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REAL-SPACE RENORMALIZATION GROUP: AN ADIABATIC METHOD OF THINNING HIGH-FREQUENCY MODES*

Shahar Ben-Menahem
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

ABSTRACT

A new adiabatic method for improving iterative blocking-truncation approximations in lattice models is presented, and applied to a test model. In this approach, the high-frequency (fast) modes are frozen into wave functions that change with the slow modes.

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1. INTRODUCTION

The method of lattice Hamiltonian blocking, or the real space renormalization group, has been successful in a variety of models, yielding results for ground-state energies, mass gaps, and correlation functions. When applied to soluble 1 + 1 dimensional models, where it can be compared with exact solutions, it succeeds in giving fairly accurate results. Improving upon these results, however, usually necessitates complicated refinements of the procedure, and even so the spatial structure of correlation functions computed in such schemes is never very accurate.

Stated simply, a blocking scheme is a systematic way to compute scale transformations in a lattice model, from one ultraviolet cutoff Λ_1 to a smaller one, Λ_2. This is done by solving approximately for those modes that have frequencies $\Lambda_2 < \omega < \Lambda_1$, and removing them from the theory.

In this paper, we report a new method of treating these higher-frequency modes using an adiabatic (Born-Oppenheimer) approximation. The usefulness of the method is explained in general, and demonstrated in the framework of a very simple model: a free Bose scalar field theory in one space dimension. We examine in detail several schemes employing the method. For the most part, we concentrate on the massless case, which is the hardest to attack by blocking techniques due to its long-range correlations. The results of the adiabatic approach are compared with those of conventional blocking schemes.

For each scheme we treat, the ground-state energy and various equal-time Green's functions are computed for the trial ground-state.[1] We find excellent agreement with the exact results; using previous methods, one must keep many more states per site and sites per block to attain accuracies that we can get using our adiabatic approach.

Our adiabatic method is closely related to, and grew out of, the recently developed Shadow Hamiltonian technique by Quinn and Weinstein[2] (QW). It is, however, more suited for treating Bose field theories. The QW technique was developed for a spin model, where one cannot define classically meaningful "fast" and "slow" modes. The distinction between these modes is, however, the cornerstone of our method. Both techniques can be applied to more complicated models than those they were tested for.

The plan of this paper is as follows: in Section 2 the adiabatic method is explained. In Section 3 our model is introduced and solved and its relevant symmetries pointed out. In Section 4 we apply the new method to the model, keeping one "slow" variable per two-site block. In Section 5 we apply it keeping two variables, again with two-site blocks. In Section 6 we summarize our conclusions. We find it useful to work in the momentum basis, due to long-range effects.

2. ITERATIVE BLOCKING-TRUNCATION METHODS

In a blocking scheme, one begins from a theory defined on a spatial lattice, and groups the lattice into blocks of two or more sites each. One then truncates away some block states or degrees of freedom. The choice of these states is based on energy: one would like to reject the (in some sense) "higher" excited states. The assumption underlying the truncation procedure is that the higher excitations will not affect the ground state by much.

Let us consider a Bose field theory. One can talk of field variables rather than states, and the problem of choosing low-lying block states becomes that of identifying "slow" and "fast" (low- and high-frequency) variables.

In the "simple-blocking" schemes,[3] the Hamiltonian is split into terms involving a single block and terms coupling the blocks. This decomposition is obviously not unique, creating a certain ambiguity in the procedure. This ambiguity will disappear when we introduce our improved blocking method; but let us proceed with the description of the conventional method. The single-block Hamiltonian is solved, exactly or approximately, and the block modes are identified. One defines "slow" and "fast" modes; the latter are frozen in their approximate ground state. All operators are next truncated by computing their expectation values with respect to these fast modes. These modes thus drop out of the theory, leaving an effective Hamiltonian in the remaining variables. This Hamiltonian is defined on a new lattice, with the blocks serving as the new sites. The new theory has a larger lattice spacing, and correspondingly a smaller ultraviolet cutoff, than the old one. Hence the theory undergoes a scale transformation. This is called a renormalization-group transformation.[4]

By truncating operators, we may express their trial ground-state expectation values, or Green's functions, in terms of their counterparts in the rescaled theory. We refer to these relations as "mappings," or renormalization group equations. The ground-state energy, for example, is computed by repeated mappings of the Hamiltonian. One repeats the mapping until the physics of $H_{effective}$ becomes either trivial, or so soft it does not affect the results any more.[5]

The difference in our new method is in its truncation prescription. Instead of using only the single-block terms in H to determine the fast-variable wave function, we allow the block-block terms to influence it. This is done by letting the fast variables in a block oscillate about mean values that change adiabatically with the slow modes in the surrounding blocks.

It is easy to understand how this can economize on the number of states we need to keep, since the adiabatic state of the fast modes is a superposition of many non-adiabatic (isolated-block) states.

The adiabatic state is a "wave packet" in the fast variables. We determine its center and shape variationally by combining the blocking with a mean-field approach. Namely, at each step we "look ahead" a certain number n_ℓ of iterations, after which some parametrized mean field trial state,

$$\prod_{\text{sites } j} |\varphi\rangle_j$$

is used. $|\varphi\rangle$ is the same state for all j. Then the parameters of $|\varphi\rangle$, and of all previous n_ℓ wave packets, are varied to minimize the trial energy $\langle t|H|t\rangle$. $|t\rangle$ is now a completely specified trial state, in the modes of the last $H_{\text{effective}}$ and in all higher-frequency modes previously frozen. But having thus determined the variational parameters, we discard the mean field state and the last $(n_\ell - 1)$ look-ahead iterations, and use only the wave-packet parameters needed for a single mapping. The mapping thus proceeds one step at a time.

The intermediate trial states, $|t\rangle$, are never used to compute the final physical quantities; these are computed in the final trial state, which we denote $|\text{trial}\rangle$. It is obtained in the limit of an infinite number of blockings (for an infinite-volume lattice).

3. THE MODEL

The model we treat has a scalar field x_r defined on each site r of the one-dimensional lattice, together with its canonically-conjugate variable p_r. The Hamiltonian is (for a system of volume L sites)

$$H = \sum_{r=0}^{L-1} \frac{1}{2} p_r^2 + \sum_{r,s=0}^{L-1} d(r - s)\, x_r x_s \tag{3.1}$$

where $d(r - s)$ is some form of the lattice Laplacian. We work in units in which the lattice spacing is unity, and use periodic boundary-conditions.

Since we shall be working in the momentum basis, we rewrite H in that basis:[6]

$$H = \sum_k \left\{ \frac{1}{2} p(k)\ p^*(k) + d(k)\ x(k)\ x^*(k) \right\} \tag{3.2}$$

$d(k)$ can be made a symmetric function of k. In particular, for a field of mass μ and a nearest-neighbor definition of the gradient, we have

$$d(k) = \frac{1}{2} \mu^2 + (1 - \cos k) \quad . \tag{3.3}$$

H has the following three symmetries which will concern us:

a) Parity invariance

$$P: \begin{cases} x(k) \to x(-k) \\ p(k) \to p(-k) \end{cases} \tag{3.4}$$

b) Field-translation invariance: this is the continuous symmetry

$$G(\tau): \begin{cases} x(k) \to x(k) + \tau \delta_{k,0} \\ p(k) \to p(k) \end{cases} \tag{3.5}$$

for arbitrary real τ. This symmetry is equivalent to the statement that the field is massless, and is valid only if $d(0) = 0$.[7]

c) Site-translation invariance: this is the invariance under the lattice symmetry of relabeling the j-th site as the j + m-th, or in momentum basis

$$T_m: \begin{cases} x(k) \to x(k)\ e^{ikm} \\ p(k) \to p(k)\ e^{ikm} \end{cases} \tag{3.6}$$

for any integer m. This symmetry can, in fact, be generalized to translations by a non-integer number of lattice spacings.

Throughout our blocking treatment of the model, we will demand that P and $G(\tau)$ be conserved by the truncation. The breaking of T_m is inherent in the blocking approach, and we will encounter it in various forms.

The solution of (3.2) is trivial: the ground-state wave function is

$$\langle\{x\}|g.s.\rangle \propto \exp\left(-\sum_k \sqrt{\frac{d(k)}{2}}\, x(k)\, x^*(k)\right) \tag{3.7}$$

and the ground-state energy density is

$$\rho_{g.s.} = \frac{1}{2\pi\sqrt{2}} \int_0^{2\pi} dk\, \sqrt{d(k)} \tag{3.8}$$

in the large-volume limit. The equal-time, two-point Green's functions we shall study are

$$\begin{aligned} G(k) &= \langle x(k)\, x^*(k)\rangle_{g.s.} = \frac{1}{2\sqrt{2d(k)}} \\ R(k) &= \langle p(k)\, p^*(k)\rangle_{g.s.} = \sqrt{\frac{d(k)}{2}} \end{aligned} \tag{3.9}$$

4. A ONE-VARIABLE ADIABATIC SCHEME

Let us group the lattice sites into blocks, so that the j-th block contains the sites 2j and 2j+1. We define sum and difference block variables,

$$\begin{aligned} x_+(j) &= \frac{1}{2}\,(x_{2j} + x_{2j+1}) \\ x_-(j) &= \frac{1}{\sqrt{2}}\,(x_{2j} - x_{2j+1}) \\ p_+(j) &= p_{2j} + p_{2j+1} \\ p_-(j) &= \frac{1}{\sqrt{2}}\,(p_{2j} - p_{2j+1}) \end{aligned} \tag{4.1}$$

To get an effective Hamiltonian with one variable per site, one of the two block modes must be frozen; in the simple blocking schemes that mode is $x_-(j)$ (essentially because it has a higher wave-number).[8]

In the momentum basis, Eqs. (4.1) assume the form (Eq. (A.3))

$$\begin{aligned} e^{-ik/2}\, x(k) &= \sqrt{2}\, \cos(k/2)\, x_+(2k) - i\, \sin(k/2)\, x_-(2k) \\ e^{-ik/2}\, p(k) &= \frac{1}{\sqrt{2}}\, \cos(k/2)\, p_+(2k) - i\, \sin(k/2)\, p_-(2k) \end{aligned} \tag{A.3}$$

The fast modes $\{x_-\}$ will be frozen to a product of Gaussians over the blocks, with a Gaussian "wave-packet" at each block having as its center a function of the slow modes:

$$\langle \{x\} | \text{trial} \rangle = \prod_j \left[\exp\, -\frac{1}{2}\, (\gamma_-)_j \left(x_-(j) - F_j(\{x_+\}) \right)^2 \right] \psi_{res} \tag{4.2}$$

where ψ_{res} depends on the residual (slow) modes. As explained in Section 1, renormalization-group calculations are concerned with unraveling, step-by-step, the physics of the various length-scales in the problem; the physics that is of concern right now is that which is interior to a block, i.e., that of the fast variables. Therefore, ψ_{res} need not be specified at this point. Nevertheless, some crude guess-form of ψ_{res} will have to be used for the purpose of variationally determining the wave-packet parameters — this is the essence of the look-ahead method (see Section 2). We shall come back to the look-ahead aspects of ψ_{res} at the end of this section.

Let us now further specify the fast-variable wave-function appearing in (4.2). F_j may, in general, be a non-linear function, and $(\gamma_-)_j$; F_j may depend explicity on j, but we will use a linear form that does not:

$$F_j(\{x_+\}) = \sum_{j'} i\, \rho(j - j')\, x_+(j') \tag{4.3}$$

and a j-independent $(\gamma_-)_j = \gamma_-$.[9] ρ is an n-dependent function, and when we wish to emphasize this dependence we will write ρ_n. We may then rewrite Eq. (4.2) in the momentum basis

$$\langle\{x\}|\text{trial}\rangle = \sum_k \exp\left[-\frac{1}{2}\gamma_- \hat{x}_-(k)\ \hat{x}_-^*(k)\right]\ \psi_{res} \tag{4.4}$$

where we have gone over to shifted variables via the canonical transformation

$$\begin{aligned} x_-(k) &= \hat{x}_-(k) + i\,\rho(k)\ \hat{x}_+(k) \\ x_+(k) &= \hat{x}_+(k) \\ p_-(k) &= \hat{p}_-(k) \\ p_+(k) &= \hat{p}_+(k) + i\,\rho(k)\ \hat{p}_-(k) \quad . \end{aligned} \tag{4.5}$$

From parity and hermiticity, $\rho(k)$ must obey

$$\rho^*(k) = -\rho(-k) = \rho(k) \quad . \tag{4.6}$$

For later convenience, we choose to rescale the slow modes,

$$\begin{aligned} \hat{x}_+(k) &= 2^{-1/4}\ \tilde{x}(k) \\ \hat{p}_+(k) &= 2^{1/4}\ \tilde{p}(k) \end{aligned} \tag{4.7}$$

The shift, rescaling and truncation together define the mapping of operators. We will only be interested here in mapping bilinear operators: xx* and pp*. Hence we only need the following rules:

$$\begin{aligned} \left(\hat{x}_-(k)\right)_{tru} &= \left(\hat{p}_-(k)\right)_{tru} = 0 \\ \left(\hat{x}_-(k)\ \hat{x}_-^*(k')\right)_{tru} &= \delta_{kk'}\,\frac{1}{2\gamma_-} \\ \left(\hat{p}_-(k)\ \hat{p}_-^*(k')\right)_{tru} &= \delta_{kk'}\,\frac{\gamma_-}{2} \end{aligned} \tag{4.8}$$

where the subscript denotes "truncated." The following mapping results:

$$\left(x(k)\ x^*(k)\right)_{tru} = \frac{1}{\sqrt{2}}\left(\sqrt{2}\cos(k/2) + \rho(2k)\sin(k/2)\right)^2 \tilde{x}(2k)\ \tilde{x}^*(2k)$$

$$+ \frac{1}{4\gamma_-}(1 - \cos k)$$

$$\left(p(k)\ p^*(k)\right)_{tru} = \frac{1}{2\sqrt{2}}(1 + \cos k)\ \tilde{p}(2k)\ \tilde{p}^*(2k) + \frac{\gamma_-}{2}\left(\frac{1}{\sqrt{2}}\cos(k/2)\ \rho(2k)\right.$$

$$\left. - \sin(k/2)\right)^2 \qquad (4.9)$$

Note that a pair of momenta k, k+π, correspond to the same wave number 2k of the new lattice; this is an umklap phenomenon, related to the artificially imposed block boundaries. We introduce a notation for sums of contributions from such an umklap pair: for any function f(k), denote

$$[f(k/2)]_u = f(k/2) + f(k/2+\pi) \qquad (4.10)$$

Then from Eqs. (3.2) and (4.9) we obtain the mapping of the Hamiltonian

$$(H)_{tru} = \frac{L}{2}c_- + \frac{1}{\sqrt{2}}\tilde{H} \qquad (4.11)$$

Here c_- is the zero-point energy per block of the frozen fast modes, L is the volume,[10] and $\tilde{H}$ is the mapped Hamiltonian:

$$8\pi c_- = \gamma_-^{(n)}\int_0^{2\pi} dk\left(1 + \frac{1}{2}\rho(k)^2\right) + \frac{1}{\gamma_-^{(n)}}\cdot\int_0^{2\pi}\left[\left(1 - \cos\frac{k}{2}\right)d\left(\frac{k}{2}\right)\right]_u$$

$$\tilde{H} = \sum_k\left\{\frac{1}{2}\tilde{p}(k)\ \tilde{p}^*(k) + \tilde{d}(k)\ \tilde{x}(k)\ \tilde{x}^*(k)\right\} \qquad (4.12)$$

$$\tilde{d}(k) = \left[\left(\sqrt{2}\cos(k/4) + \rho(k)\sin(k/4)\right)^2 d(k/2)\right]_u$$

We have mapped the Hamiltonian, and we now want to determine the wave-packet parameters characterizing the renormalization group transformation.

These parameters, $\gamma_-^{(n)}$ and $\rho(k)$, will be determined variationally via the look-ahead prescription.

First, we realize that due to the simplicity of the model under consideration, the task is easier than it could in general be. Namely, because $\tilde{H}$ does not depend on $\gamma_-^{(n)}$ but only on the shift ρ, we may ignore $\tilde{H}$ for the purpose of solving for γ_-, no matter what the look-ahead method used. (This is a feature unique to the free-field case). The point is, of course, that γ_- depends on $\rho(k)$, which <u>does</u> appear in $\tilde{H}$, and thus the fast and slow modes interact through ρ and the results are sensitive to the look-ahead scheme employed. We may, however, eliminate γ_- from our equations, and we proceed to do that. Minimizing c_- with respect to $\gamma_-^{(n)}$ gives (from (4.12))

$$\gamma_-^{(n)} = \frac{\left\{\int_0^{2\pi} \left[\left(1 - \cos(k/2)\right)\, d(k/2)\right]_u dk\right\}^{1/2}}{\left\{\int_0^{2\pi} \left(1 + \frac{1}{2}\rho(k)^2\right) dk\right\}^{1/2}} \tag{4.13}$$

$$4\pi c_- = \left\{\int_0^{2\pi} \left[\left(1 - \cos(k/2)\right)\, d(k/2)\right]_u dk\right\}^{1/2} \left\{\int_0^{2\pi} \left(1 + \frac{1}{2}\rho(k)^2\right) dk\right\}^{1/2} \tag{4.14}$$

For a single look-ahead ($n_\ell = 1$ in the notation introduced in Section 2), we use the Gaussian mean-field guess for ψ_{res}

$$\psi_{res}\{\tilde{x}\} = \prod_j \exp\left\{-\frac{1}{2}\gamma_n'(\tilde{x}_j)^2\right\} = \prod_k \exp\left\{-\frac{1}{2}\gamma_n' \tilde{x}(k)\,\tilde{x}^*(k)\right\} \tag{4.15}$$

The intermediate trial state $|t\rangle$ for this scheme is obtained by substituting (4.15) in (4.2) or (4.4). The residual energy expectation value $\langle\tilde{H}\rangle$ in the state ψ_{res} is

$$\langle \psi_{res} | \tilde{H} | \psi_{res} \rangle = \left(\frac{\gamma_n'}{4} + \frac{1}{2\gamma_n'} \frac{1}{2\pi} \int_0^{2\pi} \tilde{d}(k)\, dk \right) \frac{L_n}{2} \tag{4.16}$$

Combining with (4.11) and (4.14) we find the total trial energy for this step (in the $n_\ell = 1$ scheme);

$$\langle t|H|t\rangle = \langle \psi_{res} | (H)_{tru} | \psi_{res} \rangle$$

$$= \frac{L_n}{2} \left\{ \frac{1}{4\pi} A^{1/2} \left(\int_0^{2\pi} \left(1 + \frac{1}{2}\rho(k)^2\, dk \right)^{1/2} + \frac{\gamma_n'}{4\sqrt{2}} + \frac{1}{2\sqrt{2}\,\gamma_n'\, 2\pi} \int_0^{2\pi} \tilde{d}(k)\, dk \right) \right\} \tag{4.17}$$

where the dependence of $\tilde{d}$ on ρ is given in (4.12), and A is a constant in this variational problem:

$$A = \int_0^{2\pi} \left[\left(1 - \cos(k/2) \right) d(k/2) \right]_u dk \tag{4.18}$$

Minimizing $\langle t|H|t\rangle$ with respect to γ_n' is trivial; the remaining minimization with respect to $\rho(k)$ depends on how long-range we wish to make the shift. Finally, if a number $n_\ell > 1$ of look-ahead steps is desired, the mapping should simply be carried out n_ℓ times before a mean-field Gaussian is used in the residual variables. All intermediate $\gamma_-^{(n)}$ values are given by (4.13), and one minimizes $\langle t|H|t\rangle$ with respect to all intermediate $\rho_n(k)$-s.

From here on the variational problem is essentially numerical. We have solved it for various look-ahed schemes and various forms for the initial lattice-Laplacian, $d^{(0)}(k)$. Before reporting the results, we assume that the mapping parameters have been determined, and turn to the mapping of Green's functions.

The recursion relations for Green's functions follow from Eq. (4.9) by taking expectation values in the state $|\text{trial}\rangle$. Defining for the n-th iteration

$$G^{(n)}(k) = \langle x^{(n)}(k)\ x^{(n)*}(k)\rangle_{\text{trial}}$$
$$R^{(n)}(k) = \langle p^{(n)}(k)\ p^{(n)*}(k)\rangle_{\text{trial}} \tag{4.19}$$

We get

$$G^{(n)}(k) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\cos(k/2) + \rho^{(n)}(2k)\sin(k/2)\right)^2 G^{(n+1)}(2k) + \frac{1}{4\gamma_-^{(n)}}(1-\cos k)$$

$$R^{(n)}(k) = \frac{1}{2\sqrt{2}}(1+\cos k)\ R^{(n+1)}(2k) + \frac{\gamma_-^{(n)}}{2}\left(\frac{1}{\sqrt{2}}\cos(k/2)\ \rho^{(n)}(2k) - \sin(k/2)\right)^2 \tag{4.20}$$

If these recursion relations are iterated ad infinitum, we get for $G^{(0)}$ and $R^{(0)}$ pathological functions of k that are nowhere smooth; this is just a reflections of the block umklap problem mentioned earlier.[11] What is happening is that an uncareful application of (4.20) causes the scaling behavior of G and R to be masked by these artificial umklap singularities; the way out of this problem is to consider $G^{(n)}(k)$, $R^{(n)}(k)$ only at the discrete momenta values $k_r = \pi \cdot 2^{-r}$. The recursion relations (4.20) are then solved on this subset of the real axis. This is possible because they relate $G^{(n)}$ and $R^{(n)}$ at k_r to $G^{(n+1)}$ and $R^{(n+1)}$ at k_{r-1}, and moreover this trick causes the recursion to terminate when r reaches 0, $k_0 = \pi$. A demonstration of how we have solved recursion relations of this type is furnished in Appendix C.

Once the power laws for the Green's functions have been found at the discrete momentum values k_r, we can interpolate to all k values; this is a convenient way to smooth the non-analytic behavior of G and R. Note that the discrete sequence $\{k_r\}$ has an accumulation point at $k = 0$, which

is the region of interest (long-range). Physically, $\{k_r\}$ are the elementary harmonics quantized in the blocks of sizes 2^r.[12]

We next present the results of this one-variable adiabatic method for various forms of $\rho(k)$, $d(k)$ and look-ahead procedure.

The Results For One-Variable Schemes

The results of several one-variable schemes, of the type described above, are summarized in Tables I and II. In these tables, n_{shift} is the range of the shift in number of blocks; for a given range, the most general shift function compatible with parity is (see Eq. (4.6)).

$$\rho(k) = \sum_{j=1}^{n_{shift}} r_j \sin(jk) \tag{4.21}$$

For schemes with nearest-neighbor shifts, $n_{shift} = 1$, we recorded the limit

$$r_\infty = \lim_{n \to \infty} r_1^{(n)} \tag{4.22}$$

of the shift parameter as the number of iterations tends to infinity; it is a measure of the amount of adiabatic shifting in the large-scale limit. $\rho_{g.s.}$ is the ground-state energy density, while γ_G and γ_R are the asymptotic large-distance exponents as computed at the points $k_r = \pi \cdot 2^{-r}$ from (4.20):

$$G(k_r) \sim (k_r)^{\gamma_G}$$

$$R(k_r) \sim (k_r)^{\gamma_R}$$

as $r >> 1$. Since $\gamma_R = -\gamma_G$ for all the schemes we considered, we recorded only γ_G.

Both tables refer to a model with the nearest neighbor lattice derivative given in Eq. (3.3). Setting the shifts $\rho(k)$ to zero gives the simple-blocking scheme with mean field, i.e., with a one-step look-ahead; the no-

mean-field scheme results when we determine the Gaussian parameters $\gamma_-^{(n)}$ by diagonalizing the block Hamiltonian. Both these schemes give $\gamma_G = -1/2$,[13] but the energy density is better with mean field than without it.

We see that allowing nearest-neighbor shifts improves both main physical quantities (energy and asymptotic exponent). Taking more look-ahead steps further improves them (except for $n_\ell = 2$), until beyond $n_\ell = 4$ the results change very little. Increasing the range of the shift improves them even more, but only when n_ℓ is at least 2.[14] The best scheme we have tried was that with a range-two shift and three look-ahead steps, giving $\gamma_G = -.998$.

Table II shows ratios of the trial energy density and Green's functions to their exact values, for various masses and momenta. The nearest-neighbor gradient (Eq. (3.3)) was again used. For large masses, where simple blocking works well because the field is localized, the shift does not change the results much, but as the mass decreases the new method becomes superior both for $\rho_{g.s.}$ and Green's functions.[15]

Finally, we have checked the sensitivity of the above results to changes in the form of lattice gradient used (including for the so-called "SLAC" derivative that is infinite-range and is designed to give a relativistic spectrum). Both the exact and trial energy densities changes, and the shift again improves the agreement, even though on the whole blocking is slightly less successful for longer-ranged derivatives. The asymptotic shifts and γ_G do not depend on the form of d(k). The fixed form of the gradient is given in Eq. (A.7); why this form is of range 4 lattice sites is also explained there.

5. TWO-VARIABLE SCHEMES

A. A Two-Variable Formulation of the Model

In the last section we saw how a nearest-neighbor adiabatic shift much improves on the simple Gaussian-truncation blocking scheme. The scheme can be made even more accurate by keeping more variables per site in the effective Hamiltonian, thus truncating away less of the dynamics. We will demonstrate this for a particular class of schemes in which two variables are kept per site. First we will cast the model in its two-variable form, by blocking once without truncation.

We start from Eq. (3.2), and decompose $x(k)$, $p(k)$ in terms of block variables according to Eq. (A.3). We redefine for convenience

$$\begin{aligned} y(k) &= x_+(k), \qquad z(k) = x_-(k)/\sqrt{2} \\ p_y(k) &= p_+(k), \qquad p_z(k) = \sqrt{2}\, p_-(k) \end{aligned} \tag{5.1}$$

H then assumes the form

$$H = \sum_k \left\{ \frac{1}{4} p_y p_y^* + \frac{1}{4} p_z p_z^* + (y \quad z) \begin{pmatrix} d_{yy} & -id_{yz}/2 \\ id_{yz}/2 & d_{zz} \end{pmatrix} \begin{pmatrix} y^* \\ z^* \end{pmatrix} \right\} \tag{5.2}$$

here $d_{yy}(k)$, $d_{yz}(k)$ and $d_{zz}(k)$ are functions given in Eq. (A.4).

We now take Eq. (5.2) as our starting point. It is an effective Hamiltonian, but is equivalent to the original one. It is defined on a new lattice of volume L/2 and with a two-component scalar field at each site -- a parity-even component y and a parity-odd component z. Under the symmetry $G(\tau)$ ((Eq. (3.5)), we have

$$G(\tau): \begin{cases} y(k) \to y(k) + \tau\delta_{k,0} \\ z(k) \to z(k) \end{cases} \tag{5.3}$$

(p_y, p_z unchanged)

Since z does not transform, it is allowed to have an effective mass term without violating any symmetry, and indeed there is such a term from the outset (Eq. (A.5)).

B. A Two-Variable Adiabatic Scheme

We group the sites in pairs to form blocks (which are 4-site "superblocks" in the original sites), and define four block variables: the sum and difference variables for y and z, denoted as $y_\pm$ and $z_\pm$ (see Eq. (A.6) for their exact definition). y_-, z_+ and z_- are the three fast block modes, and y_+ is the slow mode (intuitively; since it corresponds to a $k = 0$ mode inside the superblock).

Within one block there can occur a mixing between the four modes. Parity invariance implies that the two even variables, y_+ and z_-, may only mix with each other, and so can the odd variables y_- and z_+. We will keep one even mode, and one odd mode which is a mixture of y_- and z_+. The other two modes will be truncated to adiabatic wave packets, with centers that are shifted to track the retained modes. Note that y_+ and z_- cannot actually mix (at least not orthogonally), because that would give a mass to y_+ and break the $G(\tau)$ invariance; hence we take the retained slow mode as simply y_+.

Let ζ be the mixing angle between the odd modes, defined such that the fast and slow mixed modes are, respectively,

$$z_+^M \equiv cz_+ - sy_-$$

$$y_-^M \equiv sz_+ + cy_-$$

Here c and s are shorthand for $\cos\zeta$ and $\sin\zeta$, respectively. We now invoke the most general linear adiabatic shifts among the four modes

y_+, y_-^M; z_+^M, z_- that are consistent with the symmetries of the model. The fast modes z_+^M, z_- each shift by a linear combination of the two slow modes y_+, y_-^M; as in the one-variable calculation, we work in momentum basis and the shift coefficients are k dependent. We thus define

$$\left.\begin{aligned} \hat{y}_+ &= y_+ \\ \hat{y}_- &= y_-^M \end{aligned}\right\} \quad \text{(unshifted slow modes)}$$

$$\left.\begin{aligned} \hat{z}_+ &= z_+^M - i\rho_1 y_+ - \rho_2 y_-^M \\ \hat{z}_- &= z_- - \rho_3 y_+ - i\rho_4 y_-^M \end{aligned}\right\} \quad \text{(shifted fast modes)}$$

Disentangeling the original variables in terms of $\hat{y}_\pm$, $\hat{z}_\pm$ we find

$$\begin{aligned} y_+ &= \hat{y}_+ \\ y_- &= (c - s\rho_2)\,\hat{y}_- - i\rho_1\, s\hat{y}_+ - s\hat{z}_+ \\ z_+ &= c\hat{z}_+ + i\rho_1\, c\hat{y}_+ + (s + \rho_2\, c)\,\hat{y}_- \\ z_- &= \hat{z}_- + \rho_3\hat{y}_+ + i\rho_4\hat{y}_- \end{aligned} \tag{5.4}$$

where the shift coefficients ρ_i are functions of k. $\hat{z}_+$ and $\hat{z}_-$ are the fast modes that we truncate away.

If we assume that the shifts are all nearest-neighbor, then parity and $G(\tau)$ symmetries restrict them to the following forms, parametrized by five real numbers:

$$\begin{aligned} \rho_1(k) &= r_1 \sin k \\ \rho_2(k) &= r_2 + r_3 \cos k \\ \rho_3(k) &= r_4(1 - \cos k) \\ \rho_4(k) &= r_5 \sin k \end{aligned} \tag{5.5}$$

The six mix-shift parameters ζ, $\{r_i\}$ will be determined variationally.

The truncation is done with a product-Gaussian state:

$$\langle\{\hat{y},\hat{z}\}|\text{trial}\rangle = \prod_k \left(e^{-\frac{1}{2}\gamma_1|\hat{z}_+(k)|^2} \; e^{-\frac{1}{2}\gamma_2|\hat{z}_-(k)|^2} \right) \psi_{\text{res}}(\{\hat{y}_\pm\}) \qquad (5.6)$$

where ψ_{res} is the wave function in the remaining variables, and γ_1, γ_2 are two more variational parameters. The guess-form we use for a one-step look-ahead is a product over single-site Gaussians, $\prod_k \exp\left\{-\frac{1}{2}\gamma_3|\hat{y}_+(k)|^2 - \frac{1}{2}\gamma_4|\hat{y}_-(k)|^2\right\}$. To complete the definition of the mapping we rescale

$$\begin{aligned} \tilde{y} &= 2^{1/4}\, y_+ \\ \tilde{z} &= 2^{-1/4}\, \hat{y}_- \\ \tilde{p}_y &= 2^{-1/4}\, \hat{p}_y^+ \\ \tilde{p}_z &= 2^{1/4}\, \hat{p}_y^- \end{aligned} \qquad (5.7)$$

We have studied three equal-time Green's functions,

$$\begin{aligned} G_1(k) &= \langle y(k)\; y^*(k)\rangle_{\text{trial}} \\ G_2(k) &= i\langle y(k)\; z^*(k)\rangle_{\text{trial}} \\ G_3(k) &= \langle z(k)\; z^*(k)\rangle_{\text{trial}} \end{aligned} \qquad (5.8)$$

which are arranged as a vector of rank 3, and their canonically conjugate counterparts, the vector $\vec{R}(k)$. The mapping of bilinear operators gives a tensor version of the renormalization group equations encountered in the one-variable case; e.g.,

$$\vec{G}^{(n)}(k) = \overleftrightarrow{\Gamma}^{(n)}(k) \cdot \vec{G}^{(n+1)}(2k) + \vec{\sigma}^{(n)}(k) \quad . \qquad (5.9)$$

where the matrix $\overleftrightarrow{\Gamma}^{(n)}$ and the vector $\vec{\sigma}^{(n)}$ depend on the mix-shift parameters. This equation is easily obtained from Eqs. (5.8), (A.6), (5.4) and (5.6) upon integrating out the fast "shift-mixed" variables, $\hat{z}_\pm$.

We iterated these equations for the momenta $k = k_r$, using parameter values obtained variationally through a look-ahead procedure. The mapping of the Hamiltonian is of the form Eq. (4.11), with the new Hamiltonian being again of the form Eq. (5.2). The fixed form of H has functions d_{yy}, d_{yz}, d_{zz} of range four lattice spacings, for the same reason the range of $d^{(n)}$ was 4 in Section 4 (see Eq. (A.7)). As in the one-variable calculations, this fixed form is independent of the exact form of the lattice gradient we start from, and so are the values of the Green's function exponents, defined as follows

$$G_i(k_r) \sim (k_r)^{\gamma_G^i}$$
$$R_i(k_r) \sim (k_r)^{\gamma_R^i} \qquad (5.10)$$

where $1 \leq i \leq 3$ and $r >> 1$.

Table III summarizes the results and compares them with the simple-blocking scheme: without mean field (Ref. 16) and with mean field. The adiabatic scheme is for $n_\ell = 2$, and we see that γ_G^1, γ_G^2 and γ_R^1 agree with the exact values up to ~1 part in 10^4. The other exponents are incorrect, as in the conventional schemes. We expect them to, because they involve the z mode, which is "faster" than y and so the truncation affects it more. This is again traceable to the lack of translational (T_m) invariance in blocking schemes; for in any trial state that obeys that invariance, one can prove that

$$\gamma_G^3 - 2 = \gamma_G^2 - 1 = \gamma_G^1$$
$$\gamma_R^3 - 2 = \gamma_R^2 - 1 = \gamma_R^1 \quad . \qquad (5.11)$$

For the case of only one look-ahead step, an anomalous phenomenon occurs: the $\hat{z}_+$ mode, that in the no-shift schemes lies above $\hat{y}_-$, crosses below it. Therefore, choosing $\hat{y}_-$ repeatedly as a "slow mode" has a disastrous effect —— it causes the $z^{(n)}$ oscillator to keep increasing its effective mass with n, and it eventually decouples from $y^{(n)}$. Thus, the $n_\ell = 1$ scheme reproduces, in effect, the results of the one-variable scheme. This could be avoided by allowing $\hat{y}_-$ to lower its frequency too, by shifting with the slowest mode $\hat{y}_+$. But for $n_\ell \geq 2$ the level ordering

$$E_{y_+} < E_{y_-} < E_{z_-} < E_{z_+} \tag{5.12}$$

remains intact throughout the iterations.

We also applied the scheme in which the three fast modes in a block are adiabatically shifted by the y_+ mode; that did not give a much better $\rho_{g.s.}$ than the one-variable schemes, and did not improve γ_G^i and γ_R^i.

C. Possible Improvements

After having demonstrated that the results of the one-variable adiabatic scheme can be improved impressively by keeping two variables per site, we briefly discuss the generality of the procedure used in B. The two even modes in a block, y_+ and z_-, are not allowed to mix orthogonally if one of them (i.e., z_-) is then truncated by a Gaussian, since that would give y_+ a fictitious mass. But y_+ could still shift by an amount proportional to z_- (i.e., a non-orthogonal mixing is allowed.). One may also choose a different rescaling than Eq. (5.7).[17] A no-shift, several-look-ahead calculation we did indicates that in such an approach the $\hat{z}_-$ mode crosses below $\hat{y}_-$, suggesting that perhaps the two even modes should be kept per block. To avoid such breakdowns in a blocking scheme as this

or what happens for $n_\ell = 1$, one should not predetermine at all which of the four modes are to be truncated. Their energies should be allowed to determine that anew at each step of the iteration. One should also allow for more general shifts, as discussed in Section B. But the calculations presented here suffice to make the point that systematic improvement within our approach is possible.

6. CONCLUSIONS

We have seen how the adiabatic truncation method can be used to improve the accuracy of real-space renormalization group techniques on the lattice for the case of a free scalar field. The improvement is especially noteworthy for large-distance behavior of trial Green's functions. The same method can be generalized to interacting field theories, and to any number of dimensions. We expect it to yield better results for phase transition locations, critical exponents, etc., than was possible with conventional methods.

7. ACKNOWLEDGEMENTS

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APPENDIX A

In going from position to momentum basis we used the following relations:

$$\begin{aligned} d(r) &= \frac{1}{L}\sum_k e^{ikr}\, d(k) \\ x(r) &= \frac{1}{\sqrt{L}}\sum_k e^{-ikr}\, x(k) \\ p(r) &= \frac{1}{\sqrt{L}}\sum_k e^{-ikr}\, p(k) \end{aligned} \tag{A.1}$$

where L is the volume of the system in the current iteration, and k ranges over the values $2\pi\, m_k/L$, $0 \le m_k \le L - 1$. The canonical commutation relations are

$$[x(k),\ p^*(k)] = i\ \delta_{k,k'} \tag{A.2}$$

and since x is a real field, $x^*(k) = x(-k)$, and likewise for p.

To get the blocking relations in momentum basis, start from Eq. (4.1). Multiply both sides of each equation by e^{+2jik} and sum over $0 \le j \le L/2 - 1$. using for the block variables $x_{\pm}$, $p_{\pm}$ Fourier transformations similar to Eq. (A.1), but with L replaced by the decimated volume L/2, we find the relations

$$\begin{aligned} e^{-ik/2}x(k) &= \sqrt{2}\,\cos(k/2)\ x_+(2k) - i\,\sin(k/2)\ x_-(2k) \\ e^{-ik/2}p(k) &= \frac{1}{\sqrt{2}}\cos(k/2)\ p_+(2k) - i\,\sin(k/2)\ p_-(2k) \end{aligned} \tag{A.3}$$

For the two-variable schemes, the "Laplacian matrix" appearing in Eq. (5.2) has the components

$$\begin{aligned} d_{yy}(k) &= \Big[\big(1 + \cos(k/2)\big)\ d(k/2)\Big]_u \\ d_{yz}(k) &= -2\Big[\sin\ (k/2)\ d(k/2)\Big]_u \\ d_{zz}(k) &= \Big[\big(1 - \cos(k/2)\big)\ d(k/2)\Big]_u \end{aligned} \tag{A.4}$$

Using the definition (4.10) for the umklap sum. For the zero-mass nearest-neighbor gradient, $d(k) = 1 - \cos k$ and

$$\begin{aligned} d_{yy}(k) &= 1 - \cos k \\ d_{yz}(k) &= 2 \sin k \\ d_{zz}(k) &= 3 + \cos k \end{aligned} \tag{A.5}$$

The definitions of the four block variables and their canonical momenta are

$$\begin{aligned} e^{-ik/2}\, y(k) &= \sqrt{2}\, \cos(k/2)\, y_+(2k) - i \sin(k/2)\, y_-(2k) \\ e^{-ik/2}\, z(k) &= \cos(k/2)\, z_+(2k) - i \sin(k/2)\, z_-(2k) \\ e^{-ik/2}\, p_y(k) &= \frac{1}{\sqrt{2}} \cos(k/2)\, p_y^+(2k) - i \sin(k/2)\, p_y^-(2k) \\ e^{-ik/2}\, p_z(k) &= \cos(k/2)\, p_z^+(2k) - i \sin(k/2)\, p_z^-(2k) \quad . \end{aligned} \tag{A.6}$$

Fixed forms: In the one-variable scheme with $n_{shift} = n_\ell = 1$, any model with a gradient of range four lattice spacings or less has the fixed-form gradient

$$d^{(n)}(k) \underset{n\uparrow\infty}{\longrightarrow} C_0 \left[(1 - \cos k) - .24 \left(1 - \cos(2k)\right) + 0.35 \left(1 - \cos(3k)\right) + .001 \left(1 - \cos(4k)\right)\right] \tag{A.7}$$

where C_0 decreases exponentially with n. It is easy to prove that, starting with a lattice Laplacian $d^{(0)}(k)$ of range ≤ 4, $d^{(n)}(k)$ reaches the maximal range of 4. An interaction term of range r lattice sites becomes, upon blocking, of the range $\left\lfloor \frac{r-2}{2} \right\rfloor + 1$, where $\lfloor x \rfloor$ is the largest integer smaller or equal to x; this range rapidly iterates to nearest-neighbor, 1. But with a nearest-neighbor adiabatic shift, $\tilde{x}_{j-1}$ knows about $\tilde{x}_{j+1}$ through $\hat{x}_-(j)$ so we must add 2 to the iterated range:

$$r_{it} = \left\lfloor \frac{r-1}{2} \right\rfloor + 3$$

which iterates to 4 for all $r_{initial} \leq 4$.

APPENDIX B

A QUALITATIVE EXPLANATION FOR THE ANOMALOUS MASS GAPS

Simple-blocking schemes tend to give the massless model fictitious mass-gaps, decreasing as the -1/2 power of the block volume; this is responsible for asymptotic exponents (γ_G in the one-variable schemes and γ_G^1 in the two-variable schemes) coming out to be -1/2. Why this occurs may be understood in a qualitative way as follows: consider a superblock, formed after n iterations, of volume $V = 2^n$ sites; we look at the part H_V of the original Hamiltonian that involves only fields inside this superblock, and add to it two symmetric surface terms to represent the effects of the rest of the system:[18]

$$H_V = \sum_{j=1}^{V} \frac{1}{2} p_j^2 + \sum_{j=1}^{V-1} \frac{1}{2} \left(x_{j+1} - x_j\right)^2 + \frac{\alpha^2}{2} \left(x_1^2 + x_V^2\right) \quad . \qquad (B.1)$$

The freedom in choosing the surface terms stems from the arbitrary nature of the decomposition into "block energy" and "block-block interaction," pointed out in Section 2. The surface terms will cause the field x_j quantized in this volume to have an effective mass $\alpha\sqrt{2/V}$, which scales with the anomalous power -1/2 (α being of order unity). The particular choice $\alpha = 1$ corresponds to the choice of block Hamiltonian of Ref. 19.

APPENDIX C

We now demonstrate the method used to solve the recursion relations (4.20) for the values of Green's functions at $k = k_r$. The example to be worked out in detail is the no-shift, one look-ahead case, i.e., $\rho_n(k) \equiv 0$. For a nearest-neighbor gradient $d^{(0)}(k) = 1 - \cos k$ (Eq. (3.3) for $\mu = 0$), we find from (4.13)

$$\gamma_-^{(n)} \equiv \sqrt{3} \tag{C.1}$$

In this case, Eq. (4.20) simplifies to

$$\begin{aligned} G^{(n)}(k) &= \frac{1}{\sqrt{2}}(1+\cos k)\, G^{(n+1)}(2k) + \frac{1}{4\sqrt{3}}(1-\cos k) \\ R^{(n)}(k) &= \frac{1}{2\sqrt{2}}(1+\cos k)\, R^{(n+1)}(2k) + \frac{\sqrt{3}}{4}(1-\cos k) \end{aligned} \tag{C.2}$$

Using $1+\cos k = \tfrac{1}{2}\left[\sin^2 k/\sin^2(k/2)\right]$, we find upon iterating (C.2) ad infinitum

$$\begin{aligned} G(k) = G^{(0)}(k) &= \frac{1}{4\sqrt{3}}\left\{1-\cos k + \sum_{m=1}^{\infty}\left[1-\cos(2^m k)\right]\left(\frac{1}{2\sqrt{2}}\right)^m \frac{\sin^2(2^{m-1}k)}{\sin^2(k/2)}\right\} \\ &= \frac{1}{4\sqrt{3}}\left\{1-\cos k + \frac{4}{1-\cos k}\sum_{m=1}^{\infty}(8)^{-m/2}\sin^4(2^{m-1}k)\right\} \end{aligned} \tag{C.3}$$

$$R(k) = R^{(0)}(k) = \frac{\sqrt{3}}{4}\left\{1-\cos k + \frac{4}{1-\cos k}\sum_{m=1}^{\infty}(32)^{-m/2}\sin^4(2^{m-1}k)\right\}$$

These are non-analytic functions, but at $k = k_r = \pi \times 2^{-r}$ the sums terminate and we obtain

$$G(k_r) = \frac{1}{4\sqrt{3}} \left\{ 1 - \cosk_r + \frac{4}{1 - \cosk_r} \sum_{m=1}^{r} (8)^{-m/2} \sin^4\left(\pi \times 2^{m-1-r}\right) \right\}$$

$$R(k_r) = \frac{\sqrt{3}}{4} \left\{ 1 - \cosk_r + \frac{4}{1 - \cosk_r} \sum_{m=1}^{r} (32)^{-m/2} \sin^4\left(\pi \times 2^{m-1-r}\right) \right\} \qquad (C.4)$$

These are the types of sums that appear in all our truncation schemes. For large r, the first terms (low m) behave approximately as a geometric series, with some ratio α. If $\alpha > 1$, the large-m part of the sum dominates, whereas if $\alpha < 1$ the low-m terms dominate. For the case at hand, $\alpha > 1$ for both sums (C.4), so they are dominated by m = r:

$$G(k_r) \sim \frac{1}{(k_r)^2} (8)^{-r/2} \sim (k_r)^{-1/2}$$

$$R(k_r) \sim \frac{1}{(k_r)^2} (32)^{-r/2} \sim (k_r)^{1/2}$$

(see Table I).

REFERENCES

1. This state is obtained by repeated application of the scale transformation; eventually all modes become "fast" and have their dynamics specified by the truncation.
2. H. R. Quinn and M. Weinstein, SLAC-PUB-2795 (to be published in Phys. Rev. D 15, March issue, 1982).
3. That is how we refer to the conventional blocking procedure.
4. The difference between it and its continuum counterpart is that there, the rescaled theory was equivalent to the original one, whereas here we discarded some of the physics by truncating away the fast modes.
5. In the massless theory in an infinite volume, one must block ad infinitum.
6. See Eq. (A.1).
7. It becomes a U(1) symmetry if the model is made compact, e.g., if it is made into an XY model.
8. To conserve the symmetries, this is the only block mode one can choose; x_+ and x_- have even and odd parities, respectively, so they cannot mix, and x_+ itself cannot be truncated since that would break $G(\tau)$.
9. In interaction models we expect the optimal F to be non-linear; this may even be true for the free-field case. An explicit j dependence is desirable to study kinks.
10. We denote the volume at the n-th iteration, defined as the number of effective lattice sites, by L_n; $L_0 = L$ and due to the blocking of site pairs, $L_{n+1} = L_n/2$.

11. The umklaps also cause the breaking of the T_m symmetry, i.e., non-conservation of momentum. The non-smoothness of G and R, inherent in blocking schemes, is evident in position-basis treatments as well (see for example Eq. (3.10) of J. Richardson and R. Blankenbecler, Phys. Rev. D 20, 1351 (1979), where the mapping involves the binary digits of the position).

12. This is equivalent to the usual position basis practice of computing Green's functions at spatial separations equal to block sizes.

13. See Appendix B for a qualitative explanation of this anomaly.

14. It should be remembered that, unlike in a simple Rayleigh-Ritz variational approach, simply varying over more parameters in a look-ahead scheme is not guaranteed to improve even the energy, let alone the wave function. However, improvement in energy will occur if n_ℓ is sufficiently large.

15. For any finite mass μ, the trial Green's functions are exact at $k = 0$ for both shifted and non-shifted schemes; the real test is for $\mu << k << \pi$.

16. S. D. Drell and M. Weinstein, Phys. Rev. D 17, 3203 (1978).

17. Unlike the one-variable case, that could change the results. It is equivalent to making the odd-mode mixing non-orthogonal. The rescaling Eq. (5.7) is the optimal one in the case of the simple-blocking scheme in Ref. 16.

18. This argument is in the spirit of the Shadow Hamiltonian approach of Ref. 2, in which the parameter α would be varied over.

19. S. D. Drell, M. Weinstein, S. Yankielowicz, Phys. Rev. D 16, 1769 (1977).

Table I. Results from one-variable schemes in the massless model.

Scheme	n_ℓ	n_{shift}	$\rho_{g.s.}$	γ_G	r_∞
Simple-blocking without mean field	0	0	.773	-.5	0
Simple-blocking with mean field	1	0	.670	-.5	0
Adiabatic truncation schemes	1	1	.643435	-.984	-.384
	2	1	.643447	-.979	-.388
	3	1	.643425	-.987	-.380
	4	1	.643424	-.989	-.378
	5	1	.643424	-.989	-.378
	1	2	.643933	-.91	--
	1	∞	.643878	-.93	--
	2	2	.643399	-.997	--
	3	2	.643377	-.998	--
Exact values	--	--	.6366	-1	--

Table II. Ratios of trial to exact quantities for various masses and momenta $k_r = \pi\, 2^{-r}$. a) For the adiabatic truncation scheme with $n_{shift} = n_\ell = 1$, and b) for the no-shift scheme, i.e., simple blocking, with a mean-field look-ahead.

a)

$\frac{1}{2}\mu^2$	$\rho_{g.s.}(\text{trial})/\rho_{g.s.}(\text{exact})$	$G_{trial}(k_r)/G_{exact}(k_r)$	
		r = 5	r = 14
10	1.000095	.990	.99998
10^{-6}	1.01	.518	.828
10^{-9}	1.01	.518	.472

b)

$\frac{1}{2}\mu^2$	$\rho_{g.s.}(\text{trial})/\rho_{g.s.}(\text{exact})$	$G_{trial}(k_r)/G_{trial}(k_r)$	
		r = 5	r = 14
10	1.00025	.995	.99999
10^{-6}	1.05	.249	.082
10^{-9}	1.01	.249	.011

Table III. Results for the various two-variable schemes. The numbers in the first column are the ratios of trial to exact ground-state energy densities. ζ is the mixing angle of the two odd-parity modes within a block, and the γ parameters are the asymptotic exponents, defined in Eq. (5.10).

Scheme	$\rho_{g.s.}(\text{trial})/\rho_{g.s.}(\text{exact})$	ζ	γ_G^1	γ_G^2	γ_G^3	γ_R^1	γ_R^2	γ_R^3
Two-look-ahead shift-mix adiabatic truncation	1.002	66.7^o	-1.00022	-.00022	0	.99991	1	0
No shift, one-look-ahead, mixing	1.04	17^o	-.5	1	0	.5	0	0
Simple-blocking, no look-aheads	1.06	30^o	-1	0	0	.5	0	0
Exact exponents			-1	0	1	1	2	3