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HAS $\langle G^2_{\mu\nu} \rangle \neq 0$ BEEN PROVEN IN MASSLESS QCD?*

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ABSTRACT

It is shown that a recent proof that $\langle G^2_{\mu\nu} \rangle \neq 0$ in QCD with massless quarks is in error.

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A number of authors have considered the possibility that the vacuum of quantum chromodynamics (QCD) is a nonperturbative vacuum with $\langle G^2_{\mu\nu}\rangle \neq 0$.[1] Fukuda and Kazama[2] have offered an ingenious proof of this conjecture for QCD with massless quarks. Unfortunately, a key part of this proof is in error due to an incorrect manipulation involving the Legendre transform.

In their proof, Fukuda and Kazama introduce a constant external source J coupled to $G^2_{\mu\nu}$ in the Lagrangian so that (for bare quantities)

$$\frac{\partial W(J)}{\partial J} = \Omega \, G^2_{\mu\nu} \tag{1}$$

where $\exp(iW)$ is the generating functional and Ω the spacetime volume. They point out that the (bare) Lagrangian with the term $J \cdot G^2_{\mu\nu}$ added is equivalent to a (bare) Lagrangian without such a term if one rescales the (bare) field and coupling constant:

$$\begin{aligned} g_J^2 &= \frac{g^2}{1+J} \\ A_J^\mu &= A^\mu \sqrt{1+J} \quad . \end{aligned} \tag{2}$$

They further identify $W(J)/\Omega$ as $\varepsilon(J)$, the vacuum energy density, to obtain

$$\frac{d\varepsilon(J)}{dJ} = \left\langle G^2_{\mu\nu} \right\rangle \quad . \tag{3}$$

They then employ the trace anomaly equation for the stress-energy tensor of Collins et al.,[3] to conclude that

$$\varepsilon(J) = \frac{\beta(g_J)}{g_J}\left(\frac{1+J}{2}\right)\frac{d\varepsilon}{dJ} \quad . \tag{4}$$

This equation is the key to their argument: by a solely mathematical argument without any other physical input, they use the Legendre transform to argue that (4) implies that $d\varepsilon/dJ$ [which equals $\langle G^2_{\mu\nu}\rangle$ by (3)] must be nonzero. We have no quarrel with (4) itself: it is the mathematical manipulation of (4) which we will show to be in error.

Before discussing this error, we must mention that the Fukuda-Kazama derivation just described has generated criticism concerning the identification of ε with W [and therefore Eq. (4)] and concerning the passage from bare to renormalized quantities.[4] These criticisms can largely be avoided by an alternative derivation: (3) and (4) taken to refer to renormalized quantities can be shown to be logically equivalent to the trace anomaly equation combined with the assumption that $\varepsilon(J)$ scales according to its naïve engineering dimension.

Let $g_J^2 = g^2/(1+J)$ where g and g_J are now renormalized coupling constants defined at the same renormalization point Λ_o. Now, rescaling the physical g^2 at constant Λ_o is equivalent to holding g^2 fixed while changing the renormalization point Λ: that is, going from g^2 to g_J^2 is equivalent to simply changing the characteristic length scale Λ. (J is now just a rescaling factor and is not assumed to have any physical meaning.)

Now, suppose that $\varepsilon(J)$ is the (possibly zero) vacuum energy density. Since, in a massless theory, Λ is the only length scale, by dimensional analysis we must have, if ε is a physical quantity with the correct naive dimension,

$$\varepsilon = \frac{K}{\Lambda^4} \quad . \tag{5}$$

Thus,

$$\frac{d\varepsilon}{d\Lambda} = -\frac{4K}{\Lambda^5} = -\frac{4\varepsilon}{\Lambda}$$

and

$$\frac{d\varepsilon}{dJ} = \Lambda\frac{d\varepsilon}{d\Lambda}\ \frac{1}{\left(\Lambda\dfrac{dg_J}{d\Lambda}\right)}\ \frac{dg_J}{dJ}$$

$$= 2\varepsilon\,\frac{1}{\beta(g_J)}\ \frac{g_J}{1+J} \quad . \tag{6}$$

But the trace-anomaly equation gives

$$\varepsilon(J) = \frac{2\beta(g_J)}{g_J}\left\langle G^2_{\mu\nu J}\right\rangle \quad . \tag{7}$$

We see that (6) and (7) imply

$$\varepsilon(J) = \frac{\beta(g_J)}{2g_J}\ (1+J)\ \frac{d\varepsilon(J)}{dJ} \tag{8}$$

$$\left\langle G^2_{\mu\nu J}\right\rangle = (1+J)\ \frac{d\varepsilon(J)}{dJ} \tag{9}$$

which are superficially the same as (3) and (4)--except that (8) and (9) refer directly to renormalized quantities. One can trivially reverse the reasoning to show that (8) and (9) imply both the trace-anomaly equation and the statement that ε scales with its naïve engineering dimension.

We do <u>not</u> assert that this equivalence proves (8) and (9): one[4] might deny that $\varepsilon(J)$ is a physical quantity with the correct engineering dimension or question whether the trace-anomaly equation is correct or even meaningful for a nonperturbative vacuum. [Such strictures also apply to Fukuda and Kazama's proof for (8) and (9).]

We are agnostic as to whether (8) and (9) are true. The above alternative derivation of (8) and (9), aside from being simpler and less open to criticism than Fukuda and Kazama's, will be of use in showing that, even if (8) and (9) are true, (8) and (9) do not imply $\langle G_{\mu\nu}^2\rangle \neq 0$ as Fukuda and Kazama assert.

Fukuda and Kazama now form the Legendre transform

$$V\left(\frac{d\varepsilon}{dJ}\right) = \varepsilon(J) - J\ \frac{d\varepsilon(J)}{dJ} \tag{10}$$

so that they can use the standard formula

$$V'\left(\frac{d\varepsilon}{dJ}\right) = -\,J \tag{11}$$

to eliminate J from (8).

This elimination of J via (11) produces an equation all solutions of which can be shown to have $d\varepsilon/dJ \neq 0$ for $J = 0$.

One can easily show that there must be an error in this reasoning. For, since it involves a purely mathematical transformation of (8) without any additional input, any solution of (8) must, if the reasoning is valid, be also a solution of the transformed equation: (8) alone cannot be used to show that one of its solutions is not a solution. Yet, $\varepsilon(J) \equiv 0$ is manifestly a solution of (8) but does not satisfy the criterion $d\varepsilon/dJ \neq 0$ at $J = 0$ which is supposedly derived from (8). Therefore, the argument which derives

$$\frac{d\varepsilon}{dJ} \neq 0$$

at $J = 0$ from (8) must be invalid.

While this suffices, strictly speaking, to prove that Fukuda and Kazama's argument is in error, we should also point out which specific step is wrong and why.

Since the purpose of the proof is to show that $d\varepsilon/dJ \neq 0$, any step which requires one to assume $d\varepsilon/dJ \neq 0$ is an invalid step. We will now show that Eq. (11) is such a step.

Now, from (10),

$$dV = d\varepsilon - dJ \frac{d\varepsilon}{dJ} - J\, d \frac{d\varepsilon}{dJ}$$

$$= -J\, d \frac{d\varepsilon}{dJ} \quad . \tag{12}$$

If $d \frac{d\varepsilon}{dJ}$ does not vanish, one can divide through by $d \frac{d\varepsilon}{dJ}$ and conclude with Fukuda and Kazama that $V' = -J$. If $d \frac{d\varepsilon}{dJ}$ does vanish, dividing through by $d \frac{d\varepsilon}{dJ}$ is dividing by zero and is impermissible. In short, one can obtain (11) only if one is allowed to assume that $d \frac{d\varepsilon}{dJ}$ does not vanish.

However, one can easily show that to assume that $d \frac{d\varepsilon}{dJ}$ is nonzero requires one also to assume that $d\varepsilon/dJ\big|_{J=0}$ is nonzero--thereby begging the point to be proven.

For, if $d\varepsilon/dJ\big|_{J=0}$ vanishes, by (8) we have $\varepsilon(g^2) = \varepsilon(J=0) = 0$. Therefore, K in (5) is zero, and $\varepsilon(J)$ and hence $\frac{d\varepsilon}{dJ}$ and $d \frac{d\varepsilon}{dJ}$ are identically zero for all J: $d\varepsilon/dJ\big|_{J=0} = 0$ implies that $d \frac{d\varepsilon}{dJ}$ vanishes for all J. By contraposition, $d \frac{d\varepsilon}{dJ} \neq 0$ for any J implies $d\varepsilon/dJ\big|_{J=0} \neq 0$; to assume $d \frac{d\varepsilon}{dJ} \neq 0$ is therefore to assume $d\varepsilon/dJ\big|_{J=0} \neq 0$, thus begging the question.

Indeed, unless one assumes that $d\varepsilon/dJ\big|_{J=0} \neq 0$, one does not even know that (11) is meaningful; for, if $\varepsilon(J) \equiv 0$ (which is so unless $d\varepsilon/dJ\big|_{J=0} \neq 0$), $d\varepsilon/dJ$ assumes only a single value. There would then be only a single point in the domain of V and the derivative of V would not exist. Equation (11) cannot be shown to even be meaningful without assuming the point to be proved.

Fukuda and Kazama's error in their derivation of (11) from (8) and (9) is not simply a technical loophole in the proof which might somehow be plugged: we have proved that (8) and (9) alone cannot imply $\langle G_{\mu\nu}^2 \rangle \neq 0$. Only if one knew on some other basis that $\langle G_{\mu\nu}^2 \rangle \neq 0$ might one make use of (8) and (9) to calculate $\langle G_{\mu\nu}^2 \rangle$.

We have argued that, while (8) and (9) can be put on a firmer footing, $\langle G_{\mu\nu}^2 \rangle \neq 0$ cannot be derived from (8) and (9). Of course, it is still possible, though unproven, that $\langle G_{\mu\nu}^2 \rangle \neq 0$: it remains an open question.

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