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CROSS SECTIONS FROM OBSERVABLE FIELDS*

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ABSTRACT

The observable fields of QED are used to construct operators which project onto states with prescribed configurations of incoming or outgoing observable particles. These operators are used to define the physical cross sections of QED. It is heuristically argued that the cross sections thus defined are free of infrared divergences and in fact equal the traditional results.

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1. INTRODUCTION

Cross sections in QED are typically calculated by applying LSZ scattering theory[8] to the gauge-noninvariant photon and electron fields A_μ and ψ. The resulting infrared-divergent S-matrix elements are then used to compute cross sections. In this paper we describe a procedure for computing cross sections (in principle, at least) which does not use gauge-noninvariant fields or infrared-divergent quantities. The basic idea is to use gauge-invariant fields[11] (e.g., the energy-momentum tensor $T^{\mu\nu}$ and the electric current J^ν) to define operators which project onto states whose observable outgoing (or incoming) particles lie in some prescribed region of phase space. These phase space projection operators (PSPO's) are then used to define cross sections.

Note that S-matrix elements are not defined in this approach. In this and in other ways the procedure resembles results of Araki and Haag.[10] However, it differs by being applicable to QED, by including a definition of cross sections, and, regrettably, by disregarding rigor.

The PSPO's are defined in Section 2. Their infrared finiteness will be obvious on physical grounds since they are physically observable operators. (Explicit verification of the infrared finiteness of the PSPO's found in certain model theories is given, somewhat disguised, in Ref. 2.)

A definition of cross sections based on PSPO's is given in Section 3. This definition does not correspond all that closely to actual laboratory procedures and is therefore not manifestly infrared finite. Section 3 therefore contains a heuristic argument for the infrared finiteness at the definition. (Actually, almost all of the arguments in this paper are heuristic, some more than others. Only a few results are rigorous and there are some gaps, as will be noted, in the reasoning.)

In what follows, our attention is restricted to QED mostly for the sake of notational simplicity. With obvious modifications, most of what follows should apply to any (nongravitational) Lagrangian field theory.

2. PSPO's

In this section, we construct PSPO's which analyze outgoing particle content. Trivial modifications yield PSPO's for incoming particles.

Given a field theory to which LSZ scattering theory is applicable, asymptotic states $|k_1 \ldots k_n \text{ out}\rangle$ may be defined, where $k_1,\ldots,k_n$ denote particle momenta. The PSPO's densities are then given by

$$P^{\text{out}}(k_1,\ldots,k_n) \equiv |k_1 \ldots k_n \text{ out}\rangle\langle k_1 \ldots k_n \text{ out}| \quad .$$

These, when integrated over a region R of n-particle phase space, clearly project onto states whose particle contents lie in R. However, LSZ scattering theory does not apply to QED, and we must proceed differently. To illustrate our approach, we now briefly and sloppily construct the PSPO's for a scalar field theory with a single massive spinless particle.

Outgoing particles can have various velocities. Given a region $\Omega \subset \mathbb{R}^3$, purely geometric reasoning yields that a particle with velocity in Ω will for large enough times t lie in the spatial region $t\Omega \equiv \{t\vec{v}\,|\,\vec{v} \in \Omega\}$. Therefore, the energy-momentum carried by such particles should equal

$$P_\mu^{\text{out}}(\Omega) \equiv \lim_{t\to\infty} \int_{t\Omega} d\vec{x}\; T_\mu^0(t,\vec{x}) \quad ,$$

where $T^{\mu\nu}$ is the energy-momentum tensor. (I apologize for the dual use of the letter P.)

We define a no-particle (in Ω) state -- that is, a state with possibly several particles but none with velocity in Ω -- to be a state which is annihilated by $P_\mu^{out}(\Omega)$. The projection operator onto all such states we denote by $E_o(\Omega)$.

Now, a single-particle state is typically defined as a (normalizable) eigenstate of the mass operator $\sqrt{-P^2}$. (P_μ is the total energy-momentum, and our metric is -+++.) The physical reasoning behind this definition may be found in Ref. 9. For similar reasons, we define a single-particle (in Ω) state -- that is, a state with possibly several particles but only one with velocity in Ω -- to be a normalizable eigenstate of $\sqrt{-P^{out}(\Omega)^2}$. The projection operator onto all such states we denote by $E(\Omega)$.

We are now ready to define PSPO's. Let $\Omega_1,\ldots,\Omega_n$ be disjoint regions in $\mathbb{R}^3$. Then $R \equiv \Omega_1 \times \ldots \times \Omega_n$ is a region in n-particle phase space, and we define

$$P^{out}(R) \equiv E(\Omega_1) \ldots E(\Omega_n)\, E_o(\sim \Omega_1 \cup \ldots \cup \Omega_n) \quad .$$

(x, ~, and ∪ denote respectively Cartesian product, complement, and union.) Since the E's and E_o commute (as will be argued later), $P^{out}(R)$ is in fact a projection operator. Furthermore, any state that it projects onto clearly has one particle in each velocity region Ω_i and no other particles. Thus, $P^{out}(R)$ is the desired PSPO, projecting onto states whose outgoing particle contents lie in the phase space region R.

We are not quite finished. Sets R of the type defined above are rather special subsets of phase space. We would like to define $P^{out}(R)$ for arbitrary measurable R. To do this, several approaches might be taken. For example, PSPO densities could be defined by

$$P^{out}(\vec{v}_1,\ldots,\vec{v}_n) \equiv \lim_{\mu(\Omega_i)\to 0} \frac{P^{out}(\Omega_1 \times \ldots \times \Omega_n)}{\mu(\Omega_1)\ \ldots\ \mu(\Omega_n)} \quad ,$$

where $v_i \varepsilon \Omega_i$ and $\mu(\Omega_i)$ is the volume of Ω_i. (We expect the limit to exist weakly, yielding a measurable function.) The desired PSPO's could then be obtained by integrating the densities. Alternately, given a disjoint collection of sets R_k each of the above type, one can define

$$P^{out}\left(\bigcup_k R_k\right) \equiv \sum_k P^{out}(R_k) \quad .$$

It is not difficult to show that this definition is meaningful and has all the properties a PSPO ought to have. One could then try to define $P^{out}(R)$ for arbitrary measurable R by approximating (in some sense) R by sets of the above form $\bigcup_k R_k$.

We assume that at least one of the above approaches can be made to work. This completes our construction of the PSPO's for the scalar field theory.

Turning to QED, we immediately encounter a difficulty. $P^{out}(\Omega_1 \times \ldots \times \Omega_n)$ clearly projects onto states with only n particles, but states in QED often have an infinite number of photons. We therefore adopt the following strategy. The preceding procedure will be used to define PSPO's P_+^{out} which analyze outgoing massive particles, but are inclusive with respect to massless particles. By an entirely different method, we will then define PSPO's P_0^{out} which analyze outgoing observable massless particles but are inclusive with respect to both massive particles and unobservable massless particles. The product

$P^{out} \equiv P_+^{out}\, P_0^{out}$ will then analyze all observable particles, but be inclusive over unobservable massless particles.

We now turn to the construction of P_+^{out}. Our starting point is again the equation

$$P_\mu^{out}(\Omega) = \lim_{t\to\infty} \int_{t\Omega} d\vec{x}\; T_\mu^o(t,\vec{x}) \quad , \tag{1}$$

[In a mathematical sense, $\lim_{t\to\infty} \int_{t\Omega} d\vec{x}\; T_\mu^o(t,\vec{x})$ is poorly defined even before the limit is taken. It can be made meaningful in a natural way (which we do not present) but certain assumptions are required. We do not discuss these assumptions except to assert that they are physically reasonable. In any case, we shall use the present, heuristic form.]

We now argue that $\left[P_\mu^{out}(\Omega_1), P_\nu^{out}(\Omega_2)\right] = 0$. First, we expect the result to hold if $\Omega_1 \cap \Omega_2 = \phi$, since then, for all $t > 0$, the two sets $\{(t,\vec{x})\,|\,\vec{x}\in t\Omega_i\}$ $(i=1,2)$ are space-like separated, and

$$\left[\int_{t\Omega_1} d\vec{x}_1\; T_\mu^o(t,\vec{x}_1)\;,\; \int_{t\Omega_2} d\vec{x}_2\; T_\nu^o(t,\vec{x}_2)\right] = 0 \quad .$$

Assuming some sort of strong convergence as $t\to\infty$ (which we do not describe), the result follows. It suffices, therefore, to consider only the case $\Omega_1 = \Omega_2$. Next, we assume that $P_\mu^{out}(\Omega)$ is independent of the position of the origin of our space-time coordinates, from which it follows that $[P_\mu^{out}(\Omega), P_\nu] = 0$. Finally, from $P_\nu = P_\nu^{out}(\mathbb{R}^3)$, we obtain

$$\left[P_\mu^{out}(\Omega), P_\nu^{out}(\Omega)\right] = \left[P_\mu^{out}(\Omega), P_\nu - P_\nu^{out}(\sim\Omega)\right] = 0 \quad .$$

Let $B = \{\vec{v}\in\mathbb{R}^3 \,|\, |\vec{v}| < 1\}$. Henceforward we tacitly require that all Ω lie in B. Furthermore, until we return to the subject of massless particles, the term "particle" will mean "massive particle."

We may now proceed rapidly. We define a no-particle (in Ω) state to be a state which is annihilated by $P_\mu^{out}(\Omega)$, and the projection operator onto such states we denote by $E_0(\Omega)$. A single-particle (in Ω) state is a normalizable eigenstate of $\sqrt{-P^{out}(\Omega)^2}$ with nonzero eigenvalue, and the projection operator onto such states is $E(\Omega)$.

We now decompose the space of single-particle (in Ω) states into orthogonal subspaces each containing a different type of particle (in Ω). We begin by diagonalizing $\sqrt{-P^{out}(\Omega)^2}$, yielding subspaces corresponding to particles (in Ω) of different mass. On these subspaces, we diagonalize

$$Q^{out}(\Omega) \equiv \lim_{t\to\infty} \int_{t\Omega} d\vec{x}\, J^o(t,\vec{x}) \quad ,$$

where J^μ is the electric current, yielding subspaces corresponding to particles (in Ω) of different mass and charge. (This diagonalization is possible only because $Q^{out}(\Omega)$ commutes with $P_\mu^{out}(\Omega)$, as may be established by mimicking similar considerations above.) Finally, we further diagonalize the spin operator $S_3^{out}(\Omega)$, yielding subspaces corresponding to particles (in Ω) of different mass, charge, and spin orientation.

[The subject of spin merits a paper in itself, and we do not define $\vec{S}^{out}(\Omega)$ here. We nevertheless assume that some natural definition can be given, that $\vec{S}^{out}(\Omega)$ commutes with $P_\mu^{out}(\Omega)$ and $Q^{out}(\Omega)$, and that $[S_i^{out}(\Omega),S_j^{out}(\Omega)] = i\,\varepsilon_{ijk}\, S_k^{out}(\Omega)$.[12] The reader dissatisfied with this is free to refrain from making the $S_3^{out}(\Omega)$ diagonalization. Almost all of the following considerations will still apply, and we shall point out those which do not.]

We label the various subspaces obtained by the final diagonalization above by the index i, and we let $E_i(\Omega)$ project onto the i-th subspace. In QED, we expect $i = 1,\ldots,4$, corresponding to two spin states for the electron and two spin states for the positron. For those skipping the $S_3^{out}(\Omega)$ diagonalization, $i = 1,2$. In either case, we define the PSPO's

$$P^{out}_{+i_1\ldots i_n}(\Omega_1 \times \ldots \times \Omega_n) \equiv E_{i_1}(\Omega_1)\ldots E_{i_n}(\Omega_n)\; E_0(B \sim \Omega_1 \cup \ldots \cup \Omega_n)\ , \qquad (2)$$

where the Ω_i are disjoint and $B \sim \Omega \equiv \{\vec{v} \in B \mid \vec{v} \notin \Omega\}$. The PSPO densities and $P^{out}_{+i_1\ldots i_n}(R)$ for arbitrary measurable R are obtained as in the previously considered scalar field theory.

This completes our derivation of the PSPO's P_+^{out} which analyze massive particles. Before we turn to the construction of P_0^{out}, we first switch over from velocity variables to momentum variables; that is, in the PSPO densities, we switch over from the variable $\vec{v}$ to the variable $\vec{p} = m\vec{v}/\sqrt{1-\vec{v}^2}$. Furthermore, we shall use the covariant integration measure $\overline{dp} \equiv d\vec{p}/(2\pi)^3\, 2p^o$ and the covariant δ-function $\overline{\delta}(p-q) \equiv (2\pi)^3\, 2p^o\, \delta(\vec{p}-\vec{q})$, where $p^o = \sqrt{\vec{p}^2 + m^2}$.

The construction of P_0^{out} proceeds along entirely different lines from the above construction of P_+^{out}. In fact, it resembles the LSZ formalism.

We define a massless particle state to be both annihilated by $\sqrt{-P^2}$ and orthogonal to the vacuum. In QED, the massless particles are just photons. We next need a field with nonvanishing matrix elements between massless particle states and the vacuum. We choose the electromagnetic field $F_{\mu\nu}$. By taking appropriate strong limits in <u>lightlike</u> directions, the free field $F_{\mu\nu}^{out}$ may be constructed.[13] (It follows that $F_{\mu\nu}^{out}$ commutes with P_+^{out}.) Given moderate assumptions, one can then extract from $F_{\mu\nu}^{out}$ annihilation operators $a_\lambda^{out}(k)$ and define $n_\lambda^{out}(k) = a_\lambda^{out}(k)^\dagger\, a_\lambda^{out}(k)$ away from $k=0$.[14] (λ labels photon helicity.)

For a given experiment, some photons will have too little energy to be observed. The observable photons are called hard, and we take our photon phase space to include only hard momenta.

P_0^{out} is uniquely defined if we require

$$\sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\lambda_1\ldots\lambda_n} \int_{\text{hard}} \overline{dk}_1 \ldots \overline{dk}_n\, P^{out}_{0\lambda_1\ldots\lambda_n}(k_1,\ldots,k_n) = 1$$

and

$$n_\lambda^{out}(k)\; P^{out}_{0\lambda_1\ldots\lambda_n}(k_1,\ldots,k_n) = \sum_{j=1}^{n} \delta_{\lambda\lambda_j}\, \bar{\delta}(k-k_j)\; P^{out}_{0\lambda_1\ldots\lambda_n}(k_1,\ldots,k_n) ,$$

where all momenta shown are hard. From these requirements it follows by straightforward combinatorics that

$$: n_{\lambda_1}^{out}(k_1) \ldots n_{\lambda_m}^{out}(k_m) :$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma_1\ldots\sigma_n} \int_{\text{hard}} \overline{d\ell}_1 \ldots \overline{d\ell}_n\; P^{out}_{0\lambda_1\ldots\lambda_m\sigma_1\ldots\sigma_n}(k_1,\ldots,k_m,\ell_1,\ldots,\ell_n) .$$

This equation may be solved for P_0^{out}, yielding

$$P^{out}_{0\lambda_1\ldots\lambda_m}(k_1,\ldots,k_m)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{\sigma_1\ldots\sigma_n} \int_{\text{hard}} \overline{d\ell}_1 \ldots \overline{d\ell}_n :n_{\lambda_1}^{out}(k_1)\ldots n_{\lambda_m}^{out}(k_m)\, n_{\sigma_1}^{out}(\ell_1)\ldots n_{\sigma_n}^{out}(\ell_n): . \tag{3}$$

Finally, if R_1 and R_2 are phase space regions for the (observable) massless and massive particles respectively, we define $P^{out}(R_1 \times R_2) \equiv P_0^{out}(R_1)\, P_+^{out}(R_2)$. P_0^{out} and P_+^{out} commute (since $F_{\mu\nu}^{out}$ and P_+^{out} commute), so P^{out} is in fact a projection operator, and it can be shown to have all the properties desired of a PSPO. The definition of $P^{out}(R)$ for arbitrary R is again assumed to proceed as in the previously considered scalar field theory.

3. CROSS SECTIONS

One of the key difficulties in defining cross sections in QED is the preparation of the initial state. Consider, for example, electron-electron scattering. Ideally, we would like an initial state consisting of a pair of electrons and nothing else. However, charged particles are inevitably accompanied by soft photons.[7] Furthermore, it is not possible to construct a charged state by applying observable fields to the vacuum.[1] Thus, to consider states of nonzero charge, we must depart from the context of Wightman field theory. This we choose not to do. It follows that the nicest initial state we can prepare for the purpose of electron-electron scattering contains two electrons, two positrons, and an infinite number of soft photons.

Thus, we are faced with the problem of unwanted particles in the initial state. Before attacking this problem, we first consider theories which, unlike QED, have an S-matrix. For such theories, we provide a derivation of cross sections from S-matrix elements which perhaps improves upon those found in the literature.[3-6] After demonstrating that this derivation yields the usual results, we rephrase it in terms of PSPO's.

We then reconsider the problem of unwanted particles in the initial state and propose a solution. We conclude with a heuristic argument that the proposed solution encounters no infrared divergences and in fact yields the traditional results.

We now present our derivation of cross sections from S-matrix elements. In any quantum mechanics experiment, one prepares a state and measures some observable. In our case, the observable $\mathcal{O}$ to be measured is just whether or not the outgoing particles lie in a given region R of phase space. That is,

$$\mathcal{O} = \int_R d\alpha \; |\alpha \text{ out}\rangle \langle \alpha \text{ out}| \quad .$$

To prepare the initial state, we begin with almost any two particle state

$$\rangle = \int \overline{dk} \; \overline{d\ell} \; \varphi(k,\ell) \; |k \; \ell \text{ in}\rangle \quad .$$

(We have suppressed the indices that label particle type. Also, the meaning of "almost any" will be discussed later.) Anticipating the collision of an incident particle with momentum K against a target particle with momentum L, we restrict the $\overline{dk}\;\overline{d\ell}$ integration to a small region $\Delta = \Delta_1 \times \Delta_2 \subset \mathbb{R}^6$ with $(\vec{K},\vec{L}) \in \Delta$. The state obtained by this restriction we call $|\Delta\rangle$.

In a real scattering experiment, one uses a beam of incoming particles with differing impact parameters. To incorporate various "impact parameters" in our initial state, we need operators that translate the incident particle transverse to the beam direction while leaving the target particle alone. Without going deeply into the matter, we simply make the reasonable assertion that the operator that generates translations of a collection of particles is just the momentum of those particles. In this case, the incident particle is translated along the vector a by the operator

$$\exp\left[-i\, a^{\mu} \int_{\Delta_1} \overline{dk}\; k_{\mu}\; n^{in}(k)\right] \qquad .$$

Applying this operator to $|\Delta\rangle$ yields a factor $e^{-ik\cdot a}$ in the integrand, and we call the resulting state $|\Delta a\rangle$.

To form a beam, we must average over vectors a transverse to the beam direction. For definiteness, we take $\vec{K}$ and $\vec{L}$ to lie along the z-direction, so we want $a = (0,a^1,a^2,0)$. Averaging over an area A in the x-y plane yields (using "A" for both a region in $\mathbb{R}^2$ and its measure)

$$A^{-1} \int_A d^2a \; \frac{\langle \Delta a | \mathcal{O} | \Delta a\rangle}{\langle \Delta | \Delta\rangle} \qquad .$$

The incident "flux" is clearly one particle per area A, so dividing by the flux simply eliminates the factor A^{-1}. We may now allow A to be all of $\mathbb{R}^2$. Finally, we are interested in the limit as the initial particles become momentum eigenstates, which we achieve by shrinking the region Δ down to a point (always keeping $(\vec{K},\vec{L}) \in \Delta$). Calling this limit $\lim\limits_{\Delta\to 0}$, we obtain

$$\sigma = \lim_{\Delta\to 0} \int d^2a \; \frac{\langle \Delta a | \mathcal{O} | \Delta a\rangle}{\langle \Delta | \Delta\rangle} \qquad . \tag{4}$$

That Eq. (4) yields the usual results is easily demonstrated. Assuming (as we must) that the phase space regions R and Δ are disjoint yields that

$$\langle \alpha \text{ out} | k\ell \text{ in}\rangle = i(2\pi)^4\, \delta(P_\alpha - k - \ell)\; T_{\alpha,k\ell}$$

where P_α' is the total momentum of the phase space point α and $T_{\alpha,k\ell}$ is the usual T-matrix element.[3] $\langle \Delta a|\mathcal{O}|\Delta a\rangle$ then equals

$$\int_\Delta \overline{dk}\,\overline{d\ell} \int_\Delta \overline{dk'}\,\overline{d\ell'}\; e^{-ia\cdot(k-k')}(2\pi)^4\,\delta(k+\ell-k'-\ell')\,\varphi(k',\ell')^*\varphi(k,\ell)$$

$$\times \int_R d\alpha\;(2\pi)^4\,\delta(P_\alpha-k-\ell)\;T^*_{\alpha,k'\ell'}\;T_{\alpha,k\ell} \quad .$$

The d^2a integration replaces $e^{-ia\cdot(k-k')}$ with $(2\pi)^2\delta(k^1-k'^1)\delta(k^2-k'^2)$ which, combined with $(2\pi)^4\delta(k+\ell-k'-\ell')$, is just

$$\bar{\delta}(k-k')\quad \bar{\delta}(\ell-\ell')\;\frac{1}{4}\left|k^3\ell^0-k^0\ell^3\right|^{-1} \quad .$$

The $\overline{dk'}\,\overline{d\ell'}$ integrals are now trivial. Dividing the result by $\langle\Delta|\Delta\rangle = \int_\Delta \overline{dk}\,\overline{d\ell}\,|\varphi(k,\ell)|^2$ and letting $\Delta\to 0$ yields

$$\sigma = \frac{1}{4}\left|K^3L^0 - K^0L^3\right|^{-1}\int_R d\alpha\;(2\pi)^4\,\delta(P_\alpha - K - L)\,\left|T_{\alpha,KL}\right|^2 \quad .$$

Since $\left|K^0L^3 - K^3L^0\right| = \sqrt{(K\cdot L)^2 - K^2L^2}$, this is the desired result.[3]

[In the preceding argument, we have tacitly assumed that the wave-function φ behaves reasonably at (K,L). Letting Δ denote both a region and its measure, it suffices to require

$$0 < \lim_{\Delta\to 0}\frac{1}{\Delta}\int_\Delta \overline{dk}\,\overline{d\ell}\,|\varphi(k,\ell)|^2 < \infty.] \qquad (5)$$

Equation (4) may readily be expressed in terms of PSPOs. $\mathcal{O}$ is of course just $P^{out}(R)$,

$$|\Delta a\rangle = \int_\Delta \overline{dk}\,\overline{d\ell}\;e^{-ik\cdot a}\;P^{in}(k,\ell)\rangle \quad ,$$

and Eq. (5) may be expressed as

$$0 < \lim_{\Delta\to 0}\frac{1}{\Delta}\langle\Delta|\Delta\rangle < \infty \quad .$$

We how attack the problem of extra particles in the initial state by considering electron-electron scattering. As noted earlier, we must consider states containing not only a pair of electrons, but also a pair of positrons and an infinite number of photons. Intuitively, we should solve the problem by sending the unwanted particles far away ("behind the moon"), but it is not clear how to implement this idea.

Perhaps the first idea to come to mind is to use behind-the-moon ideas to construct charged <u>states</u> containing only the desired incoming particles. Without going deeply into the matter, we note that the idea is ruined by infrared divergences: long range interactions between the unwanted and desired particles result in soft photon emissions and Coulomb distortion factors which do not settle down as the unwanted particles go off to infinity. To lessen these difficulties we proceed as follows: First, we define $\mathcal{O}$ and $|\Delta a\rangle$ as before except that we include the unwanted particles. They, <u>after</u> forming the quantity $\int d^2a \, \langle \Delta a | \mathcal{O} | \Delta a \rangle$, we translate the unwanted particles infinitely far away. This procedure lessens the infrared difficulties for the following reasons. First, soft photon emissions cause no problem because $\mathcal{O}$ is insensitive to them. (All our PSPOs are inclusive with respect to unobservable particles.) Second, Coulomb distortion should have less effect on a beam than on a single particle, since a beam is to some extent translation invariant.

Remarkably enough, the above procedure not only avoids some infrared difficulties, but in fact seems to avoid them altogether. To see this, we first describe the procedure in more detail, step by step.

1. Choose for the incoming electrons K, L, and Δ as before. Choose also $\overline{\Delta} = \overline{\Delta}_1 \times \overline{\Delta}_2 \subset \mathbb{R}^6$ for the incoming positrons.

2. Given $\rangle \in \mathscr{H}$, define

$$|\Delta\rangle = \int_{\Delta} \overline{dk}\ \overline{d\ell} \int_{\overline{\Delta}} \overline{dp}\ \overline{dq}\ P^{in}(k,\ell,p,q)\rangle$$

[We have suppressed the indices which label particle type. The first two particles are electrons and the second two positrons.] Find a state $\rangle$, obtained by applying observable fields to the vacuum, such that

a) $0 < \lim_{\Delta \to 0} \frac{1}{\Delta} \langle\Delta|\Delta\rangle < \infty$

b) the energy available to incoming photons in $|\Delta\rangle$ is less than the minimum energy of an observable particle. (This may be arranged by limiting the sizes of Δ, $\overline{\Delta}$, and the range of energies in $\rangle$.)

3. We take our experimental question to be whether or not the collision products of the two electrons lie in a "phase space" region R. Choose R. Make sure that no point in R contains an electron in either Δ_1 or Δ_2 or a positron in either $\overline{\Delta}_1$ or $\overline{\Delta}_2$. Define the phase space region R_0 to consist of all particle configurations obtained by adding a positron pair in $\overline{\Delta}$ to a particle configuration in R. Set $\mathscr{O} = P^{out}(R_0)$.

4. As before, translate the incident electron along the vector a by inserting the factor $e^{-ik\cdot a}$ into the integrand of $|\Delta\rangle$. Also insert $e^{-i\lambda a_1 \cdot p}$ and $e^{-i\lambda a_2 \cdot q}$, which translate the positrons along the non-collinear spacelike vectors a_1 and a_2. Finally, insert $e^{-i\lambda a_3 \cdot (k+\ell)}$, which simultaneously translates the two electrons. (This is simpler than translating the soft photons and, since $\mathscr{O}$ is translation invariant, physically equivalent.) Call the result $|\Delta a\rangle$.

5. $\sigma = \lim_{\Delta \to 0} \left(\lim_{\lambda \to \infty} \int d^2a \, \frac{\langle\Delta a|\mathscr{O}|\Delta a\rangle}{\langle\Delta|\Delta\rangle} \right)$

We now present a heuristic argument that the cross section σ thus defined is finite, independent of $\overline{\Delta}$, $>$, and a_i, and is in fact the same as the result traditionally obtained using S-matrix elements with an infrared cutoff.

The gist of our argument is as follows. Although the PSPOs are unambiguously defined in Section 2, the definitions are, at present, unsuitable for practical calculations. On the other hand, given LSZ scattering theory, the PSPOs are trivially obtained. (See the second paragraph of Section 2.) Therefore, for the purposes of calculation only (and only in the context of a heuristic argument), we introduce an infrared cutoff and use LSZ scattering theory to compute the PSPOs. We may therefore calculate σ in terms of cut diagrams. (After each step of the following procedure, we will argue that the infrared cutoff may be released without encountering divergences. Unfortunately, however, in the procedure itself, the infrared cutoff is to be left on until the final result is obtained.)

To begin, $\langle \Delta a | \mathcal{O} | \Delta a \rangle$ is expressed in terms of cut diagrams in Figure 1. The right-hand dot with particles coming out is sympolic for $\langle k\ell\ pq\ \gamma\ \text{in}|\rangle$. From bottom to top, the lines coming out symbolize a pair of electrons, a pair of positrons, and a collection of soft photons. (We use γ to label both the collection of photons and its total momentum.) The rightmost cut represents the integrals $\int_\Delta \overline{dk}\ \overline{d\ell}$, $\int_{\overline{\Delta}} \overline{dp}\ \overline{dq}$, and an integral over all photon momenta (this being a consequence of requirement b. in step 2 above.) T is the usual infrared cutoff T-matrix element associated with $\langle \alpha\ \text{out}|k\ell\ pq\,\gamma\ \text{in}\rangle$. The middle cut $\int d\alpha$ runs over the observable particle phase space region R_0 and is inclusive over soft photons. T*, the leftmost cut, and the left-hand dot are defined similarly to their right-hand counterparts. Suppressed in the diagram are the factors

$$(2\pi)^4\ \delta(P_\alpha - k - \ell - p - q - \gamma)\ (2\pi)^4\ \delta(k + \ell + p + q + \gamma - k' - \ell' - p' - q' - \gamma')$$
$$e^{-i(k-k')\cdot a}\ e^{-i\lambda a_1\cdot(p-p')}\ e^{-i\lambda a_2\cdot(q-q')}\ e^{-i\lambda a_3\cdot(k+\ell-k'-\ell')}\quad .$$

As noted above, it is reasonable to expect that no infrared divergences arise in the construction of the PSPOs.[2] Hence $\langle\Delta a|\mathcal{O}|\Delta a\rangle$, and consequently Figure 1 summed over all contributions, is infrared finite.

We would next like to do the $\int d^2a$ integration, but we must first argue that $\langle\Delta a|\mathcal{O}|\Delta a\rangle$ falls off rapidly enough as $a^2 \to \infty$ for the integral to converge. Our argument proceeds on physical grounds: Because the outgoing phase space region R contains no electrons in Δ_1, the incident electron must interact if $\langle\Delta a|\mathcal{O}|\Delta a\rangle$ is to be nonvanishing. As a^2 gets large, the incident electron moves away from all the incoming particles, so the question is whether or not the interactions are sufficiently long range to prevent the d^2a integral from converging. To test this, we consider nonrelativistic Coulomb scattering. Appropriately modifying the above definition of σ so that it applies to potential scattering (which isn't difficult), we ask whether or not the d^2a integral converges. Using the results of Ref. 15 and 2, it can be rigorously proved that the integral converges and, in fact, that the usual exact Coulomb cross section[16] is obtained.

So, we expect the d^2a integration to be well defined. The resulting δ-function, combined with the factor $(2\pi)^4\ \delta(k + \ell + p + q + \gamma - k' - \ell' - p' - q' - \gamma')$, may be used to do the $\overline{dk'}\ \overline{d\ell'}$ integrals. To remind ourselves that the integrals have been done, we put a pair of dots on the relevant cut lines. Furthermore we use the factor $(2\pi)^4\ \delta(P_\alpha - k - \ell - p - q - \gamma)$ to partially do the $d\alpha$ integral, denoting this graphically with a "δ" near the $d\alpha$ cut. This yields the equation

$$\int d^2a\ \langle \Delta a|\mathscr{O}|\Delta a\rangle = \text{Figure 2} \quad .$$

Before taking the behind-the-moon limit $\lambda \to \infty$, it is useful to decompose Figure 2 into the sum of two contributions: those graphs of the form shown in Figure 3 plus those graphs which are not. For the moment, let us consider the graphs in Figure 3. With these, the behind-the-moon limit is trivial: the graphs contain δ-functions setting $p = p'$ and $q = q'$, and the dots set $k' = k$ and $\ell' = \ell$. Thus the λ dependent phase factors are all unity. Furthermore, by the unitarity of the cutoff S-matrix, we may replace Figure 3 by Figure 4. But Figure 4 is simply

$$\int_\Delta \overline{dk}\ \overline{d\ell} \int_{\overline{\Delta}} \overline{dp}\ \overline{dq}\ \langle P^{in}(k,\ell,p,q)\rangle\, \frac{1}{4}\, |k^3\ell^0 - k^0\ell^3|^{-1}\, W(k,\ell) \quad ,$$

where $W(k,\ell)$ is defined graphically in Figure 5. Now, the graphs in Figure 3 were chosen precisely because they contain the δ-functions required to eliminate the λ-dependent phase factors. The remaining contributions to Figure 2 do not have this property and vanish as $\lambda \to \infty$, as may be shown by using Riemann-Lebesgue arguments. (This is true only because we required in step 3 that no point in R contain an electron in Δ_2. Without that requirement, graphs of the form shown in Figure 6 would contribute to Figure 2. Such graphs are λ-independent but are not of the form shown in Figure 3.)

Thus, the $\lambda \to \infty$ limit leaves us with Figure 4, which is infrared finite. (The infrared finiteness of $W(k,\ell)$ is proved for example in

Ref. 17.) Finally, dividing by

$$\langle\Delta|\Delta\rangle = \int_{\Delta} \overline{dk}\ \overline{d\ell} \int_{\overline{\Delta}} \overline{dp}\ \overline{dq}\ \langle P^{in}(k,\ell,p,q)\rangle$$

and letting $\Delta \to 0$ yields

$$\sigma = \frac{1}{4} |K^3L^0 - K^0L^3|^{-1} W(K,L) \quad ,$$

which is the usual result.

This completes our heuristic argument that σ defined above is finite, independent of $\overline{\Delta}$, $>$, and a_i, and equal to the traditional result. Two comments:

Had we considered (perturbative) quark-quark scattering rather than electron-electron scattering, the above argument would have failed. In particular, the QCD analog of $W(k,\ell)$ is infrared divergent.[18] Thus, we have failed to define σ for quark-quark scattering. This is as it should be: physical arguments indicate that the quark-quark cross section simply does not exist. (We should perhaps point out that the results of Lee and Nauenberg[19] do not guarantee the existence of cross sections or, for that matter, of any physically observable quantity.)

Also, those readers who refrained from making the $S_3(\Omega)$ diagonalization in Section 2 will obtain results slightly different from those above. Without the $S_3(\Omega)$ diagonalization, the PSPOs cannot be used to specify the spins of the incoming electrons. Thus, cross sections which are sensitive to the spins of the incoming electrons will be $>$-dependent.

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FIGURE CAPTIONS

Fig. 1. $\langle \Delta a|\mathcal{O}|\Delta a\rangle$ expressed diagrammatically.

Fig. 2. $\int d^2a\, \langle \Delta a|\mathcal{O}|\Delta a\rangle$ expressed diagrammatically.

Fig. 3. Some contributions to Fig. 2.

Fig. 4. Figure 3 reexpressed.

Fig. 5. Definition of $W(k,\ell)$.

Fig. 6. A graph which fortunately does not contribute to Fig. 2.

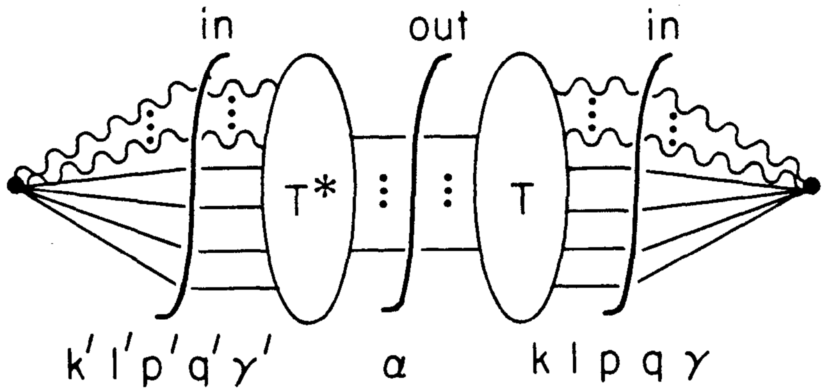

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Fig. 1

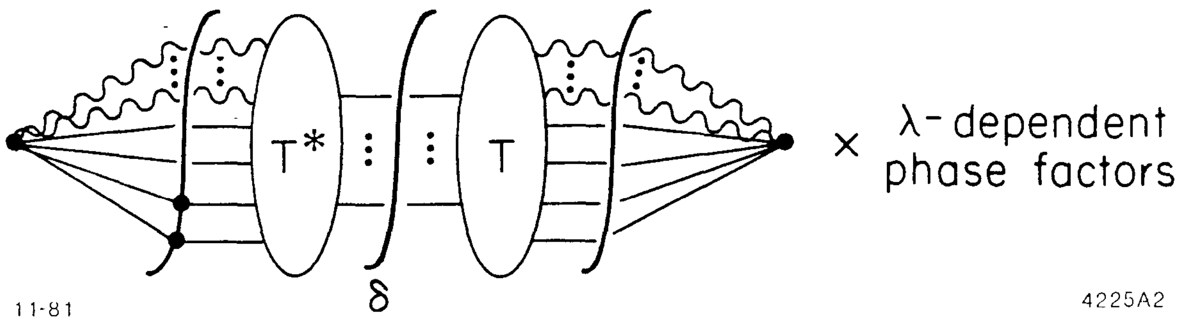

Fig. 2

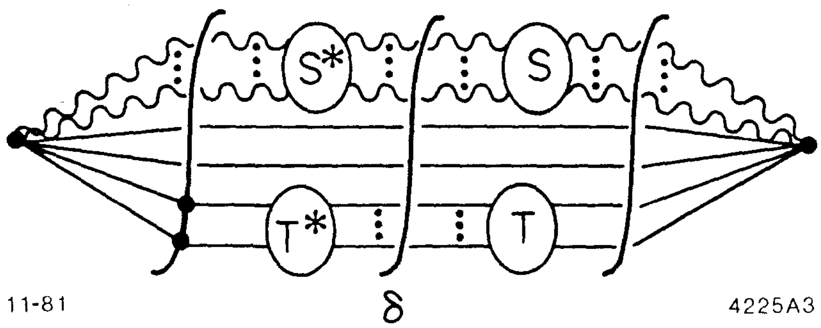

Fig. 3

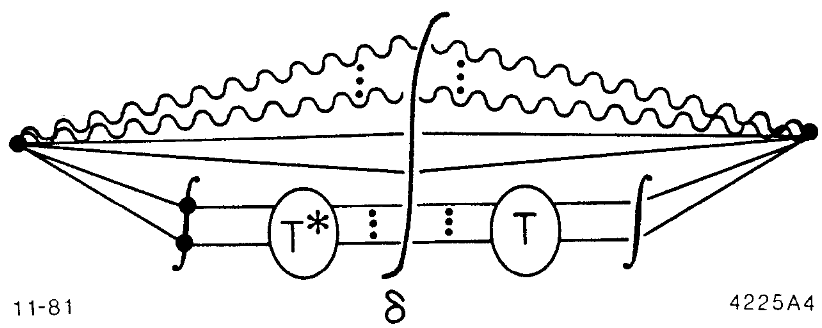

Fig 4

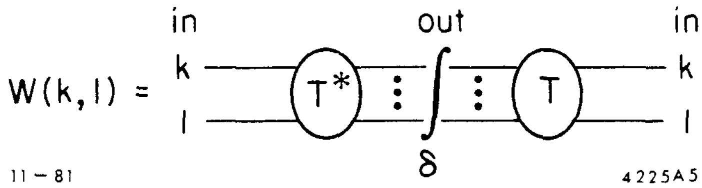

Fig. 5

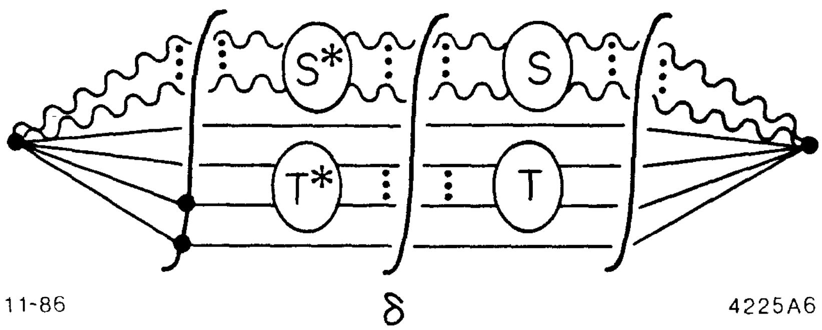

Fig. 6