lie algebra $s\ell(3,\mathbb{R})$, which contains, in addition to the five Π_μ the three angular momentum operators $\underset{\sim}{L}$. The corresponding non-compact Lie group is the group of volume-preserving rotations and shearing deformations of three-space: precisely the collective motions one would associate with nuclei or hadrons if composed of incompressible fluid matter.

Let us make a series of remarks in place of an extended discussion:[20]

Remark 1: Tomonaga showed that in a Taylor series approach to a general collective Hamiltonian it was always possible to ensure that

$$\left[H - T_{\text{collective}}, Q_\mu\right] = 0 \quad , \tag{3.3}$$

where $T_{\text{collective}}$ is the collective kinetic energy and Q_μ the collective coordinate.

This is equivalent[20] to the Gell-Mann "anti-contraction" postulate:

$$\Pi_\mu \equiv \frac{i}{\hbar}\left[H, Q_\mu\right] \quad , \tag{3.4}$$

which defines the deformation generators as time derivatives of quadrupole collective coordinates.

Remark 2: The Gell-Mann -- Tomonaga result [Eqs. (3.2), (3.3) and (3.4)] is not an arbitrary choice but is in fact *essential* for a physically meaningful result! This can be seen group theoretically in this way: consider the quadrupole operators Q_μ and rotation operators $\underset{\sim}{L}$ as generators of a group having the algebra A_2.

If the group is *compact* [SU(3)] then $L \cdot L + Q{:}Q$ is an invariant, so that the matrix elements of Q *necessarily decrease*, and finally cutoff, in any given irrep, as L increases.

If the group is *non-compact*, then L•L − Q:Q is invariant and the matrix elements of Q *increase* without limit as L increases! Since the (mass) quadrupole Q is proportional to the *charge* quadrupole, such an increase [in BE(2) values with L] would be in flat contradiction to experiment.

Note how the Gell-Mann -- Tomonaga prescription neatly avoids the dilemma: the quadrupole group generator Π is *not* Q but rather $\frac{i}{\hbar}[H,Q]$ and hence has for matrix elements:

$$\langle \Pi \rangle \approx (\Delta E) \langle Q \rangle \quad . \tag{3.5}$$

Thus the rise in ⟨Π⟩ with L is opposed by the increase in ΔE with L, and the resulting competition can conform to experiment. (The rigid body limit can be obtained precisely for example.)

Remark 3: It would be reasonable to add to the algebra of $s\ell(3,\mathbb{R})$ generated by {L,Π} the operators Q_μ also, but this *won't work!* [Group theoretically one sees this from the fact that $s\ell(3,\mathbb{R})$ has *no* (non-unitary) five dimensional irrep.] *One must adjoin the element* $Q_0 \equiv \frac{1}{2} \sum_{n,i} \left(x_i^{(n)} \right)^2$, (the trace of Q), *to obtain a closed algebra.* This yields the collective motion group, CM(3), introduced by Cusson.[42]

Since Q_0 is a scalar under rotations, we see that we *are forced purely group theoretically* to generalize[20] from "d-bosons" (Q_μ) by adding monopolar (Q_0) "s-bosons."

Remark 4: The *collective flow* corresponding to Q_0 is radial; the associated generator is the *dilation operator:*

$$\Pi_0 = \sum_{\substack{i=1,2,3 \\ n=1}}^{N} \left(x_i^{(n)} \, p_i^{(n)} + \text{h.c.} \right) \quad . \tag{3.6}$$

Thus, step-by-step, symmetry techniques force one to consider a 15-parameter collective motion group, $\mathbb{R}^6 \circledS GL(3,\mathbb{R})$, as sketched in the following diagram:

$$\begin{pmatrix}\text{quadrupole}\\ \text{moments:}\\ Q_\mu\end{pmatrix} \xrightarrow[\text{Tomonaga}]{\text{Gell-Mann}} \begin{pmatrix}\text{shearing}\\ \text{generators:}\\ \Pi_\mu\end{pmatrix} \xrightarrow[\text{commutation}]{} \begin{pmatrix}s\ell(3,\mathbb{R})\\ \text{algebra}\end{pmatrix}$$

$$\xrightarrow[\text{action on } Q_\mu]{} \begin{pmatrix}Q_0\end{pmatrix} \xrightarrow[\text{Tomonaga}]{\text{Gell-Mann}} \begin{pmatrix}\text{dilation}\\ \text{generator:}\\ \Pi_0\end{pmatrix}$$

The operators $\{Q_0, Q_\mu, \Pi_\mu, \Pi_0\}$ generate $\mathbb{R}^6 \circledS GL(3,\mathbb{R})$.

Remark 5: Let us omit the dilations and consider the 14 generators of $CM(3) \equiv \mathbb{R}^6 \circledS SL(3,\mathbb{R})$. There are two invariants of CM(3): a $(\text{volume})^2 \equiv \Lambda$ and a vortex-spin v. [These invariants are analogs of the $(\text{mass})^2$ and intrinsic spin s of the Poincaré group.]

What is the physical meaning of the vortex-spin?

To answer this, let us note that the $SL(3,\mathbb{R})$ group is realized by the set of all 3×3 real unimodular matrices under matrix multiplication. A generic element M of the group may be written as:

$$M = R\,\Delta\,S \quad , \qquad (3.7)$$

where R and S are real 3×3 rotation matrices ($\tilde{R} = R^{-1}$, $\tilde{S} = S^{-1}$) and Δ is 3×3, diagonal and unimodular. (This form implies that M is unimodular.)

Space-fixed rotations, generated by the angular momentum operators $\underset{\sim}{L}$, correspond to multiplication of M on the left by the matrices R. Body-fixed rotations correspond to multiplication on the right by S; *the vortex-spin operators* $\underset{\sim}{\mathscr{L}}$ *are the generators of these rotations.*

Clearly the transformations generated by $\underset{\sim}{L}$, and by $\underset{\sim}{\mathscr{L}}$, commute; moreover each obeys the commutation relations of an angular momentum (possibly with reversed sign).

Operationally the vortex-spin eigenvalue v is generated (in analogy with the intrinsic spin of the Poincaré group) by rotating to the intrinsic frame, deforming to sphericity, and measuring the angular momentum (v) in the resulting analog to the rest frame.

Remark 6: The smallest simple group containing the group $\mathbb{R}^6 \circledS GL(3,\mathbb{R})$ is the symplectic group $SP(6,\mathbb{R})$ (or better its covering, the metaplectic group) of canonical transformations[43] in three-space. This is the group studied in the collective motion context by Rowe and Rosensteel,[44] by Sternberg,[45] and by Gulshani and Volkow.[46]

IV. RELATIONSHIP BETWEEN THE TWO SYMMETRY APPROACHES

At first glance, it might appear that the two symmetry approaches are completely unrelated, for the symmetry group in the standard approach is compact whereas in the kinematic approach the group is non-compact. Yet on closer analysis, there are some relationships.

The relation between the Bohr Model and the Collective motion group CM(3) has been discussed by Tomonaga[38] (in two dimensions) and by Weaver et al.[20] more generally. This relationship is that of *group contraction*. In the limit in which the shear generators become large, and the operator Q_0 as well, one finds that: (1) Q_0 commutes with everything; and (2) the operators Π_μ and Q_μ/Q_0 become conjugates. Thus we have recovered the known result: *a contraction limit of CM(3) yields quadrupole bosons ("d-bosons"), and their conjugates*, which are the basis of the Bohr model.

What is the relation to the Interacting Boson Model? Here we must obtain some new results.

Let us recall that in CM(3) we were forced to adjoin Q_0 to the five quadrupole operators, Q_μ. Correspondingly the flow associated with the operator Q_0 forced the adjunction of the generator Π_0, the dilation operator (volume changes) so that one obtained the algebra $\mathbb{R}^6 \circledS GL(3,\mathbb{R})$.

It is not quite straightforward to find the contraction limit now. First we must take Π_0 to be large ($\sim\varepsilon^{-1}$). Secondly, (taking a hint from the non-relativistic limit of P_0 in the Poincaré algebra) we must take the limit of Q_0 in the form: $Q_0 = \varepsilon^{-1}\xi_0 + \varepsilon s$, where ξ_0 is a *c-number*, s is an operator, and $\varepsilon \to 0$ in the limit.

The operator Π_0 commutes with the $s\ell(3,\mathbb{R})$ algebra, so these commutation relations are unchanged in the limit. Thus only the commutation relations with the Q_μ, Q_0 are at issue. For these we have:

$$\left[\Pi_0, Q_m\right] = i\hbar\, Q_m \quad , \quad m = 0,1,\ldots,5 \quad . \tag{4.1}$$

Multiplying by ε (for $m = 1,\ldots,5$) and by 1 (for $m = 0$) we obtain in the limit:

$$\left[\Pi_0, Q_\mu\right] = 0 \quad , \tag{4.2}$$

$$\left[\Pi_0 , s\right] = i\hbar\, \xi_0 \quad . \tag{4.3}$$

(It is crucial to note the $[\Pi_0, \varepsilon^{-1}\xi_0] = 0$ for ξ_0 a c-number.)

Thus we obtain, in the limit, *six* bosons and their conjugates. (Π_μ and Q_μ/ξ_0) -- d-bosons and (Π_0 and s/ξ_0) -- the s-boson.

We conclude: *a contraction limit of the collective motion group with dilations* ($\mathbb{R}^6 \circledS GL^+(3,\mathbb{R})$) *yields precisely the s and d boson operators of the IBM.*

This is an interesting -- even if not unexpected! -- result for it shows that the interacting boson model *simply relaxes the incompressibility condition* in the original Bohr-Mottelson treatment of collective nuclear flow.

Let us conclude this section with two remarks:

Remark 1: The symmetry group of the Bohr-Mottelson unified model (rigid rotation, adiabatic model) is a subgroup of the collective motion group $CM(3) = \mathbb{R}^6 \circledS SL(3,\mathbb{R})$, that is, to say, the group $\mathbb{R}^5 \circledS SU(2)$ is contained as a subgroup. (The adjunction or not of dilations is of no concern for the rigid body limit.)

Remark 2: There is a quite different way to find a relationship between the two symmetry approaches, this time in terms of the spectrum. The condition that (in the kinematic approach) the Hamiltonian *split but not mix* the states means, in effect, that both approaches deal with the same set of states but organize the spectrum differently. Consider SU(3) symmetry vs the kinematic $SL(3,\mathbb{R})$ symmetry. Both symmetries deal with the same abstract set of angular momentum states: in the harmonic oscillator SU(3) shell model these are the familiar $SU(3) \supset SO(3)$ states for [n 0 0]. $SL(3,\mathbb{R})$ acting on these same states, organizes them "vertically": all $L=0$ states are made into coherent states, similarly for $L=2$, $L=4,\ldots$. This yields an $SL(3,\mathbb{R})$ irrep: the band 0,2,4,... with a continuous quadrupole (invariant labeling) parameter. The odd angular momenta became the irrep: 1,3,5,... . This relationship between Elliott SU(3) and kinematic $SL(3,\mathbb{R})$ shows how the interacting boson model is to be related to the kinematic symmetry approach of $Sp(6,\mathbb{R})$: both approaches are based on the harmonic oscillator shell model states, the IBM using compact SU(6) symmetry, while the kinematic approach embeds $Sp(6,\mathbb{R})$ as a subgroup in the non-compact SU(3,3) group. Note that the covering group is *spinorial* in the latter approach so that half-integer excitations are obtainable as well.

V. THE DIRECT APPROACH TO COLLECTIVE MOTION

The direct approach to nuclear collective motion attempts to introduce collective coordinates into the nuclear Hamiltonian via a (possibly implicit) coordinate transformation. The prototype for this is the transformation to center-of-mass coordinates.

A very elegant realization of this approach was developed by Brian Buck in the early 70's, but was not published[§] until 1979. We will sketch these developments in order to show, first, how nicely they accord with the symmetry approaches of Sections II and III and, second, how the concept of vortex-spin clarifies the problem of the moment of inertia.

The key to Buck's development is to regard the coordinates of N particles in three-space as a *rectangular* $3 \times N$ *matrix:* $M = (M_{i\mu}) = (r_i^{(\mu)})$ that is, the matrix element $(M_{i\mu})$ is the i^{th} coordinate of the particle μ. Such a matrix allows one to define *two "quadrupoles":* (a) $Q \equiv M \tilde{M}$ and (b) $\mathcal{Q} \equiv \tilde{M} M$.

The 3×3 matrix Q is precisely the usual quadrupole array (whose elements are sums over the N particles) with tr $Q = Q_0$ as defined earlier.

The $N \times N$ matrix $\mathcal{Q}$ is a "quadrupole" matrix in "particle label space." (The matrix elements of $\mathcal{Q}$ are sums over the three spatial coordinates.)

As real, symmetric, matrices both Q and $\mathcal{Q}$ can be brought to diagonal form by a real similarity transformation. The three eigenvalues of Q are just the three quadrupole moments $\{\lambda_\alpha\}$ defined in the intrinsic frame. The eigenvalues of $\mathcal{Q}$ are surprisingly simple! *They are just the three eigenvalues of Q with all other eigenvalues zero.* We orient particle-

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§ The vortex-spin concept was developed and added in the interim.

label space so that the three non-zero eigenvalue axes coincide with the three intrinsic axes of Q.

The new coordinates are now seen to be:

(a) The three eigen-moments, $\{\lambda_\alpha\}$. It is convenient to use $+(\lambda_\alpha)^{1/2} \equiv \mu_\alpha$ as the actual variables.

(b) The three Euler angles defined by the rotation into the body-fixed (intrinsic) frame of Q. The generators for this rotation are $\underset{\sim}{L}$.

(c) The $3(N-2)$ angles that specify the coordinates of particle label space relative to the intrinsic frame.

The six coordinates, (a) and (b), are collective; the former explicit, the latter (Euler angles) implicit.

At this point, we note that we have neglected the three center-of-mass collective coordinates. These are easily taken into account by using *relative vectors* $(\underset{\sim}{r}^{(n)} - \underset{\sim}{R}_{CM})$ in the matrix M. This replaces (c) by $3(N-3)$ angles.

The $3(N-3)$ internal angular coordinates correspond to rotation of the N-1 dimensional label space (one dimension is removed by the center-of-mass collective coordinates) relative to the three orthonormal vectors defining the intrinsic frame. Thus we have $[(N-1)(N-2)]/2$ angles specifying a general orientation of label space from which we subtract $[(N-4)(N-5)]/2$ angles corresponding to the irrelevant orientation of the (N-4) dimensions defined by null eigenvalues of $\mathcal{Q}$. This yields $3(N-3)$ angles.

Group-theoretically this structure is that of a coset space of the rotation group SO(N-1) with respect to the subgroup SO(N-4), that is, SO(N-1)/SO(N-4). Motion in this space is generated by the $3(N-3)$ operators:

$$\mathscr{L}_{\gamma} = \mathscr{L}_{\alpha\beta} = -\mathscr{L}_{\beta\alpha} \quad , \quad (\alpha,\beta,\gamma = 1,2,3 \text{ and cyclic}) \tag{5.1}$$

$$\mathscr{L}_{\alpha\kappa} = -\mathscr{L}_{\kappa\alpha} \quad , \quad (\alpha = 1,2,3 \;;\; \kappa = 4,5,\ldots,N-1). \tag{5.2}$$

[Acting on the coset space the remaining operators $\mathscr{L}_{\kappa\kappa'}$ of SO(N-1) vanish.]

The three operators $\mathscr{L}_{\gamma}$ are distinguished since they generate rotations of the three dimensions singled out by non-vanishing eigenvalues of $\mathscr{Q}$.

A surprising, and important, result[47] *is that the three operators* $\mathscr{L}_{\gamma}$ *are precisely the vortex-spin operators found in the kinematic symmetry approach to collective motion in Section III.*

In order to see the importance of this result let us record the form of the classical Hamiltonian expressed in terms of the new variables:

$$\begin{aligned} H = {} & \frac{P^2}{2MN} + \sum_{\alpha} \frac{p_{\alpha}^2}{2M} + \sum_{\alpha,\kappa} \frac{\mathscr{L}_{\alpha\kappa}^2}{2M\lambda_{\alpha}} \\ & + \sum_{\alpha<\beta} \frac{\lambda_{\alpha}+\lambda_{\beta}}{2M(\lambda_{\alpha}-\lambda_{\beta})^2} L_{\alpha\beta}^2 + \sum_{\alpha<\beta} \frac{\lambda_{\alpha}+\lambda_{\beta}}{2M(\lambda_{\alpha}-\lambda_{\beta})^2} \mathscr{L}_{\alpha\beta}^2 \\ & + \sum_{\alpha<\beta} \frac{4\mu_{\alpha}\mu_{\beta}}{2M(\lambda_{\alpha}-\lambda_{\beta})^2} L_{\alpha\beta}\mathscr{L}_{\alpha\beta} + V(\xi) \quad . \end{aligned} \tag{5.3}$$

[Here $\underset{\sim}{P}$ is the C.M. momentum operator, $\underset{\sim}{p}_{\alpha}$ the conjugate operator to μ_{α} and $V(\xi)$ denotes the potential expressed in terms of the new coordinates.] The quantal Hamiltonian corresponding to Eq. (5.3) is given in Ref. 47.

One notes that only the vortex-spin operators $\{\mathscr{L}_{\gamma}\}$ are coupled via the Hamiltonian to the angular momentum, $\{L_{\gamma}\}$. This fact (and the vortex-spin itself) are crucial[47] to the "moment of inertia problem": *If* $\mathscr{L}_{\gamma} \rightarrow 0$

(this is a condition of the space of states) *then the moments of inertia take on the "liquid" value:*

$$I_\gamma^{liq} = \frac{M(\lambda_\alpha - \lambda_\beta)^2}{\lambda_\alpha + \lambda_\beta} \quad , \quad (\alpha\beta\gamma \text{ cyclic}) \quad . \qquad (5.4)$$

This condition is responsible for the liquid moments of inertia that necessarily arise in the Bohr model.

By contrast the rigid body moments arise[47] if the velocities conjugate to the $\{\mathscr{L}_\gamma\}$ are set to zero: $(\dot{\theta}_\gamma = 0) \Longrightarrow I = I_\gamma^{rigid} = M(\lambda_\alpha + \lambda_\beta)$, ($\alpha\beta\gamma$ cyclic). Setting velocities to zero is a *dynamical condition*, and one sees that the moment of inertia problem *cannot be resolved without an understanding of the nuclear potential.* (Indeed it is an empirical fact that atoms do not possess rotational spectra whereas many nuclei do: the long-range character of the Coulomb interaction accounts for this difference.)

The *explicit* introduction of collective coordinates, in terms of which the work of Buck et al. is just the beginning, is an important task to which the UNAM group is now making contributions.[48]

The angular momentum operators, $\{L_\gamma\}$, and the vortex-spin operators, $\{\mathscr{L}_\gamma\}$, are common both to the kinematic symmetry approach and the direct approach, and, as mentioned, are the key to the problem of moments of inertia. Let us discuss these operators further, especially since the vortex-spin operator has recently been re-investigated critically by Louck.[49]

The commutation relations obeyed by these operators are:

$$\left[L_\alpha , L_\beta\right] = -i\, \varepsilon_{\alpha\beta\gamma}\, L_\gamma \quad , \quad (\alpha\beta\gamma = 123 \text{ cyclic}) \tag{5.5}$$

$$\left[L_\alpha , \mathscr{L}_\beta\right] = 0 \quad , \tag{5.6}$$

$$\left[\mathscr{L}_\alpha , \mathscr{L}_\beta\right] = -i\, \varepsilon_{\alpha\beta\gamma}\, \mathscr{L}_\gamma \quad , \quad (\alpha\beta\gamma = 123 \text{ cyclic}) \quad . \tag{5.7}$$

(The minus sign results from the fact that these operators are referred to the intrinsic frame, by projection with the $\hat{S}_\alpha$ unit vectors of the intrinsic frame.)

A local definition of the angular momentum can be given in the form:

$$L_\gamma = L_{\alpha\beta} = \sum_{n=1}^{N} \left[\left(\hat{S}_\alpha \cdot \underset{\sim}{r}^{(n)}\right)\left(\hat{S}_\beta \cdot \underset{\sim}{p}^{(n)}\right) - \left(\hat{S}_\beta \cdot \underset{\sim}{r}^{(n)}\right)\left(\hat{S}_\alpha \cdot \underset{\sim}{p}^{(n)}\right)\right] . \tag{5.8}$$

A rather similar appearing form for the vortex-spin operators can also be given:

$$\mathscr{L}_\gamma = \mathscr{L}_{\alpha\beta} = -\sum_{n=1}^{N} \left[\left(\frac{\mu_\beta}{\mu_\alpha}\right)\left(\hat{S}_\alpha \cdot \underset{\sim}{r}^{(n)}\right)\left(\hat{S}_\beta \cdot \underset{\sim}{p}^{(n)}\right) - \left(\frac{\mu_\alpha}{\mu_\beta}\right)\left(\hat{S}_\beta \cdot \underset{\sim}{r}^{(n)}\right)\left(\hat{S}_\alpha \cdot \underset{\sim}{p}^{(n)}\right)\right] . \tag{5.9}$$

This form for the vortex-spin shows two important features:

(1) The vortex-spin is a *non-local quantity* (since the μ_α's depend on the instantaneous positions of all particles).

(2) For a system classically constrained to have $\mu_\alpha = \mu_\beta$ the vortex-spin $\mathscr{L}_{\alpha\beta}$ becomes numerically $-L_{\alpha\beta}$. The existence of a distinct vortex-spin operator is thus intimately connected with deformations. Quantum-mechanically, because of fluctuations, μ_α never equals μ_β; vortex-spin is always distinct from angular momentum.

Louck[49] has recently criticized the commutation relation, Eq. (5.7), for vortex-spin; he finds the right hand side to be multiplied by: $f_\alpha f_\beta / f_\gamma$, where $f_\alpha = (\lambda_\beta + \lambda_\alpha)/2\mu_\beta\mu_\gamma$.

One can verify, however, from the $SL(3,\mathbb{R})$ realization of the vortex-spin -- as given by Eq. (3.7) -- that Eq. (5.7) is correct. What goes wrong is apparently that the internal angular momentum calculated by Louck is *not* the vortex-spin; the internal angular momentum defined "*in* the intrinsic frame" (as opposed to "*referred to*" the intrinsic frame) has complicated commutation relations, and no relation to the vortex-spin.

Let us make one remark on the moment of inertia problem. The difficulty, as discussed by Buck, is not only dynamical, but closely related to implementing the Pauli principle. Recently Robson[50] has shed new light on the problem, for nuclei, by explicitly introducing quark degrees of freedom, which make the nucleus look far more like a rigid body rotationally. It is interesting to note that not only does the bag model of individual hadrons clearly involve $SL(3,\mathbb{R})$ degrees of freedom, but the bag model applied to *nuclei* suggests a sort of "pomegranate" structure (of many deformed bags with domain-like walls) which might verify Robson's concept of a tetrahedrally deformed alpha particle sub-structure as important in nuclei!

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