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A FINITE, COVARIANT, AND UNITARY EQUATION FOR SINGLE QUANTUM EXCHANGE AND PRODUCTION*

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ABSTRACT

By requiring the mass of the "bound state" of particle and quantum to have the mass of the particle we derive a fully covariant theory of single quantum exchange and production with physical unitarity, the correct nonrelativistic limit of Yukawa or Coulomb "potential" scattering, and scalar "Thompson scattering."

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It has been shown by one of us (JVL)[1] that the three particle zero range scattering theory, first developed in a nonrelativistic context,[2] using relativistic kinematics and an appropriate simple unitary and covariant two particle scattering amplitude as the driving term allows the calculation of unitary three particle amplitudes for elastic scattering, rearrangement, breakup, coalescence and 3-3 scattering. For three equal mass particles it has been shown[1] that these equations predict an infinite logarithmic accumulation of three particle bound states in the nonrelativistic kinematic region in quantitative agreement with the Efimov effect[3] calculated using a separable nonrelativistic Hamiltonian.[4] The theory therefore is proved to have the correct limit in nonrelativistic scattering theory in this case. In this communication we prove that by appropriately defining one of the three particles as a "quantum" we have a covariant model which predicts unitary scattering amplitudes for two particles due to single quantum exchange and single quantum production in a two particle system, with physical flux conservation. In the nonrelativistic two particle elastic channel we recover Yukawa or Coulomb "potential" scattering. The two quantum-one particle sector is also correctly described.

In our zero range scattering theory starting from three free particles with masses m_a, m_b and m_c and momenta $\underline{p}_a^{(0)}$, $\underline{p}_b^{(0)}$ and $\underline{p}_c^{(0)}$ in the three particle zero 3-momentum system, the physical three particle on-shell amplitude M_{ab} has the general form

$$M_{ab}\left(\underline{p}_a,\underline{p}_b^{(0)};M\right)-\delta_{ab}\tau_a\left(s_a\right)\delta^3\left(\underline{p}_a-\underline{p}_b^{(0)}\right)\sqrt{p_a^2+m_a^2}=\tau_a\left(s_a\right)Z_{ab}\left(p_a,p_b^{(0)};M\right)\tau_b\left(s_b\right) \tag{1}$$

where M^2 is the invariant four-momentum squared and s_a is the invariant two particle four momentum squared. For our model we use the covariant generalization of the nonrelativistic scattering length model $q\,\mathrm{ctn}\delta=-1/a$, which is

$$\tau_a\left(s_a\right)=\frac{\sqrt{s_a}\;\theta\left(s_a\right)}{(2\pi)^2\left(\sqrt{-\left|q_a^2\left(\mu_a\right)\right|}-\sqrt{-\left|q_a^2\left(s_a\right)\right|}\right)}\equiv\frac{g_a^{(0)2}}{s_a-\mu_a^2-i0^+}+\hat{\tau}\left(s_a\right) \tag{2}$$

with

$$q_a^2\left(s_a\right)=\frac{\left[s_a-\left(m_{a+}+m_{a-}\right)^2\right]\left[s_a-\left(m_{a+}-m_{a-}\right)^2\right]}{4s_a}\quad. \tag{3}$$

The Faddeev equation for M_{ab} is given diagramattically in Fig. 1a. Thanks to Eq. (1), all the physical amplitudes can be recovered by solving the single set of coupled equations

$$\begin{aligned} Z_{ab}&\left(\underline{p}_a,\underline{p}_b;M\right)+\bar{R}_{ab}\left(\underline{p}_a,\underline{p}_b^{(0)};M\right) \\ &=-\sum_{c=a\pm}\int_0^{\left(M^2-m_c^2\right)/2M}\frac{d^3p_c}{\varepsilon_c}\;\bar{R}_{ac}\left(\underline{p}_a,\underline{p}_c;M\right)\,\tau_c\left(s_c\right)\,Z_{cb}\left(\underline{p}_c,\underline{p}_b;M\right) \\ &=-\sum_{c=a\pm}\int_0^{\left(M^2-m_c^2\right)/2m}\frac{d^3p_c}{\varepsilon_c}\;Z_{ac}\left(\underline{p}_a,\underline{p}_c;M\right)\,\tau_c\left(s_c\right)\,\bar{R}_{cb}\left(\underline{p}_c,\underline{p}_b;M\right) \end{aligned} \tag{4}$$

where

$$\begin{gathered}\bar{R}_{ab}=\left(1-\delta_{ab}\right)\left(\varepsilon_a+\varepsilon_b+\varepsilon_{ab}-M-i0^+\right)^{-1}\;\varepsilon_{ab}^{-1} \\ \varepsilon_a=\sqrt{m_a^2+p_a^2}\qquad\varepsilon_b=\sqrt{m_b^2+p_b^2}\qquad\varepsilon_{ab}=\sqrt{m_c^2+\left(\underline{p}_a+\underline{p}_b\right)^2}\end{gathered} \tag{5}$$

and the equivalence of the two forms of the equation for Z_{ab} can be proved by iteration. This immediately establishes the time reversal invariance of the theory. Unitarity has been proved in Ref. 1, and for essentially the same model by Brayshaw.[5] It also follows from the algebraic form of the equation for M_{ab} and two particle unitarity, as has been proved by Freedman, Lovelace and Namyslowski.[6]

To isolate the 2,2 + 2,3 coupled channels in which we are interested, we follow the discussion of primary singularities in M_{ab} given by Osborn and Bollé.[7] Calling their amplitudes $\mathcal{K}_{ab}$, $\mathcal{G}_{ab}$ and $\mathcal{B}_{0b}$ in the zero range limit K_{ab}, G_{ab} and B_{0b} respectively, we find by comparison of OB Eqs. (IV.7) and (IV.8) with our Eqs. (1) and (2) that the physical elastic scattering and rearrangement amplitude $K_{ab} = g_a Z_{ab} g_b$, that $G_{ab} = \hat{\tau}_a Z_{ab} g_b$ and that the physical breakup amplitude is given by

$$B_{0b}\left(\underline{p}_a, \underline{q}_a; p_b^{(0)}; M\right) = \sum_a \left[\hat{\tau}_a\left(s_a\right) Z_{ab}\left(p_a, p_b^{(0)}; M\right) g_b + \frac{g_a^2 Z_{ab}\, g_b}{s_a - \mu_a^2}\right] \qquad (6)$$

With an obvious generalization to relativistic kinematics, the cross sections for elastic scattering, rearrangement and breakup follow from OB Eqs. (III.8), (III.15) and the total cross sections from (IV.37) and (IV.39).

In the nonrelativistic case when $\tau_a = -N_a^2(\tilde{p}_a^2 - \varepsilon_a - W)^{-1} + \hat{\tau}_a$ with ε_a the binding energy of m_b and m_c and W the three particle energy normalized to zero at breakup threshold, the asymptotic form of the bound state wave function is $N_a \exp\{-\sqrt{2\mu_a\varepsilon_a}\, y/ y$. Taking $N_a^0 = 2(2\mu_a\varepsilon_a)^{\frac{1}{2}}$ corresponds to aassuming that the asymptotic form holds for all y greater than zero (our zero range assumption) and to assuming that the bound state contains exactly two particles. However, as is well understood in the nonrelativistic case, if the bound state has internal structure, N_a will not have that value, but will instead be a measure of what fraction of that composite structure will break up into the bound state and whatever part of the remaining three-body system carries off the momentum required by momentum conservation. N_a can either be computed from a microscopic theory or determined from experiments which contain the bound state asymptotically, for example by extrapolating observed cross sections to the bound state pole; in that context N_a^2 is called the "reduced width". Thus for phenomenological purposes N_a becomes an empirical parameter and it is a task for data analysis and microscopic theory to prove that it is a constant independent of the reaction mechanism in which the bound state is produced. In a context close to ours, Aaron, Amado and Yam[8] have exploited this freedom in Amado's "non-relativistic field theory" for n-d scattering, as have Barton and Phillips[9] in their dispersion theoretic approach to the same problem. In that context N_a measures how much of the deuteron is a composite of neutron and proton, and how much is "elementary". Thus, if we do not restrict ourselves to the value $g_a^{(0)2}$ defined by Eq. (2) and replace it by a parameter g_a^2 to be determined by experiment, we claim we are not departing from accepted practice. For us $f_a^2 = \left[g_a^{(0)2} - g_a^2\right]/g_a^{(0)2}$ tells us what fraction of the physical particle m_a is a composite of $m_a + m_Q$, while $1 - f_a^2$ tells us what fraction of the physical particle is "bare" in our two-particle one-quantum sector. In terms of the density matrix, the parameter f_a^2 measures the incoherent mixing of two pure states in different spaces which construct a mixed physical state of mass m_a.

Once this is understood we can specialize the general treatment to the case of most immediate interest in establishing a covariant theory with the proper nonrelativistic limit. We assume that channels m_a and m_b represent particles and that channel m_c is a quantum of mass m_Q. For the initial state, if the spectator is m_b we assume the other system is a "bound state" of $m_a + m_Q$ with mass $\mu_b \equiv m_a$, and visa versa. We further assume that there is no direct particle-particle scattering, since all such scatterings are to be generated by single quantum exchange; this is easily accomplished by taking $\tau_c \equiv 0$. The resulting coupled equations, illustrated diagramatically in Fig. 1b, are then

$$K_{ab}\left(\underline{p}_a,\underline{p}_b^{(0)};M\right) = -g_a R g_b - \int \frac{d^3p_b}{\varepsilon_b}\left[\frac{g_a R g_b K_{bb}}{s_b - m_a^2 - i0^+} + g_a R G_{bb}\right]$$

$$K_{bb}\left(\underline{p}_a,\underline{p}_b^{(0)};M\right) = -\int \frac{d^3p_a}{\varepsilon_a}\left[\frac{g_b R g_a K_{ab}}{s_a - m_b^2 - i0^+} + g_b R G_{ab}\right] \qquad (7)$$

$$a,b = 1,2 \; ; \qquad R_{ab} = R_{ba} = R \quad ,$$

where the corresponding equations for G_{ab} and G_{bb} are obtained by replacing g_a by $\hat{\tau}_a$ in the first equation and g_b by $\hat{\tau}_b$ in the second. These are our proposed covariant equations for the scattering of two scalar particles of mass m_a and m_b generated by the exchange of a single scalar quantum coupled to the production of that quantum starting from the same system.

If we close the production channel by taking $\hat{\tau}_a = 0 = \hat{\tau}_b$ and recall that we are in the zero momentum system so that on shell $\underline{p}_a = -\underline{p}_b$ the equations (with G = 0) describe elastic scattering with the on-shell amplitude $T(\underline{p}_a,\underline{p}_a^{(0)};M) = K_{ab}(\underline{p}_a,\underline{p}_a^{(0)};M) + K_{bb}(-\underline{p}_a,-\underline{p}_a^{(0)};M)$.

Further, since $s_a = (M-\varepsilon_a')^2 - (k_a')^2$ and $M = \varepsilon_a^{(0)} + \varepsilon_b^{(0)}$ we find that $s_a - m_b^2 = 2M[\varepsilon_a^{(0)} - \varepsilon_a']$ and the equations become a two-channel covariant equation of the Lippmann-Schwinger type due to single quantum exchange. For equal masses they reduce to a single equation for T, but for unequal masses we must retain the coupled form in order to have the proper relativistic kinematics for "recoil". Our equations differ from the Bethe-Salpeter equation[10] in the ladder approximation in that they are on-shell or single-time equations. They differ from the Blankenbecler-Sugar[11] reduction of the Bethe-Salpeter equation in that they have no spurious singularities. In the nonrelativistic limit the "propagators" become $[k^{(0)2} - k'^2 - i0^+] = -G_0$ and $T = -V + VG_0T$ with $V = g_a R\, g_b$. If we make the "adiabatic approximation"[12] in R in which the energies of the particles are assumed large compared to the energy of the exchanged quantum, V corresponds to the exchange of a virtual quantum with energy $[m_Q^2 + (\underline{k} - \underline{k}')^2]^{\frac{1}{2}}$ and in the nonrelativistic limit we have the conventional equation for the scattering due to a "Yukawa potential". There is no singularity in the potential for $m_Q \neq 0$ thanks to our assumption that $m_a = \mu_b$ and $m_b = \mu_a$, but for $m_Q = 0$ and $g^2 = e^2$ we develop the usual Coulomb singularity. Thus we predict at low energy Rutherford scattering, and for charges of opposite sign, the Bohr bound state spectrum, in agreement with experiment to the accuracy expected. Restoring the production channel and using the covariant equation we can predict elastic scattering coupled to the production of a single scalar quantum at any energy.

The theory can be extended to the two quantum-one particle sector simply by taking $m_a = m_Q = m_b$, $m_c = m$, and $\mu_a = m = \mu_b$. Covariance and unitarity are preserved. This allows us to discuss quantum-particle scattering. To order g^2 we obtain the two diagrams of Fig. 1c, and hence go to the correct "Thompson limit" for our scalar theory.[13]

Since our driving terms now have the same form as the lowest order perturbation diagrams of quantum field theory (of course only if we drop $\hat{\tau}$), the inclusion of spin is straightforward, if tedious. If only one of the particles has spin we obtain a relativistic generalization of the Dirac equation, but with full covariant kinematics; the conventional one-particle Dirac equation is recovered in an appropriate limit. So we obtain fine structure splitting and the magnetic moment of the electron to lowest order. Introducing a vector quantum and two spinor particles or two vector quanta and one spinor particle is, we believe, equally straightforward. We will present details on another occasion. To "cross" our theory and obtain a unitary equation for particle-antiparticle annihilation to two quanta might take us into a four-particle sector of the theory. Since the four particle version of the zero range theory already exists in the nonrelativistic case,[14] we are confident that this can be accomplished. The real test of the theory will be whether this allows us to predict the Lamb shift in a finite theory without renormalization.

Once we have included spinors and vector and pseudoscalar quanta, we can obviously construct a covariant "one boson exchange" model for nuclear forces and single meson production in nucleon-nucleon collisions. We can also explore the $q\bar{q}$ spectrum of quantum chromodynamics in the quark-antiquark-gluon approximation. Another obvious application is to problems in which Ze^2 is less than 1 which, so far as we can see, is already included in what we have achieved. And so on.

It remains to ask why this superficially simple route to an elementary particle theory with full unitarity in sectors restricted to finite particle and quantum number was not developed long ago. We believe part of the answer can be seen by examing Eq. (7). There we see that what plays the role of the "potential" in the equation is the relativistic propagator of the virtual state, while what is normally called the "propagator" or "Green's Function" in the nonrelativistic limit comes from the intermediate state in the two particle scattering amplitude, which has a pole when this intermediate state is "on-shell". It was natural enough in a Hamiltonian field theory to identify the "potential" with the "matrix element of an interaction", but this breaks the connection with the particles and distributes that energy all over space. Thanks to "renormalization" and herculean efforts an enormous amount of correct physics has been obtained from that approach, but contact with nonrelativistic quantum mechanics became extremely difficult. By insisting on using physical asymptotic free particle wave functions (covariantly normalized) rather than field functions as our basis we have avoided all these difficulties and recover the physically correct nonrelativistic limit without difficulty. This possibility was suggested by Serber long ago,[15] but so far as we know his suggestion has never previously been successfully implemented. Another part of the answer was the realization by one of us[16] that the Faddeev summation convention in the multiple scattering series automatically excludes "self energy diagrams" while guaranteeing exact unitarity in a finite particle sector. It was faith in this insight that kept the program going in spite of many setbacks.

In conclusion we emphasize that the coupling constant e^2 which we introduce in the Coulomb example is the physical coupling constant as determined by the Bohr spectrum and Rutherford scattering, subject only to obvious small and finite corrections due to fine structure splitting and the like. Similarly the g^2 of a meson theory is physical, and not in principle defined perturbatively. The masses of stable particles are physical and have no renormalization — though again precise values obtained from experiment for these and for unstable masses will have finite corrections due to multiparticle sectors when it comes to comparing theory and experiment. The real test of this theory will come when we go into the two-quantum — two-particle sector and attempt to obtain the e^4 results of renormalized perturbation theory.

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References

1. James V. Lindesay, Ph.D. thesis, Stanford University, 1981, available as SLAC Report 243.
2. H. Pierre Noyes, "Zero Range Scattering Theory. I. Three and Four Particle Equations," SLAC-PUB-2866 (December 1981) to be submitted to Physical Review C.
3. V. Efimov, Phys. Letters 33B, 563 (1970); Nuclear Phys. 12, 589 (1973); V. N. Efimov, Sov. J. Nucl. Phys., 12, 589 (1971).
4. A. T. Stelbovics and L. R. Dodd, Phys. Letters 39B, 450 (1972); if in the nonrelativistic theory with qctn $\delta = -1/a + \frac{1}{2} r q^2$ we let $1/a \to 0$ and $\frac{1}{2} r = \hbar/mc$ comparison is possible for zero three particle binding energy.
5. D. D. Brayshaw, Phys. Rev. D18, 2638 (1978).
6. D. Z. Freedman, C. Lovelace and J. M. Namyslowski, Nuovo Cimento 43A, 258 (1966).
7. T. A. Osborn and D. Bollé, Phys. Rev. C8, 1198 (1973), hereinafter OB.
8. R. Aaron, R. D. Amado and Y. Y. Yam, Phys. Rev. 140, B1291 (1965).
9. G. Barton and A. C. Phillips, Nucl. Phys. A132, 97 (1969).
10. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951).
11. R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
12. One of a host of approximations tried in the attempt to extract a nonrelativistic "potential" from quantum field theory reviewed by M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nucl. Sci. 11, 95 (1961).
13. This necessity has often been pointed out to HPN by S. D. Drell.

14. HNP is indebted to V. Vanzani for showing that the equation developed in Ref. 2 is formally equivalent to the on-shell limit of one of the four body equations discussed by V. Vanzani in Nuovo Cimento 2A, 521 (1971).

15. R. Serber, lectures on field theory, University of California at Berkeley, 1950.

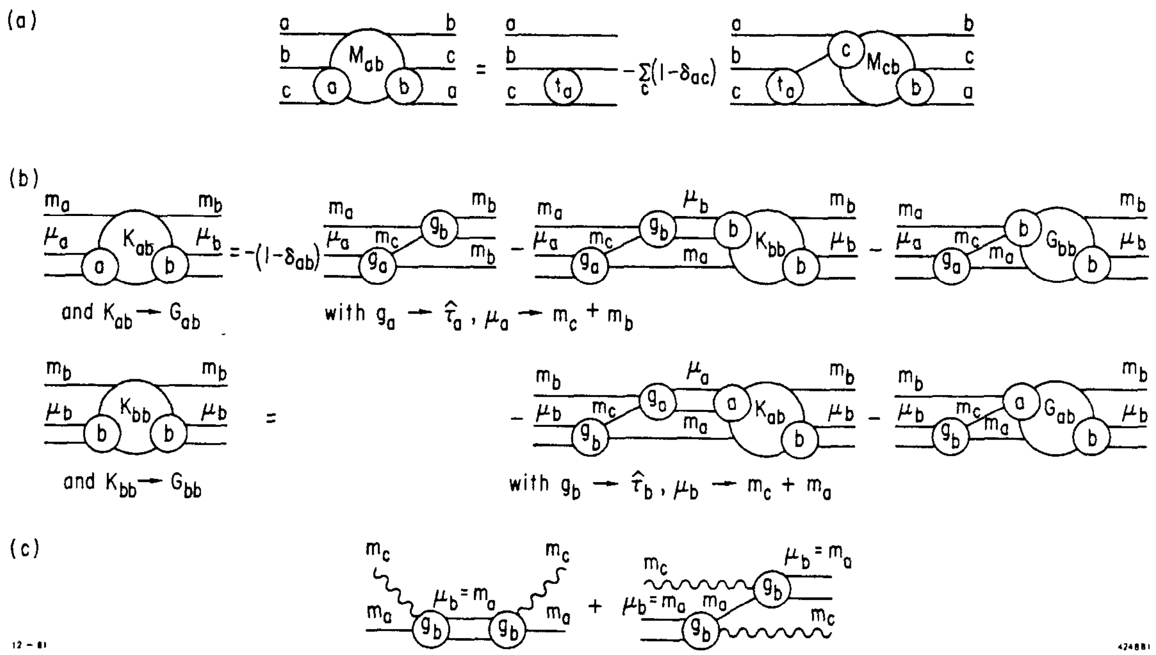
16. H. Pierre Noyes, "Three Body Forces," in Few Particles Problems, I. Slaus *et al*., eds., North Holland, Amsterdam-London-New York, 1972, p. 122; Foundations of Physics, 5, 37 (1975) [Erratum 6, 125 (1976)] and 6, 83 (1976).

Figure Caption

1a. Diagrammatic representation of the Faddeev equation.

1b. The particle-particle-quantum sector when there is no direct particle-particle scattering ($\tau_c \equiv 0$).

1c. The particle-quantum scattering to order g^2 (Thompson limit).



(a)
(b)
and $K_{ab} \to G_{ab}$
with $g_a \to \hat{\tau}_a$, $\mu_a \to m_c + m_b$
and $K_{bb} \to G_{bb}$
with $g_b \to \hat{\tau}_b$, $\mu_b \to m_c + m_a$
(c)
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Fig. 1