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LOOP DIAGRAMS IN BOXES*

C. Peterson†#
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

T. H. Hansson††
Institute for Theoretical Physics
University of California, Santa Barbara, California 93103##

and

NORDITA
Bledgamsvej 17, DK-2100 Copenhagen, Denmark

K. Johnson†††
Institute for Theoretical Physics
University of California, Santa Barbara, California 93103

ABSTRACT

Methods for computing loop-diagrams in confined scalar and gauge theories are developed. We construct propagators with different types of boundary conditions in boxes (rectangular cavities) using the method of image charges, which allows for a separation of the short-distance singularities. The techniques are illustrated by a calculation of the Casimir effect in a box using a covariant gauge, and of the energy-shift of a photon in confined scalar QED. The relevance of such a calculation for a proposed model for the QCD vacuum is discussed.

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##Address after 1 September 1981.
†††On leave from Center for Theoretical Physics, MIT, Cambridge, Massachusetts 02139.

I. Introduction

In some applications of quantum field theory, the fields must be defined, not in whole space, but in finite cavities and with non-periodic boundary conditions on the walls. The presence of boundaries can give rise to new quantum phenomena, the classical example being the Casimir effect in QED.[1] More recently the bag model for hadrons[2] has increased the interest in confined field theories.

In this paper we develop techniques for computing quantum (i.e., loop) corrections in charged scalar and gauge theories. The relevance to the bag model of a scheme for calculating loop corrections in confined QCD is obvious. Since our technique is restricted to rectangular cavities and does not permit spinor fields, it is, however, not directly applicable to most phenomenological bag calculations. It will, though, enable us to indirectly check the validity of approximate methods frequently used to handle loop diagrams in the bag.

The main physical motivation for this work is however not to improve on bag phenomenology, but rather to understand qualitative features of a recently proposed model in which the QCD vacuum is densely filled by $J^{PC} = 0^{++}$ two gluon glueballs.[3] The reason for believing in such a glueball "condensate" is the large negative colormagnetic interaction energy of two gluons in a 0^{++} color singlet state. In fact, by calculating the magnetic part of the diagrams of Fig. 1a,b in a static spherical cavity with radius R, one obtains after correcting for the c.m.s. motion of the gluons[3,4]

$$M^2_{0^{++}} = \left(2 \times \frac{2.74}{R} - \alpha_s(R)\,\frac{1.58}{R}\right)^2 - 2 \times \left(\frac{2.74}{R}\right)^2 \qquad (1)$$

Since the strong coupling constant increases with R we expect $M^2(R)_{min} < 0$. The presence of a tachyon signals an instability of the perturbative vacuum against creation of 0^{++} glueballs. Of course these will interact, but as shown in Ref. 3 the overlap is small, and we are thus led to the picture of the QCD vacuum as a condensate of 0^{++} glueballs. It is important to check whether the minimum of Eq. (1) is stable against variations of the shape of the glueball; it might, for example, occur for a tubelike configuration. Secondly, Eq. (1) is based on the approximation of keeping only the magnetic contribution to the interaction energy[5] but to be more precise we also need the electric part of Fig. 1a,b as well as the self energy diagrams of Fig. 2. We hope that the methods developed in the next sections will provide means to handle these problems.

The plan of the paper is as follows. Section II begins with a general discussion of confined perturbation theory. We then construct propagators for scalar fields in a rectangular cavity with various boundary conditions. Next gauge fields are treated and ghosts with proper boundary conditions are introduced. A first application of our methods is given in Section III where we calculate the Casimir effect in a box using a covariant gauge. The second application is to compute the energy shift of the lowest confined photon mode in scalar QED in a cube, which is done in Section IV. Some integrals needed for this calculation are listed in Appendix C, and some intermediate results in Appendix D. Appendix A deals with properties of the cavity propagators and in Appendix B we derive the boundary conditions for the ghost fields using functional techniques.

II. Cavity Propagators

i. General Considerations

To do perturbation theory, one needs vertices and propagators. The vertices are local functions and will not be changed by confining the theory to a cavity. The propagators, however, are modified so that the fields from a source fulfill the prescribed boundary conditions on the cavity walls. A straightforward method to construct the confined Green functions is by summing over cavity eigenmodes.[6] So is, for example, the Feynman propagator for a real scalar field given by

$$i\Delta(x,y) = \int d\omega \sum_N \frac{\varphi_N(x)\ \varphi_N(y)}{\omega^2 - \omega_N^2 - i\varepsilon}\, e^{i\omega(x_0-y_0)} \tag{2}$$

where N labels the solution $\varphi_N(x)$ to the field equation with suitable boundary conditions, and ω_N is the energy of the mode. This way of writing the cavity propagator is unfortunately not very well suited for calculations of loop diagrams. The reason is that in all such calculations one has to deal with divergences arising from the singular short distance behavior of the propagator. In unconfined field theory these infinities appear as divergent momentum integrals which are handled by the usual regularization and renormalization procedures. In the case of a confined theory using the propagator in Eq. (2) we instead encounter divergent sums which are much harder to deal with. The way to overcome this difficulty, as will be discussed and exemplified in Sections III and IV is by separating the propagator into two parts:

$$\Delta(x,y) = \Delta_0(x,y) + \Delta_B(x,y) \tag{3}$$

Here the first term, which is the free propagator, contains the usual short distance singularity whereas the boundary term $\Delta_B(x,y)$ is regular at $x = y$ except perhaps on the boundary itself. For an arbitrary geometry, we do not know how to construct Δ_B, but in the case of rectangular cavities the problem can be solved by summing over mirror images as shown below.

Before discussing the various propagators in detail, we comment on the relevance of considering rectangular cavities. Although the presence of sharp edges and corners are unphysical features, we have reason to believe that these are not very important. So is, for example, the energy of the lowest mode of the e.m. field in a cubical conducting cavity of volume V

$$E_0^{cube} = \frac{4.44}{V^{1/3}} \tag{4}$$

while the corresponding lowest energy in a sphere is

$$E_0^{sphere} = \frac{4.42}{V^{1/3}} \tag{5}$$

Also if we compare the "inside" (cf. the discussion in Section IV) contribution to the Casimir energy in the two cases

$$E_C^{cube} = \frac{0.091}{V^{1/3}} \tag{6}$$

$$E_C^{sphere} = \frac{0.104}{V^{1/3}} \tag{7}$$

the similarity is striking, and strongly suggests that the finite part of the energy is insensitive to the shape of the cavity.

ii. Scalar Fields

In order to construct the propagator for a scalar field, confined in a rectangular cavity, we use the image charge method introduced by Lukosz.[7] Here, the boundary conditions are fulfilled by adding the contributions of direct propagation from the source point to those from an infinite number of mirror sources obtained by reflections in the walls. This is a particularly simple case of the more general method of multiple scattering expansion of propagators. Because of the simple geometry we can obtain analytic expressions for the propagators, which is usually not possible. In Ref. 8 the electromagnetic field is treated in a general cavity using the multiple reflection method.

For a box with sides ℓ_i centered around origo, the positions of the mirror charges are given by (index i not summed over)

$$y_i^N = (-1)^{n_i} y_i + n_i \ell_i \tag{8}$$

where $N = (n_1,n_2,n_3)$, y is the position of the original source ($y_0^N = y_0$) and the n_i the number of reflections in the i^{th} direction. The solution to the Neumann boundary value problem

$$\Box_{(x)} \Delta(x,y) = -\delta^{(4)}(x-y) \tag{9a}$$

$$\frac{\partial}{\partial x_i} \Delta(x,y) = 0 \quad \text{on} \quad x_i = \pm\frac{1}{2}\ell_i \tag{9b}$$

is given by[9]

$$\Delta(x,y) = \sum_N \Delta_0^N (x - y^N) \tag{10}$$

where Δ_0 is the free propagator and the sum runs over all integers.

As it stands, the N-sum in Eq. (10) is only conditionally convergent, and a prescription must be given in order to define this and similar expressions that will occur below. Such a prescription essentially involves a redefinition of the sum by grouping several terms together in such a way that the new sum is absolutely convergent. This is described in detail in Appendix A. Similarly the Dirichlet boundary value problem

$$\Box_{(x)} \Delta(x,y) = -\delta^{(4)}(x-y) \tag{11a}$$

$$\Delta(x,y) = 0 \quad \text{on} \quad x_i = \pm\frac{1}{2}\ell_i \tag{11b}$$

is solved by

$$\Delta(x-y) = \sum_N (-1)^{n_1+n_2+n_3} \Delta_0(x-y^N) \tag{12}$$

The generalization to mixed boundary conditions is straightforward. It is obvious that Eqs. (10) and (12) separate out the free part $\Delta_0(x-y)$, since $y^{(0,0,0)} = y$. Singularities occur for $y^N = x$, which means that all terms with two or more reflections in any direction are regular. Terms with one reflection in one direction are only singular on a surface, etc. (see Fig. 3). Of course it does not matter which point is considered as source, i.e.,

$$\sum_N \Delta(x-y^N) = \sum_N \Delta(x^N-y) \tag{13}$$

iii. Gauge Fields and Ghosts

Now consider a Yang-Mills field[10] (QCD) in a covariant gauge defined by

$$\mathscr{L}_{YM} + \mathscr{L}_{GF} = -\frac{1}{4} G_{\mu\nu} G^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \tag{14}$$

where $A = A_a T^a$, $G^{\mu\nu} = G_a^{\mu\nu} T^a$, T^a being the generators of the gauge group, and

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \tag{15}$$

The field equations read

$$\partial_\mu G^{\mu\nu} + \frac{1}{\alpha}\partial^\nu \partial_\mu A^\mu = ig[G^{\mu\nu},A_\mu] \tag{16}$$

Again consider a rectangular cavity, and impose the gauge invariant "bag" boundary condition

$$n_\mu G^{\mu\nu} = 0 \qquad \text{on} \quad \Gamma \tag{17}$$

where $n_\mu = (0,\vec{n})$, $\vec{n}$ being the outside unit normal vector on the boundary Γ of the cavity. This boundary condition is "confining" in the sense that the normal component of the (color) current

$$j^\mu_a = ig[G^{\sigma\mu}, A_\sigma]_a \tag{18}$$

vanishes on the cavity walls in all gauges. Written in terms of E- and B-fields Eq. (18) reads

$$\vec{n}\cdot\vec{E} = 0 \tag{19a}$$

$$\text{on} \quad \Gamma$$

$$\vec{n}\times\vec{B} = 0 \tag{19b}$$

which are recognized as the boundary conditions on a perfect magnetic conductor. It is easy to convince oneself that for a rectangular cavity the boundary conditions Eqs. (17) or (19) can be realized in terms of potentials as

$$\vec{n}\cdot\bar{A} = 0 \tag{20a}$$

$$(\vec{n}\cdot\nabla)\,\vec{n}\times\bar{A} = 0 \qquad \text{on} \quad \Gamma \tag{20b}$$

$$\vec{n}\cdot\nabla A^0 = 0 \tag{20c}$$

The propagator is the solution of Eq. (17) for a point source, i.e.,[11]

$$\Box_{(x)} D^{\mu\nu}_{(\alpha)}(x,y) + \left(1-\frac{1}{\alpha}\right)\partial^\mu_{(x)}\,\partial^{(x)}_\sigma\, D^{\sigma\nu}_{(\alpha)}(x,y) = g^{\mu\nu}\,\delta^{(4)}(x-y) \tag{21}$$

All propagators are diagonal in color, i.e., $(D^{\mu\nu}_{(\alpha)})_{ab} = \delta_{ab} D^{\mu\nu}_{(\alpha)}$ etc. Together with the boundary conditions of Eq. (20) this completely determines $D^{\mu\nu}_{(\alpha)}(x,y)$. Now choose Fermi gauge ($\alpha = 1$), i.e.,

$$\Box_{(x)} D^{\mu\nu}(x,y) = g^{\mu\nu} \delta^{(4)}(x-y) \tag{22}$$

Using the same method as in the scalar case we get

$$D^{\mu\nu}(x,y) = -g^{\mu\nu} D^{(\mu)}(x,y) \tag{23}$$

where

$$D^{(0)}(x,y) = \sum_N \Delta_N(x-y^N) \tag{24a}$$

$$D^{(i)}(x,y) = \sum_N (-1)^{n_i} \Delta_N(x-y^N) \tag{24b}$$

From Eq. (8) we immediately have the useful identities:

$$\partial_\mu^{(x)} D^{(\mu)}(x,y) = -\partial_\mu^{(y)} D^{(0)}(x,y) \tag{25a}$$

$$\partial_\mu^{(x)} D^{(0)}(x,y) = -\partial_\mu^{(y)} D^{(\mu)}(x,y) \tag{25b}$$

To get to an arbitrary covariant gauge characterized by α we make the gauge transformation

$$\tilde{A}^\mu = A^\mu + \partial^\mu \phi \tag{26}$$

where A^μ is given in Fermi gauge. From the requirement

$$\frac{1}{2\alpha} (\partial_\mu \tilde{A}^\mu)^2 = \frac{1}{2} (\partial_\mu A^\mu)^2 \quad , \tag{27}$$

one obtains

$$\Box \phi = (\sqrt{\alpha} - 1)\, \partial_\mu A^\mu \tag{28}$$

Now it follows from Eqs. (20) and (22) that

$$(\vec{n} \cdot \nabla)\, \partial_\mu A^\mu = 0 \qquad \text{on} \quad \Gamma \tag{29}$$

so Eq. (28) can be inverted to

$$\phi = (\sqrt{\alpha} - 1)\, D^{(0)}\, \partial_\mu A^\mu \tag{30}$$

Now,

$$iD^{\mu\nu}(x,y) = \langle 0 \,|\, P(A^\mu(x)\; A^\nu(y)) \,|\, 0 \rangle \tag{31a}$$

$$iD_\alpha^{\mu\nu}(x,y) = \langle 0 \,|\, P(\tilde{A}^\mu(x)\; \tilde{A}^\nu(y)) \,|\, 0 \rangle \tag{31b}$$

where P stands for T, [,] etc., together with Eqs. (25), (26), and (30) gives after some algebra

$$D^{\mu\nu}_{(\alpha)}(x,y) = D^{\mu\nu}(x,y) + (1-\alpha)\, \partial^\mu_{(x)}\, \partial^\nu_{(y)}\; D^{(0)}(x,z)\, D^{(0)}(z,y) \tag{32}$$

Here the product $D^{(0)}D^{(0)}$ is to be understood in the sense of matrix multiplications. In a similar way we can obtain the propagator in other gauges, as for example, Coulomb gauge.

It is well known that in order to quantize a gauge theory in a covariant gauge ghosts are needed. For an Abelian theory, the ghosts do not couple to anything and are usually neglected, but in this case we must consider them as discussed in Section III. In Appendix B we show how to modify the conventional Faddeev-Popov procedure in order to take the boundary conditions Eq. (20) into account, and here only the result is stated. The ghost Lagrangian is

$$\mathscr{L}_{Gh} = -(\partial_\mu C^+)(\partial^\mu C - ig\,[A^\mu, C]) \tag{33}$$

and is now supplemented with the boundary condition

$$\vec{n} \cdot \nabla C = 0 \qquad \text{on} \quad \Gamma \tag{34}$$

giving the propagator

$$S(x,y) = D^{(0)}(x,y) \tag{35}$$

Ghost loops as usual carry a minus sign. Note that the contribution to the current from the ghostfields is

$$j^{\mu}_{Gh} = ig\,[\partial^{\mu} C^{+}, C] \tag{36}$$

so Eq. (34) is consistent with $n_{\mu} j^{\mu} = 0$ on the boundary.

We shall finally comment on another possible set of boundary conditions. As pointed out earlier the boundary condition Eq. (20) corresponds to a perfect magnetic conductor. In the case of QED the dual condition, i.e., that of a perfect ordinary conductor, is more interesting

$$n_{\mu}\,\tilde{G}^{\mu\nu} = n_{\mu}\,\varepsilon^{\mu\nu\sigma\lambda}\,G_{\sigma\lambda} = 0 \qquad \text{on} \quad \Gamma \tag{37}$$

or

$$\vec{n} \times \vec{E} = 0 \tag{38a}$$

on Γ

$$\vec{n} \cdot \vec{B} = 0 \tag{38b}$$

An analysis like the one above gives in this case the propagator

$$\tilde{D}^{\mu\nu}(x,y) = g^{\mu\nu}\, D^{(\mu)}(x,y) \tag{39}$$

where

$$\tilde{D}^{(0)}(x,y) = -\sum_{N} (-1)^{\sum_k n_k}\, \Delta(x - y^{N}) \tag{40a}$$

$$\tilde{D}^{(i)}(x,y) = -\sum_{N} (-1)^{\sum_{k\neq i} n_k}\, \Delta(x - y^{N}) \tag{40b}$$

while the ghost propagator is still given by Eq. (35). Transformations to other gauges can be carried out as above.

III. The Casimir Effect in a Box

The ground state energy of a quantum field theory is usually quartically divergent. For example, in the case of a massless scalar field with periodic boundary conditions a box with volumn V_o one has

$$E = \int_{V_o} d^3x \, \langle 0 \,|\, \mathscr{H}(\vec{x},t) \,|\, 0 \rangle = \sum_{\vec{k}} \frac{1}{2} \omega_{\vec{k}} \tag{41}$$

where $\omega_{\vec{k}}$ are the eigenfrequencies in V_o. This divergence is usually handled by normal ordering of the Hamiltonian density $\mathscr{H}(\vec{x},t)$. As first noted by Casimir (in the case of QED), this simple prescription does not work if the theory is defined in a cavity with nonperiodic boundary conditions. Here one should instead consider a regularized form of Eq. (41), e.g.,

$$E(\tau) = \sum_N \frac{1}{2} \omega_N \, e^{-\omega_N \tau} \tag{42}$$

where the eigenfrequencies ω_N depend on the boundary conditions and the geometry of the cavity. The ultraviolet $(\tau \to 0)$ divergent parts of $E(\tau)$ can by general arguments be shown to be simply related to the geometry of the cavity. For the case of QED in a box (volume V, sides ℓ_i) with boundary conditions of a perfect conductor Eqs. (20) or (39) we have[7]

$$\lim_{\tau \to 0} E(\tau) = \frac{3}{\pi^2} \frac{V}{\tau^4} - \frac{1}{4\pi} \frac{\ell_1 + \ell_2 + \ell_3}{\tau^2} + E_C^{in}(\ell_i) \tag{43}$$

where E_C^{in} is τ-independent, and can also be shown to be independent of the cutoff procedure. Except for the quartic divergence proportional to the volume, discussed above, there is a quadratic divergence related to the edges of the box.[8] (For a smooth surface the $1/\tau^2$ term is a surface integral of the mean radius of curvature.[12)] If the cavity is thought of being placed inside a large normalization volume, it is clear that the regular piece $E_C^{in}(\ell_i)$ is the "inside" contribution to the Casimir energy.[8] We shall now show how the methods of Section III allow for a simple calculation of E_C^{in}.

Choose Fermi gauge, that is, take the effective Lagrangian

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 - (\partial_\mu C^+)\,\partial^\mu C \qquad (44)$$

where $F^{\mu\nu}$ is the Abelian field tensor. From Eq. (44) we construct the Hamiltonian density

$$\mathscr{H} = \pi^\mu(\partial_0 A_\mu) + \pi_C\,\partial_0 C + \pi_C^+\,\partial_0 C^+ - \mathscr{L} \qquad (45)$$

with

$$\pi^0 = -\partial_0 A_0 \qquad (46a)$$

$$\pi^k = F^{k0} \qquad (46b)$$

$$\pi_C = -\partial_0 C^+ \qquad (46c)$$

which gives

$$\mathscr{H} = -(\partial_0 A_0)^2 + (\partial_k A_0 - \partial_0 A_k)\partial^0 A^k - 2(\partial_0 C^+)\partial_0 C +$$

$$+ \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu + \frac{1}{2}(\partial_\mu A^\mu)^2 + (\partial_\mu C^+)\partial^\mu C \qquad (47)$$

In order to relate $\mathscr{H}$ to the Greens function of the previous section, we regularize this expression by

$$A_\mu(x)\; A_\nu(x) \rightarrow A_\mu\left(\vec{x}, \frac{i\tau}{2}\right) A_\nu\left(\vec{x}, -\frac{i\tau}{2}\right)$$

$$\equiv A_\mu(x)\; A_\nu(x') \qquad (48)$$

This regularization corresponds in k-space to the one of Eq. (42) and it is easy to see from the explicit expressions that $D_{\mu\nu}(x,x')$ and its derivatives are finite for $\tau \neq 0$. From Eqs. (32) and (47) we get

$$\langle \mathcal{H}_\tau(\vec{x}) \rangle = \frac{i}{2} (\partial_0 \partial_0' + \partial_i \partial_i') \left[\sum_k D^{(k)}(x,x') + + D^{(0)}(x,x') - 2S(x,x') \right] \tag{49}$$

where ∂_μ' denotes derivative w.r.t. x'. Equation (36) gives

$$\langle \mathcal{H}_\tau(\vec{x}) \rangle = \frac{i}{2} (\partial_0 \partial_0' + \partial_i \partial_i') \left[\sum_k D^{(k)}(x,x') - D^{(0)}(x,x') \right] \tag{50}$$

and this together with Eq. (25) and the explicit expression

$$i\Delta_0(x,x') = \frac{-1}{4\pi^2 (x-x')^2} \tag{51}$$

gives for the regularized energy

$$\langle E(\tau) \rangle = \int_{\text{box}} d^3x \, \mathcal{H}_\tau(\vec{x}) \tag{52}$$

In the $\tau \to 0$ limit

$$\lim_{\tau \to 0} E(\tau) = \frac{3}{\pi^2} \frac{V}{\tau^4} - \frac{1}{4\pi} \frac{\ell_1 + \ell_2 + \ell_3}{\tau^2} - - \frac{V}{16\pi^2} \sum_M{}' \left[(m_1\ell_1)^2 + (m_2\ell_2)^2 + (m_3\ell_3)^2 \right]^{-2} + + \frac{\pi}{48} \left(\frac{1}{\ell_1} + \frac{1}{\ell_2} + \frac{1}{\ell_3} \right) \tag{53}$$

where in the summation $\sum'$ the point M = (0,0,0) is excluded. This result agrees with earlier calculations using other methods.[7] It is clear that the term V/τ^4 comes from the tree propagators which are singular in the whole volume, while the $(\ell_1 + \ell_2 + \ell_3)/\tau^2$ term arises from propagators with two reflections which are singular only on the edges, as discussed in Section II. The possible Area/τ^3 divergence from the surface is canceled between different terms. By comparison with Eq. (43) we identify E_C^{in} as the two last terms of Eq. (53).

The contributions of ghosts, with the particular boundary condition Eq. (35), was crucial in order to get the correct result[13] (taking, i.e., $C^+ = 0$ on Γ would have given a different answer). The result thus serves as a check of the formalism developed in Section II.

Since ghosts, though always formally present in covariant gauges, are usually neglected in QED calculations a comment on their importance here might be in order. The reason for not considering the ghosts in Abelian theories is that they have no coupling to the gauge field (or any other field) and thus do not contribute to S-matrix elements. There is, however, one possible ghost diagram, namely the single ghost loop, and this contributes to the vacuum energy. Usually this is subtracted away as are all other zeropoint contributions, but in the confined case it is these very terms that give rise to the Casimir effect and must be included. Physically this is not surprising that the ghost terms are important, since one has already included effects of other unphysical degrees of freedom (longitudinal gauge fields) which must be compensated for.

The above calculation clearly shows the advantage of the separation of the propagator into a free and a boundary dependent part, although in this simple case the result can be obtained by direct calculation of the sum in Eq. (42).

IV. Energy Shift of Confined Photons in Scalar QED

In this section we shall illustrate the use of the techniques developed earlier, by calculating the energy shift at the lowest cavity mode of an Abelian gauge field confined to a cube and coupled to a likewise confined scalar field (i.e., confined scalar QED).

Since the mass shift of an unconfined photon is zero due to gauge invariance, we expect only a finite energy shift of type $\delta E \sim \text{const.}/V^{1/3}$. We shall go into some detail about how to handle the short-distance singularities, and our main result in this section is an explicit demonstration that all divergences disappear and that we obtain a finite well defined result for the energy shift.[14]

Although this calculation has no direct physical relevance, the structure of the diagrams involved is very similar to that of the QCD case (cf. Figs. 2 and 4). So even though the latter is more complicated because of the tensor structure, we expect the methods used below to be applicable also to the QCD calculations.

Consider scalar QED, i.e.,

$$\mathscr{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} D_\mu \phi \, D^\mu \phi \tag{54}$$

where

$$D_\mu = \partial_\mu - i e A_\mu \tag{55}$$

The diagrams contributing to the energy shift are shown in Fig. 4. For the gauge field we take the confining boundary conditions of Eq. (21) and specifically consider one of the lowest modes given by

$$A^0(x) = -\frac{\sqrt{2}i}{a\sqrt{\omega a}} \sin\frac{\pi x_1}{a} \sin\frac{\pi x_2}{a} e^{-i\omega x_0} \tag{56a}$$

$$A^1(x) = -\frac{1}{a\sqrt{\omega a}} \cos\frac{\pi x_1}{a} \cos\frac{\pi x_2}{a} e^{-i\omega x_0}$$

$$A^2(x) = A^3(x) = 0 \tag{56b}$$

where a is the side of the cube and $\omega = \sqrt{2}\pi/a$ the energy of the mode. For the scalar field, we demand

$$\nabla \cdot \vec{j} = 0 \quad \text{on} \quad \Gamma \tag{57}$$

where the conserved current is given by

$$j^\mu = ie[\phi^*(\partial^\mu \phi) - (\partial^\mu \phi^*)\phi] - 2e^2 A^\mu \phi^*\phi \tag{58}$$

Since the boundary condition on A^μ ensures $n_\mu A^\mu = 0$ on Γ the condition Eq. (57) can be realized as either

$$\phi = 0 \quad \text{on} \quad \Gamma \tag{59}$$

or

$$\vec{n} \cdot \nabla\phi = 0 \quad \text{on} \quad \Gamma \tag{60}$$

or as a mixture of these. Either choice will do, and we shall take the second which means the propagator

$$i\Delta(x,y) = \sum_N (-1)^{\sum_i n_i} \Delta_0(x - y^N) \tag{61}$$

The energy shift δE is given by

$$4\pi\, i\, \delta(0)\, \delta E = \Pi \tag{62}$$

where

$$\Pi = \Pi^1 + \Pi^2 \tag{63}$$

$$\Pi^1 = e^2 \int_{\Box} d^4x\, d^4y\, A^\mu(x) [\partial_\mu\, i\Delta^{(1)}(x,y)\, \partial'_\nu\, i\Delta^{(2)}(x,y) -$$

$$- \partial_\mu \partial'_\nu\, i\Delta^{(2)}(x,y)\, i\Delta^{(1)}(x,y) - \partial_\mu \partial'_\nu\, i\Delta^{(1)}(x,y)\, i\Delta^{(2)}(x,y) +$$

$$+ \partial_\mu\, i\Delta^{(2)}(x,y)\, \partial'_\nu\, i\Delta^{(1)}(x,y)]\, A^{\nu *}(y) = \sum_{N^1 N^2} \Pi^1(N^1, N^2) \quad (64a)$$

$$\Pi^2 = 2ie^2 \int_{\Box} d^4x\, A^\mu(x)\, i\Delta(x,x)\, A^{\nu *}(x) = \sum_N \Pi^2(N) \quad (64b)$$

Here 1 and 2 refer to contributions from Figs. 4a and b, respectively. The image charges in $\Delta^{(1)}$, $\Delta^{(2)}$ and Δ are labeled by N^1, N^2 and N, respectively. The symbol $\Box$ denotes x and y integration over the cube and the prime derivatives w.r.t. y. It is easy to see that there are possible short-distance singularities in Π^1 for

$$|n_i^1| \le 1 \quad \text{and} \quad |n_i^2| \le 1 \quad (65)$$

and in Π^2 for

$$|n_i| \le 1 \quad (66)$$

All other terms are regular at $x = y$.[15] The tricky part of the calculation is to show that these singularities cancel. To do that, some care has to be taken to regularize the integrals without destroying gauge invariance.

Before explicitly demonstrating how this is done, we shall briefly discuss the remaining explicitly nonsingular terms. First, we notice that the sums over image charges in Eq. (64) are only conditionally convergent. If, however, the summations are carried out symmetrically around origo, and in such a way that the total image charge does not grow, the sums are well defined as discussed in Appendix A.

It turns out to be convenient not to consider the singular terms alone, but the sums Π_S^1 and Π_S^2 defined as

$$\Pi = \Pi_S + \Pi_{NS} = \Pi_S^1 + \Pi_S^2 + \Pi_{NS} \tag{67}$$

where

$$\Pi_S^1 = \sum_{(n_i^1, n_i^2) \in S_i} \Pi^1(N^1, N^2) \tag{68a}$$

$$\Pi_S^2 = \Pi^2 \tag{68b}$$

The set S_i is shown in Fig. 5. Π_S includes all the possible short-distance singularities of Π. The remaining terms are all finite, and their sum is absolutely convergent, as can be easily checked. Now go to momentum space by

$$\Delta^{(1)}(x,y) = \sum_N (-1)^{\sum_i n_i} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y^N)}}{p^2} \tag{69}$$

and similarly for $\Delta^{(2)}$. Then change to the summation variables

$$n_i^1 = n_i' \tag{70a}$$

$$n_i^2 = n_i + (-1)^{n_i} n_i' \tag{70b}$$

giving

$$\Pi_S^1 = \sum_{n_i=-2}^{2} \sum_{n_i'=-\infty}^{\infty} \Pi(N,N') \tag{71}$$

Also make the variable change

$$\hat{y}_i = (-1)^{n_i'} y_i + n_i a \tag{72}$$

which gives

$$\int_{-a/2}^{a/2} dy_i \to \int_{\left(n_i'-\frac{1}{2}\right)a}^{\left(n_i'+\frac{1}{2}\right)a} d\hat{y}_i \tag{73}$$

By using the wavefunctions in Eq. (56) one can also show

$$A^\nu(\vec{y})\,\partial_\nu' \to A^\nu(\hat{y})\,\partial_\nu^{(\hat{y})} \tag{74}$$

We can now trade the N'-summation for an extension of the y integral to all space, i.e. (again letting $\hat{y}\to\vec{y}$)

$$\begin{aligned}\Pi_S^1 = \sum_{n_i=-2}^{2} (-1)^{\sum_i n_i} e^2 \int_\Box d^4x \int_{-\infty}^{\infty} d^4y\, A^\mu(x) \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4p'}{(2\pi)^4} \\ \times \left[-(-1)^{n_\nu} p_\mu p_\nu' + (-1)^{n_\nu} p_\mu' p_\nu' + p_\mu p_\nu - p_\mu' p_\nu \right] \frac{1}{p^2 p'^2} \\ \times \exp i\left[(p+p')_i x_i - (p+(-1)^{n_i} p')_i y_i - (p+p')_0 (x-y)_0 - \right. \\ \left. - p_i' n_i a\right] A^{\nu *}(y)\end{aligned} \tag{75}$$

The x_o, y_o and p_o' integrations can easily be done, and by writing

$$A^{\nu *}(\vec{q}) = \int \frac{d^3q}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{y}} A^{\nu *}(y) \tag{76}$$

also the y and p' integrations can be performed. By using the Fourier representation of $\delta^{(4)}(x-y)$ similar manipulations can be made on the term Π_S^2. The final result is

$$\Pi_S = 2\pi e^2 \delta(0) \sum_{n_i=-2}^{2} \int_{\Box} d^3x\, A^\mu(x) \int \frac{d^3q}{(2\pi)^3} A^{\nu *}(\vec{q})\, e^{-i(-1)^{n_i} q_i x_i}$$

$$\times \int \frac{d^4p}{(2\pi)^4} \left\{ e^{i(-1)^{n_i} q_i n_i a} \left[p_\mu (2p+q)_\nu + (-1)^{n_\mu} (p+q)_\mu (2p+q)_\nu \right] \right.$$

$$\left. - 2(-1)^{n_1+n_2+n_\mu^+} (p+q)^2 g_{\mu\nu} \right\} \frac{e^{ip_i \tilde{x}_i}}{p^2 (p+q)^2} \tag{77}$$

where $n_\mu^+ = n_\mu$ if $q_\mu \neq 0$, otherwise $n_\mu^+ = 0$, and

$$\tilde{x}_i = 2n_i a \qquad \text{for} \quad n_i = 2m_i \tag{78a}$$

$$\tilde{x}_i = 2x_i - n_i a \qquad \text{for} \quad n_i = 2m_i + 1 \tag{78b}$$

The momentum integral can now be done using the formulas in Appendix C. From Eqs. (77)-(78) it is clear that we must distinguish between odd (o) and even (e) n_i and treat the different combinations N = (e,e,e),(e,o,e) etc., separately. The results are listed in Appendix D. From Eqs. (D1)-(D8) we obtain the numerical result

$$\Pi_S = \sum_N \Pi_S(N) = ie^2 \delta(0) \frac{0.37}{a} \tag{79}$$

Since the N-summation includes a substantial part of the $n_i^1 - n_i^2$ plot and the sum is supposed to converge rapidly (when performed symmetrically) we make the approximation $\Pi \simeq \Pi_S$

$$\delta E = 0.37\, \alpha \frac{1}{a} \tag{80}$$

where

$$\alpha = \frac{e^2}{4\pi} \quad .$$

We have by explicit calculation verified that all short-distance singularities disappear in the final results. In order for this to happen, the dimensional regularization method described in Appendix C is important. A typical example is the integral

$$(\mu a)^{\varepsilon} \int_{-2}^{1} dx_2 \cos x_2 \sin x_2 \frac{1}{\left(x_2^2\right)^{1-\varepsilon/2}} \tag{81}$$

which we encounter in the calculation of $\Pi^{(1,0,1)}$. This integral is convergent for $\varepsilon \geq 1$, ($\varepsilon = 4 - D$) and we define it for $\varepsilon = 0$ by analytical continuation. The only place where we pick up poles is in the term $\Pi^{(0,0,0)}$ which however give no contribution to δE.

V. Conclusions

We have shown how perturbative calculations can be done in boxes by using the image representation of the propagators. This expansion is to be preferred over mode expansions since in the image method singularities are present only in a finite number of terms, which can be handled analytically. This feature was exemplified by a recalculation of the Casimir effect in a box using a covariant gauge and by a calculation of the one loop energy shift of a confined photon in scalar QED. Although the image representation only exists in closed form for boxes, the multiple reflection expansion is its direct analog for general cavities, so the method has a wider applicability.

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Appendix A

In this appendix some properties of the cavity propagators defined in Section III are discussed. First consider the scalar propagator with Dirichlet boundary conditions on all walls

$$\Delta(x,y) = \frac{i}{4\pi^2} \sum_N (-1)^{\sum_i n_i} \frac{1}{(x - y^N)^2} \tag{A.1}$$

For large n_i we can expand

$$\frac{1}{(x - y^N)^2} = \frac{1}{|\vec{n}|^2} + 4 \frac{(x - (-1)^{n_i} y)_i \, n_i}{|\vec{n}|^4} + 0\left(\frac{1}{|\vec{n}|^4}\right) \tag{A.2}$$

When summing over N, neither of the two first terms gives an absolutely convergent series, but both are conditionally convergent. In the first one, there is a cancellation between even and odd values of n_i and in the other, an exact cancellation between the n_i and $-n_i$ terms. All remaining terms in the expansion (A.2) give absolutely convergent series. If we look at the derivatives of the propagator

$$\partial_\mu \, \Delta(x,y) = \frac{i}{4\pi^2} \sum_N (-1)^{\sum_i n_i} \frac{x_\mu - (-1)^{n_\mu} y_\mu - n_\mu \ell_\mu}{(x - y^N)^4} \tag{A.3}$$

the only possible divergence comes from the sum $\sum_N n_i / |\vec{n}|^4$ which is conditionally convergent as discussed above. Higher order derivatives of the propagator give only absolutely convergent sums. It is thus important that whenever the propagator itself or its first derivative occurs, the infinite N summation is taken simultaneously over both even and odd and both positive and negative values of n_i. This can be formally achieved by grouping these terms together, and redefining the summation which is now absolutely convergent. From a physical view, this is an obvious condition. It simply means that in order to satisfy the boundary conditions well, one must sum the contributions from all the image charges inside a large box centered around the cavity, and, for example, not all contributions from one side of the cavity. Almost everything said so far holds also for propagators with Neuman or mixed boundary conditions. There is one exception though. In the case of Neuman boundary conditions on all sides,

$$\Delta(x,y) = \frac{i}{4\pi^2} \sum_N \frac{1}{(x - y^N)^2} \tag{A.4}$$

there is no $(-1)^{\sum_i n_i}$-factor to allow for a cancellation between odd and even terms so the leading linear divergence remains. The physical origin of the divergence is clear. In this case all image charges have the same sign as the original charge, so an infinity in the field is generated by the infinite image charge, while in the Neuman or mixed cases, the average mirror charge is zero. To get rid of the infinity, we simply put in compensating charges of opposite sign

$$\Delta(x,y) = \frac{i}{4\pi^2} \sum_N \left[\frac{1}{(x - y^N)^2} - \frac{1}{|\vec{n}|^2} \right] \tag{A.5}$$

Of course, the precise way in which the infinity is subtracted plays no role since the Neuman b.c. only specifies the derivative of the field.

Appendix B

The generating functional of unconnected diagrams, $W_{\mathscr{F}}[J]$, for a non-Abelian gauge theory can by standard path integral methods be written[16]

$$W_{\mathscr{F}}[J] = \int d[A_\mu]\, \Delta_{\mathscr{F}}[A_\mu]\, \delta[\mathscr{F}_a(A_\mu)] \times \exp i \int_V d^4x[\mathscr{L}_{YM} + J_\mu A^\mu] \qquad (B.1)$$

where

$$\Delta^{-1}_{\mathscr{F}}[A_\mu] = \int d[g]\, \delta[\mathscr{F}_a({}^gA_\mu)] \qquad (B.2)$$

and g is a gauge transformation. $\mathscr{F}_a(A_\mu)$, which appears inside the δ-functional, is a gauge condition which we shall take as

$$\mathscr{F}_a(A_\mu) = B_a - \partial_\mu A^\mu_a \qquad (B.3)$$

for arbitrary constants B_a. Let $\mathscr{F}(A_\mu) = 0$, then as usual because of the δ-functional, only infinitesimal gauge transformations $g = 1 + \alpha$

$${}^gA^\mu = A^\mu - D_\mu \alpha \qquad (B.4)$$

contributes to the integration. Here $\alpha = \alpha_a T^a$ and the covariant derivative are given by

$$D_\mu \alpha = \partial_\mu \alpha - ig[A_\mu, \alpha] \qquad (B.5)$$

We can now replace d[g] by d[α] and extend the α-integration to all space. Then substitute Eqs. (B.3)-(B.5) into Eq. (B.2) and rewrite the δ-functional as a functional integral over the dummy field $\beta(x) = \beta^a(x)T^a$

$$\Delta^{-1}[A_\mu] = \int d[\alpha,\beta]\, \exp i \int_V d^4x\, \beta(x)\, \partial_\mu D^\mu \alpha(x) \qquad (B.6)$$

To this point everything follows as in the free case, but now notice that in our case only those g where ${}^gA^\mu$ satisfy the boundary conditions

Eq. (21) are to be integrated over. A necessary and sufficient condition for both A^μ and $^{(1+\alpha)}A^\mu$ to satisfy Eq. (21) is

$$(\vec{n} \cdot \nabla)\, \alpha(x) = 0 \quad ; \quad x \in \Gamma \tag{B.7}$$

To cast Eq. (B.6) into a more familiar form, we impose the same boundary conditions on the dummy field, i.e.,

$$(\vec{n} \cdot \nabla)\, \beta(x) = 0 \quad ; \quad x \in \Gamma \tag{B.8}$$

By the variable change

$$\alpha = a + b \tag{B.9a}$$

$$\beta = a - b \tag{B.9b}$$

and the subsequent contour relation $b \rightarrow ib$ we can write

$$\Delta^{-1}[A_\mu] = \int d\,[C, C^+]\ \exp\, i \int_V d^4x\ C^+\, \partial_\mu\, D^\mu\, C \tag{B.10}$$

where we identify

$$C = a + ib \tag{B.11}$$

and consequently

$$\vec{n} \cdot \nabla C = 0 \quad ; \quad x \in \Gamma \tag{B.12}$$

Using this boundary condition and integrating by parts gives:

$$\Delta^{-1}[A_\mu] = \int d\,[C, C^+]\ \exp\, i \int_V d^4x\ \mathscr{L}_{Gh}(x) \tag{B.13}$$

$$\mathscr{L}_{Gh} = -(\partial_\mu C^+)\ D^\mu\, C \tag{B.14}$$

As usual we now go from $\Delta^{-1}[A_\mu]$ to $\Delta[A_\mu]$ by treating C and C^+ as anticommuting fields which introduce a minus sign for each closed ghost loop. Since $\Delta[A^\mu]$ is independent of B_a, we can make the replacement[16]

$$\delta[B_a - \partial_\mu A^\mu_a] \rightarrow \exp\, i \int_V d^4x\ \mathscr{L}_{GF} \tag{B.15}$$

where

$$\mathscr{L}_{GF} = -\frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \tag{B.16}$$

and finally

$$W[J] = \int d[A^\mu, C, C^+] \exp i \int_V d^4x \, (\mathscr{L}_{YM} + \mathscr{L}_{GF} + \mathscr{L}_{Gh} + J_\mu A^\mu) \tag{B.17}$$

This completes the proof that to quantize a YM-theory in a covariant gauge fixed by Eq. (B.16) and with the boundary conditions Eq. (21), one needs the usual ghost Lagrangian, Eq. (B.14), together with the boundary conditions, Eq. (B.12).

Appendix C

In this appendix we list some useful integrals and discuss shortly the regularization procedure used. The following integrals in $D = 4 - \varepsilon$ euclidian dimensions can be obtained by standard methods:

$$\int \frac{d^Dp}{(2\pi)^D} \frac{e^{-ipx}}{p^2(p+q)^2} = \frac{i}{8\pi^2} (2\pi)^{\varepsilon/2} r^\varepsilon \int_0^1 d\alpha \, \tilde{K}_{-\varepsilon/2}(q_\alpha r) \, e^{i\alpha\vec{q}\cdot\vec{x}} \tag{C.1a}$$

$$\int \frac{d^Dp}{(2\pi)^D} \frac{e^{-ipx}}{p^2(p+q)^2} p_\mu = \frac{i}{8\pi^2} (2\pi)^{\varepsilon/2} r^\varepsilon \int_0^1 d\alpha \left[-\alpha \, \tilde{K}_{-\varepsilon/2}(q_\alpha r) \, q_\mu + \right.$$
$$\left. + \, i \, \tilde{K}_{1-\varepsilon/2}(q_\alpha r) \frac{x_\mu}{r^2}\right] e^{i\alpha\vec{q}\cdot\vec{x}} \tag{C.1b}$$

$$\int \frac{d^Dp}{(2\pi)^D} \frac{e^{-ipx}}{p^2(p+q)^2} p_\mu p_\nu = \frac{i}{8\pi^2} (2\pi)^{\varepsilon/2} r^\varepsilon \int_0^1 d\alpha \left\{\alpha^2 \, \tilde{K}_{-\varepsilon/2}(q_\alpha r) \, q_\mu q_\nu - \right.$$
$$- \tilde{K}_{1-\varepsilon/2}(q_\alpha r) \left[i\alpha \frac{x_\mu q_\nu + q_\mu x_\nu}{r^2} + \frac{g_{\mu\nu}}{r^2}\right] -$$
$$\left. - \tilde{K}_{2-\varepsilon/2}(q_\alpha r) \frac{x_\mu x_\nu}{r^4}\right\} e^{i\alpha\vec{q}\cdot\vec{x}} \tag{C.1c}$$

where $q^2 < 0$, $r^2 = -x^2 > 0$ for $x_0 = 0$ and

$$q_\alpha^2 = q^2 \alpha(\alpha - 1) \tag{C.2}$$

$$\tilde{K}_\nu(x) = x^\nu K_\nu(x) \tag{C.3}$$

where $K_\nu(x)$ is a Bessel function of second kind. As usual we must introduce a factor μ^ε to fix the dimensions. For an on-shell particle, as in our case, we take the limit $q^2 \to 0$, i.e., $q_\alpha^2 \to 0$ and use

$$\lim_{x\to 0} \tilde{K}_\nu(x) = \left(\frac{1}{2}\right)^{1-|\nu|} \Gamma(\nu) x^{\nu-|\nu|} \tag{C.4}$$

Then the parametric integrals in Eq. (C.1) can be explicitly performed and for $\varepsilon \lesssim 2$ we get the two important integrals

$$\lim_{q^2 \to 0} \int \frac{d^D p}{(2\pi)^4} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2 (p+q)^2} \left[(2p+q)_\mu (2p+q)_\nu - 2(p+q)^2 g_{\mu\nu}\right] =$$

$$= \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi_\varepsilon (q,\vec{x}) +$$

$$+ \frac{i}{8\pi^2} (2\pi)^{\varepsilon/2} r^\varepsilon \left\{ 2\left[1 + e^{-i\vec{q}\cdot\vec{x}} + \frac{2i}{\vec{q}\cdot\vec{x}} \left(1 - e^{-i\vec{q}\cdot\vec{x}}\right)\right] \frac{q_\mu x_\nu + x_\mu q_\nu}{(\vec{x}\cdot\vec{q}) r^2} + \right.$$

$$\left. + 4\left[e^{-i\vec{q}\cdot\vec{x}} + \frac{i}{\vec{q}\cdot\vec{x}} \left(1 - e^{i\vec{q}\cdot\vec{x}}\right)\right] \frac{g_{\mu\nu}}{r^2} + \frac{8i}{\vec{q}\cdot\vec{x}} \left(1 - e^{-i\vec{q}\cdot\vec{x}}\right) \frac{x_\mu x_\nu}{r^2} \right\} \tag{C.5a}$$

and

$$\lim_{q^2 \to 0} \int \frac{d^D p}{(2\pi)^4} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2 (p+q)^2} \left[q_\mu (2p+q)_\nu - 2(p+q)^2 g_{\mu\nu}\right] =$$

$$= \left(q_\mu q_\nu - g_{\mu\nu} q^2\right) \Pi'_\varepsilon (q,\vec{x}) +$$

$$+ \frac{i}{8\pi^2} (2\pi)^{\varepsilon/2} r^\varepsilon \left[2\left(1 - e^{-i\vec{q}\cdot\vec{x}}\right) \frac{q_\mu x_\nu}{(\vec{x}\cdot\vec{q}) r^2} + 4 \frac{g_{\mu\nu}}{r^2}\right] \tag{C.5b}$$

In the limix $\vec{x} \to 0$ one can show that $\Pi_\varepsilon(x,q^2)$ is the usual free vacuum polarization $\Pi_\varepsilon(q^2)$ and $\Pi'(q^2)$ is a closely related function.

In most of the calculations of Section V we can directly put $\varepsilon = 0$, but as discussed in the text, there are a few integrals which are defined only when $\varepsilon \gtrsim 1$ in Eq. (C.5). The final $\vec{x}$ integration, however, is essentially only over the physical subspace of the D-dimensional space (we keep $\tilde{x}$ in 3 dimensions so the extra ε dimensions contribute only an overall volume factor that goes to 1 as $\varepsilon \to 0$). After the x integration has been performed the result is analytically continued to $\varepsilon = 0$.

Appendix D

Here we list the 8 different contributions to Π_S which are obtained after a lengthy calculation using Eqs. (56) and (77) and the formulas of Appendix C.

$$\Pi_S^{(e,e,e)} = K \cdot \left[\frac{1}{2} \sum_{m_i=-1}^{1}{}' \frac{1}{\vec{m}^2} - \frac{1}{2} \sum_{m_1 m_3=-1}^{1} \frac{6m_1^2 + m_3^2}{\left(2m_1^2 + m_3^2\right)^2}\right] \tag{D.1}$$

with

$$K = \frac{ie^2\delta(0)}{2\sqrt{2}\,\pi^2 a}$$

The term $\Pi_S^{(o,o,o)}$ is excluded from the sum denoted Σ' and can be shown to vanish identically (it is essentially the free vacuum polarization contribution)

$$\Pi_S^{(e,e,o)} = K \cdot \int_{-3/2}^{3/2} dx_3 \left\{ \sum_{m_1 \neq m_2 = -1}^{1} \frac{3m_2 - m_1}{m_1 + m_2} \cdot \frac{1}{m_1^2 + m_2^2 + \left(x_3 - \frac{1}{2}\right)^2} - 4 \sum_{m_1=-1}^{1} \frac{m_1^2}{\left[2m_1^2 + \left(x_3 - \frac{1}{2}\right)^2\right]^2} \right\} \tag{D.2}$$

$$\Pi_S^{(e,o,e)} = K \cdot \sum_{m_1,m_3=-1}^{1} \int_{-3/2}^{3/2} dx_2 \cos x_2' \left[3 \cos x_2' - \right.$$

$$- \frac{1}{\pi} \sin x_2' \left(\frac{2}{m_1 + x_2} + \frac{x_2}{x_2^2 - m_1^2} \right) -$$

$$\left. - \frac{8}{\pi} \sin x_2' \frac{1}{m_1 + x_2} \frac{m_1^2}{m_1^2 + m_3^2 + (x_2 - \frac{1}{2})^2} \right] \frac{1}{m_1^2 + m_3^2 + (x_2 - \frac{1}{2})^2} \tag{D.3}$$

where $x_2' = \pi x_2$

$$\Pi_S^{(o,e,e)} = K \cdot \sum_{m_2,m_3=-1}^{1} \int_{-3/2}^{3/2} dx_1 \cos x_1' \left\{ 4 \cos x_1' - \right.$$

$$\left. - \frac{1}{\pi} \sin x_1' \frac{x_1}{x_1^2 - m_2^2} \right\} \frac{1}{m_2^2 + m_3^2 + (x_1 - \frac{1}{2})^2} \tag{D.4}$$

$$\Pi_S^{(o,e,o)} = K \cdot \sum_{m_2,m_3=-1}^{1} \int_{-3/2}^{3/2} dx_1 \left[\left(\cos^2 x_1' - \frac{1}{\pi} \cos x_1' \sin x_1' \cdot \right. \right.$$

$$\left. \left. \cdot \frac{x_1}{x_1^2 - m_2^2} \right) + \sin^2 x_1' \left(\frac{x_1^2}{x_1^2 - m_2^2} + 1 \right) \right] \frac{1}{m_1^2 + m_3^2 + (x_2 - \frac{1}{2})^2}$$

(D.5)

$$\Pi_S^{(o,o,e)} = K \cdot \sum_{m_3=-1}^{1} \int_{-3/2}^{3/2} dx_1 \int_{-3/2}^{3/2} dx_2 \left[4 \sin x_1' \cos x_2' \cdot \right.$$

$$\left(\sin x_1' \cos x_2' \left(\frac{x_1^2}{x_1^2 - x_2^2} \quad -2 \right) - \cos x_1' \cos x_2' \frac{x_1 x_2}{x_1^2 - x_2^2} \right) +$$

$$+ 2 \cos x_1' \cos x_2' \left(\cos_1' \cos x_2' - \frac{1}{\pi} \left(\sin x_1' \cos x_2' \frac{x_1}{x_1^2 - x_2^2} - \right.\right.$$

$$\left.\left.\left. - \cos x_1' \sin x_2' \frac{x_2}{x_1^2 - x_2^2} \right)\right)\right] \frac{1}{(x_1 - \frac{1}{2})^2 + (x_2 - \frac{1}{2})^2 + m_3^2} \qquad \text{(D.6)}$$

$$\Pi_S^{(e,o,o)} = K \cdot \sum_{m_1} \int_{-3/2}^{3/2} dx_2 \, dx_3 \left[\cos x_2' \left\{ 3 \cos_2' - \right.\right.$$

$$\left. - \frac{1}{\pi} \sin_2' \left(\frac{2}{m_1 + x_2} + \frac{x_2}{x_2^2 - m_1^2} \right) \right\} -$$

$$\left. - \frac{8}{\pi} \sin x_2' \frac{1}{m_1 + x_2} \frac{m_1^2}{m_1^2 + m_3^2} \right] \frac{1}{m_1^2 + x_3^2} \qquad \text{(D.7)}$$

$$\Pi_S^{(o,o,o)} = K \cdot \left[\frac{1}{2} \int_{-2}^{1} dx_1 \int_{-2}^{1} dx_2 \int_{-2}^{1} dx_3 \frac{1}{x_1^2 + x_2^2 + x_3^2} - \right.$$

$$\left. - \frac{1}{2} \int_{-2}^{1} dx_1 \int_{-2}^{1} dx_3 \frac{6x_1^2 + x_3^2}{(2x_1^2 + x_3^2)^2} \right] \qquad \text{(D.8)}$$

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10. Everything in this section trivially applies to the Abelian case (QED) by setting the gauge coupling constant $g = 0$.

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12. Strictly speaking, this interpretation requires that the edges of the box are smoothened in order for the $1/\tau^2$ divergence to cancel between the inside and outside of the box.

13. We have also performed the calculation in Coulomb gauge with the same result.

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15. One might think that an additional divergence could occur for $n_i^1 = 0$ and $|n_i^2| > 1$ or $n_i^2 = 0$ and $|n_i^1| > 1$, i.e., when one propagator is free. The only dangerous integral is:

$$\int_{\square} d^4x\ d^4y\ A^{\mu}(x)\ \partial_{\mu}\partial'_{\nu}\ i\Delta_0(x-y^N)\ \Delta_N(x,y^N)\ A^{\nu*}(y)$$

which however becomes finite after partial integration using the boundary conditions on A^{μ}.

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Figure Captions

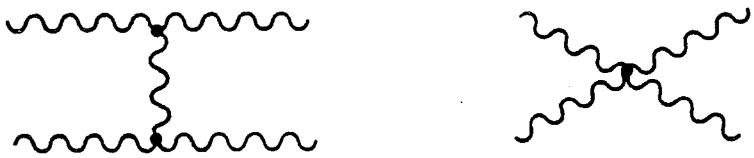
Fig. 1 Lowest order diagrams giving rise to magnetic attraction between 2 gluons.

Fig. 2 Lowest order gluon self energy diagrams.

Fig. 3 a) The free part $\Delta_0(x-y)$ of the propagator in Eq. (12); b-d) Terms in Eq. (12 with one reflection in each direction.

Fig. 4 Lowest order photon self energy diagrams in scalar QED.

Fig. 5 The $n_i^1 - n_i^2$-plane. Possible short-distance singularities occur for points in the square. The n_i and n_i^1-summation in Eq. (71) is shown by the circled and crossed straight lines.



(a) (b)

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Fig. 1

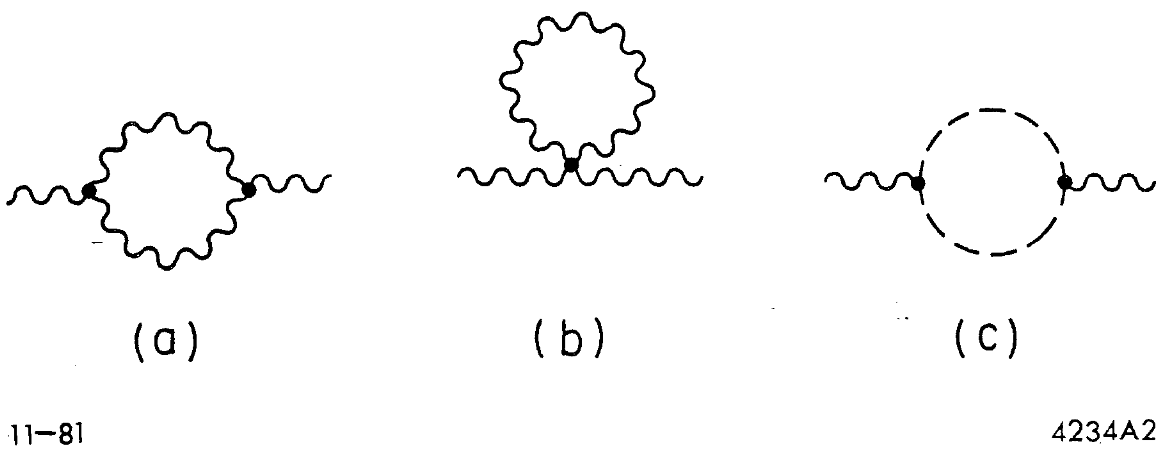

Fig. 2

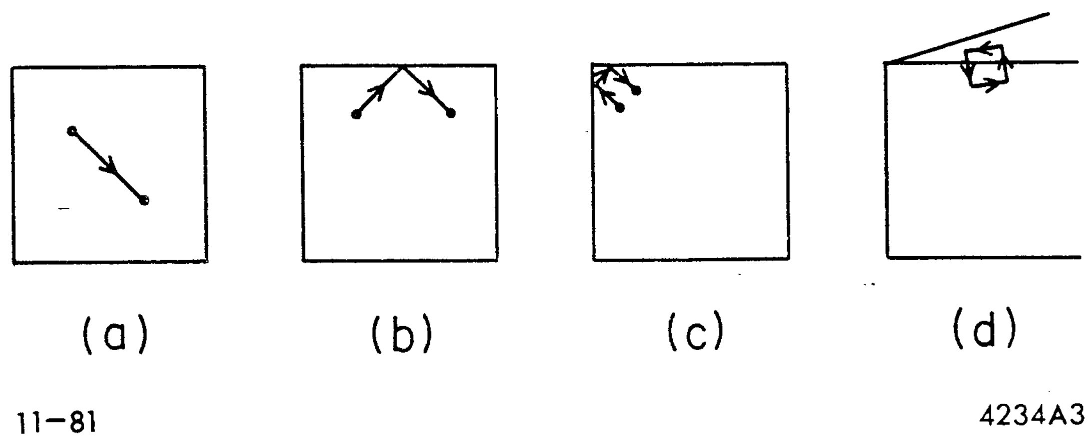

Fig. 3

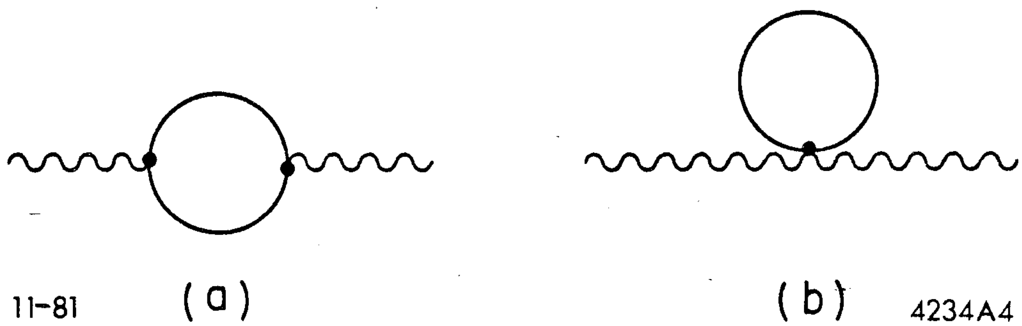

Fig. 4

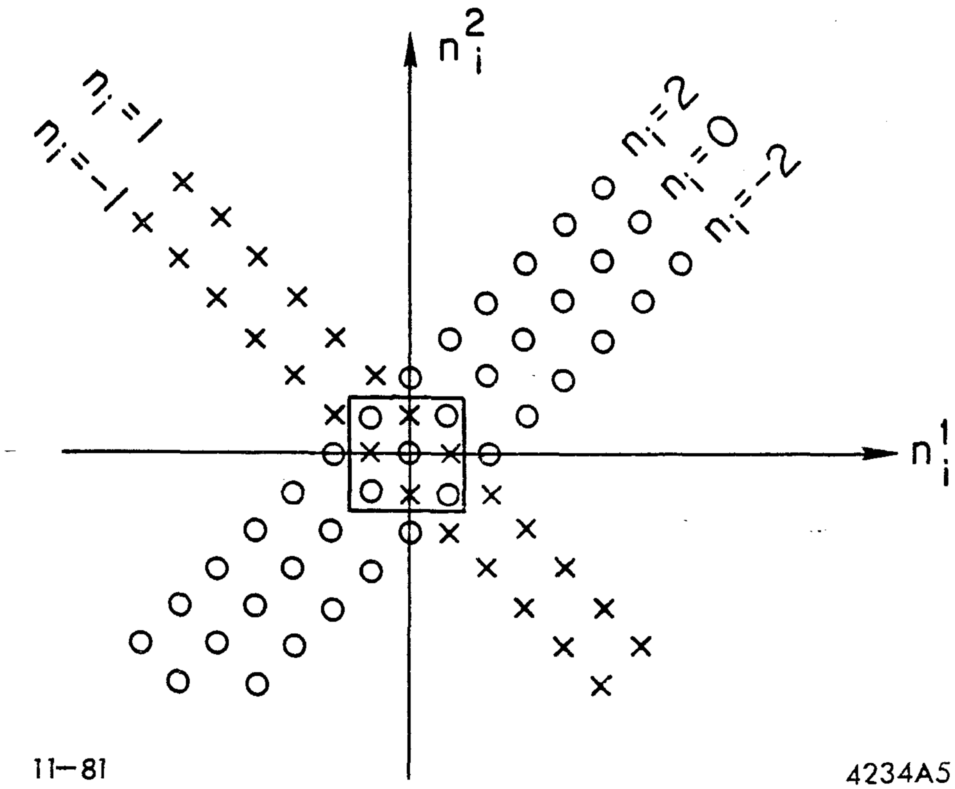

Fig. 5