

POLARIZATION OF A STORED ELECTRON BEAM\*

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ABSTRACT

Synchrotron radiation by a point charge is a familiar subject in classical electrodynamics. Perhaps less familiar are some quantum mechanical corrections to the classical results. In section I, we describe some of those quantum aspects of synchrotron radiation. One of the quantum effects leads to the expectation that electrons in a storage ring will polarize themselves to 92%--a surprisingly high value. Section II gives a semi-classical derivation of the quantum effects described in section I. An effort has been made to minimize the need of using quantum mechanics.

Results of the previous sections are put together in section III to derive a final expression of beam polarization. Conditions under which the expected 92% polarization is destroyed are found and attributed to depolarization resonances. The various depolarization mechanisms are first illustrated by an idealized example and then systematically treated by a matrix formalism. It is shown that the strength of depolarization is specified by a key quantity called the spin chromaticity. Finally as an application of the obtained results, we include an estimate of the achievable level of beam polarization for two existing electron storage rings SPEAR and PEP.

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### 1. Some Quantum Mechanical Aspects of Synchrotron Radiation

As an electron travels in a circular accelerator, it is accelerated sideways and radiates electromagnetic waves known as the synchrotron radiation. The phenomenon of synchrotron radiation is a much studied subject. For instance, one finds in textbooks<sup>1,2</sup> that the instantaneous power radiated by a relativistic electron of energy  $E$  is given by classical electrodynamics:

$$\mathcal{P}_{\text{class}} = \frac{2}{3} \frac{e^2 \gamma^4 c}{\rho^2} \quad (1)$$

with  $e$  the electron charge,  $\gamma = E/mc^2$  the Lorentz factor,  $m$  the electron mass,  $c$  the speed of light and  $\rho$  the instantaneous bending radius. The frequency spectrum of the radiation is somewhat complicated. It covers more or less all frequencies up to a critical frequency defined by

$$\omega_c = \frac{3\gamma^3 c}{\rho} \quad (2)$$

which is essentially  $\gamma^3$  times the revolution frequency of the electron.

The above results assume that the electron follows a prescribed circular trajectory and is unperturbed by its radiation. This is a good approximation if the radiation can be regarded as being continuously emitted rather than being emitted as quantized photons as dictated by quantum mechanics. A more accurate picture is in fact to imagine an electron emitting discrete photons as it circulates along. The photon energies are typically around the value  $\hbar\omega_c$  but their exact values are otherwise unpredictable. As a quantum is emitted, the electron receives a recoil. The effective energy of the electron during the emission process is thus not  $E$ , but slightly lower than  $E$  by an amount comparable to the energy of the quantum. Assuming  $\hbar\omega_c \ll E$ , this slight reduction in the effective electron energy means the synchrotron radiation power is slightly reduced from expression (1). A quantum mechanical calculation<sup>3</sup> shows in fact

$$\mathcal{P} = \mathcal{P}_{\text{class}} \left( 1 - \frac{55}{16\sqrt{3}} \frac{\hbar\omega_c}{E} \right) . \quad (3)$$

The correction term to the classical expression (1) is of the order of  $\hbar\omega_c/E$ . The fact the correction involves the Planck constant  $\hbar$  is a distinct sign of quantum mechanical considerations. Equation (3) has included only the leading term linear in  $\hbar$ ; higher order terms have been ignored.

In practical electron storage rings,  $\hbar\omega_c/E$  is very small. Take  $E = 5$  GeV and  $\rho = 25$  meters, for instance, we find  $\hbar\omega_c/E = 4.5 \times 10^{-6}$ . This means the quantum correction in Eq. (3) is not easy to detect. However, the discreteness of quantum emissions does have an easily detectable effect in practical accelerators. The equilibrium emittances of a bunch of electrons in an electron storage ring, for example, is determined by the balance between a damping effect (the "radiation damping," which is a purely classical phenomenon) and a diffusion effect (the "quantum diffusion," which is a quantum phenomenon) of the electron trajectories.<sup>4,5</sup> Should all electrons radiate continuously, the electron bunch will eventually shrink into a point bunch of zero dimensions due to radiation damping. The discreteness of photon emissions introduces noise into the electron trajectories and causes the beam dimensions to grow by diffusion, which counteracts and balances the damping at equilibrium. The fact that the beam does have a finite size in a storage ring is therefore a quantum mechanical effect. Indeed, if we take the rms energy spread of the beam for example, it does contain a factor of  $\hbar$ :

$$\left(\frac{\Delta E}{E}\right)^2 = \frac{55}{192\sqrt{3}} \frac{\hbar\omega_c}{E} \quad (4)$$

The energy spread is thus of the order of the square root of  $\hbar\omega_c/E$ . With  $\hbar\omega_c/E = 4.5 \times 10^{-6}$  as in the example mentioned above, we find  $\Delta E/E = 0.9 \times 10^{-3}$ , which is easy to detect.

In addition to the discreteness of photon emissions, there is another quantum mechanical aspect of synchrotron radiation, namely that associated with the spin of the electron. Since spin is a quantum mechanical quantity with its magnitude in units of  $\hbar$ , all spin effects of synchrotron radiation involve the Planck constant and are necessarily of quantum mechanical origin. The problem becomes complicated when spin is taken

into account; one now has to distinguish between two cases whether the electron spin stays in its initial state or flips over after emitting the synchrotron photon.

Let  $\hat{n}$  be the spin orientation in the electron's rest frame before photon emission. In case of no spin-flip, the main contribution to the synchrotron radiation power is still given by (1), but the  $\hbar$  correction term now has a spin-dependent term in addition to that given by Eq. (3):<sup>6</sup>

$$\mathcal{P} = \mathcal{P}_{\text{class}} \left[ 1 - \left( \frac{55}{16\sqrt{3}} + \frac{1}{2} \hat{n} \cdot \hat{y} \right) \frac{\hbar\omega_c}{E} \right] \quad (5)$$

where  $\hat{y}$  is the direction of magnetic field that bends the electron. If we average over all spin orientations  $\hat{n}$ , we obtain Eq. (3) as it should. Again, this spin correction term is very small in practice and is difficult to observe experimentally. However, the other spin effect that involves spin-flips does have an easily observable effect on the electron beam--that on its spin polarization. One can of course calculate the instantaneous power radiated with spin-flip and compare with Eq. (5), but the more relevant quantity here is the instantaneous transition rate that involves spin-flip,<sup>7</sup>

$$W = \frac{5\sqrt{3}}{16} \frac{e^2 \gamma^5 \hbar}{m^2 c^2 \rho^3} \left[ 1 - \frac{2}{9} (\hat{n} \cdot \hat{z})^2 + \frac{8}{5\sqrt{3}} \hat{n} \cdot \hat{y} \right] \quad (6)$$

which we note is linear in  $\hbar$ . In (6),  $\hat{z}$  is the unit vector in the direction of motion of the electron. Remembering that the power is equal to the transition rate multiplied by the energy carried by each photon  $\hbar\omega$ , we note the spin-flip power contains a factor of  $\hbar^2$  and is smaller than the classical power by a factor of  $(\hbar\omega_c/E)^2$ , which typically can be  $\sim 10^{-11}$ .

In a storage ring, the guiding magnetic field is in the vertical direction  $\hat{y}$ . If we specify  $\hat{n}$  to be either along the field (the up state) or against the field (the down state), we find that the transition rate from up state to down state,  $W_{\uparrow\downarrow}$ , is larger than that from down state to up state,  $W_{\downarrow\uparrow}$ :

$$W_{\uparrow\downarrow} = \frac{5\sqrt{3}}{16} \frac{e^2 \gamma^5 \hbar}{m^2 c^2 \rho^3} \left(1 + \frac{8}{5\sqrt{3}}\right) \quad (7)$$

$$W_{\downarrow\uparrow} = \frac{5\sqrt{3}}{16} \frac{e^2 \gamma^5 \hbar}{m^2 c^2 \rho^3} \left(1 - \frac{8}{5\sqrt{3}}\right)$$

If we inject into a storage ring an unpolarized electron beam, the imbalance between the two transition rates would cause the beam to accumulate a net polarization in the direction against the guiding field. One then observes that  $W_{\uparrow\downarrow}$  and  $W_{\downarrow\uparrow}$  are not only different but also so very different that the net polarization can potentially reach almost a full level:

$$P_o = \frac{W_{\uparrow\downarrow} - W_{\downarrow\uparrow}}{W_{\uparrow\downarrow} + W_{\downarrow\uparrow}} = \frac{8}{5\sqrt{3}} = 92.38\% \quad (8)$$

Furthermore, the time constant that this equilibrium polarization is approached by the initially unpolarized beam, inspite of being proportional to  $\hbar^{-1}$ , is short enough to be practical. The time constant is

$$\tau_o = (W_{\uparrow\downarrow} + W_{\downarrow\uparrow})^{-1} \quad (9)$$

$$= \left( \frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{m^2 c^2 \rho^3} \right)^{-1}$$

Taking again  $E = 5$  GeV and  $\rho = 25$  m, the time constant is found to be 8 minutes. One can now imagine the excitement when it was realized that the electron beam would polarize itself to a high degree and all we have to do is to inject an unpolarized beam into a storage ring and wait a quarter of an hour or so. For once, we seem to be getting something free from mother nature.

If we draw an analogy to how equilibrium emittances are established in a storage ring, saying the beam will polarize to the full value of 92% due to spin-flip radiation is the same as saying the beam will shrink into a dimensionless size due to radiation damping. What we have forgotten here is the fact that the discrete photon emissions have introduced noise into the system, and when taken into account, there is a diffusion effect on both the emittances and the spin orientations of the electrons.

The equilibrium value of beam polarization, just like the emittances, must be determined by a balance between the polarizing effect of spin-flip radiation and the depolarizing effect of quantum diffusion. The analogy is illustrated in Table 1. In particular, it is necessary to calculate the quantum diffusion rate of spin orientation. We will find then that the pleasant situation of the beam building up 92% polarization all on its own is subject to a stormy environment in a jungle of what is known as the depolarization resonances, near which the spin diffusion rate becomes large and the beam polarization can be much reduced from 92%.

Table 1

The Analogy Between the Mechanisms for the  
Orbital and Spin Equilibrium

|                |  |                   |
|----------------|--|-------------------|
|                | damping $\longleftrightarrow$ diffusion                                | equilibrium       |
| orbital motion | radiation damping $\longleftrightarrow$ quantum diffusion on orbit     | emittances        |
| spin motion    | radiative polarization $\longleftrightarrow$ quantum diffusion on spin | beam polarization |

The spin diffusion rate has been treated by several authors<sup>8,9,10,11</sup> using different methods. The one we shall adopt utilizes the beam transport matrices discussed in Courant's lecture.<sup>4</sup> The difference here is that those matrices, which describe the orbital motions of electrons, will be generalized to include spin motions as well. The advantage of using matrices is that one can put the orbital and the spin motions of an electron on equal footing. The analogy of Table 1 is then taken care of more easily. Once quantum diffusion is introduced, this matrix formalism provides the calculations for beam emittances and polarization simultaneously. How to develop those matrices and how to use them are also subjects that we want to cover.

## 2. Semi-Classical Description of Spin Effects on Synchrotron Radiation

Although spin effects are necessarily quantum mechanical, it is possible to derive most of the results of the previous section semi-classically provided we start with an effective Hamiltonian that includes a term that describes the interaction between electron spin and electromagnetic fields. These derivations will be given in this section. The purpose of doing this is not to replace the more rigorous quantum mechanical calculations<sup>7,8</sup> but to do a calculation that avoids the need of explicitly introducing the Dirac equation or the commutation relations of various operators. The procedure of such a calculation has been discussed in the literature<sup>6,12</sup> and what we will do in this Section 2 is to continue such an effort. Readers who are not interested in detailed derivations can skip Sections 2.2 to 2.7.

### 2.1 Spin Precession in an Electromagnetic Field

Spin of a particle interacts with an electromagnetic field through the magnetic moment associated with the spin. Let  $\hbar\vec{S}$  be the spin represented as a 3-dimensional vector, the associated magnetic moment is given by

$$\vec{\mu} = \frac{ge}{2mc} \hbar\vec{S} \quad (10)$$

where  $g$  is the gyromagnetic ratio of the particle. For electrons,  $g$  is very close to 2. The deviation of  $g$  from 2, attributed to an "anomalous" magnetic moment of an electron, is specified by the parameter

$$a = \frac{g-2}{2} \quad (11)$$

The value of  $a$  is approximately given by the fine structure constant  $1/137$  divided by  $2\pi$ . More accurately, it is found both theoretically and experimentally that  $a = 0.00115965$ .

Consider an electron at rest in a magnetic field  $\vec{B}$ . The precession equation of motion for the spin is

$$\frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S} \quad (12)$$



with the precession angular velocity given by<sup>13</sup>

$$\vec{\Omega} = - \frac{ge}{2mc} \vec{B} \quad (13)$$

Eqs. (12) and (13) describe the precession for a stationary electron, but we need an equation for a relativistic electron moving in an electromagnetic field  $\vec{E}$  and  $\vec{B}$ . Let  $c\vec{\beta}$  be the instantaneous velocity of the electron, it is obvious that we need to make a Lorentz transformation to the electron's rest frame. When doing so, the form of the precession equation remains to be (12); only  $\vec{\Omega}$  needs to be transformed. Note that we are not Lorentz transforming  $\vec{S}$ , so in the final equation,  $\vec{S}$  will be a quantity in the electron's rest frame, while all other quantities  $t$ ,  $\vec{E}$  and  $\vec{B}$  refer to the laboratory frame. One may find it necessary to stretch his imagination somewhat here. A covariant description does exist,<sup>1</sup> but for our purpose, it is not necessary.

The magnetic field in the rest frame is given by a Lorentz transformation from the laboratory frame:

$$\vec{B}_R = \gamma \vec{B}_\perp + \vec{B}_\parallel - \gamma \vec{\beta} \times \vec{E} \quad (14)$$

with  $\vec{B}_\perp$  and  $\vec{B}_\parallel$  the components of  $\vec{B}$  perpendicular and parallel to  $\vec{\beta}$ . The angular velocity  $\vec{\Omega}$  in the laboratory frame consists of two terms. The first term is

$$- \frac{1}{\gamma} \frac{ge}{2mc} \vec{B}_R \quad (15a)$$

where we have included a factor of  $1/\gamma$  to take care of the time dilation. The second term is due to Thomas precession\* which contributes an additional term to the angular velocity when the electron is accelerated sideways:

$$- \frac{(\gamma - 1)}{\beta^2} \vec{\beta} \times \dot{\vec{\beta}} = - \frac{e\gamma}{mc(\gamma + 1)} (\vec{\beta} \times \vec{E} - \beta^2 \vec{B}_\perp) \quad (15b)$$

---

\*Two successive Lorentz transformations along  $\vec{\beta}_1$  and  $\vec{\beta}_2$  can be combined into one single Lorentz transformation only if  $\vec{\beta}_1 \parallel \vec{\beta}_2$ . Otherwise, the two Lorentz transformations can be combined into a Lorentz transformation plus a rotation. The additional rotation needed here is the origin of the Thomas precession.

Adding the two terms and substituting Eq. (14) into the result, we obtain

$$\vec{\Omega} = -\frac{e}{mc} \left[ \left( a + \frac{1}{\gamma} \right) \vec{B} - \frac{a\gamma}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{B}) - \left( a + \frac{1}{\gamma+1} \right) \vec{\beta} \times \vec{E} \right] \quad (16)$$

which, when substituted into Eq. (12), is called the BMT equation,<sup>14</sup> where BMT stands for Bargman, Michel, and Telegdi.

To describe the spin motion in a storage ring, it is more convenient to change the time variable  $t$  into the distance travelled by the electron  $s = ct$ . In a storage ring, we apply several types of electric and magnetic fields to confine the electrons. These fields as seen by a circulating electron are periodic in  $s$  with the period equal to the circumference  $2\pi R$  of the storage ring. Many of those applied fields, such as those provided by quadrupole and sextupole magnets, have effects on a particle only if its trajectory deviates from the designed circular orbit. An ideal electron travelling along the designed orbit sees only the guiding magnetic field and the accelerating electric field. The accelerating field does not cause spin precession on the ideal electron because the electric field is parallel (or anti-parallel, rather, for negatively charged electrons) to the velocity  $\vec{\beta}$  and the precession is, according to (16), proportional to  $\vec{\beta} \times \vec{E}$ . The guiding field  $\vec{B} = B_0(s) \hat{y}$ , with  $\vec{B}_0(s + 2\pi R) = \vec{B}_0(s)$ , on the other hand, does give rise to a precession

$$\frac{d\vec{S}}{ds} = -\frac{eB_0(s)}{mc} \left( a + \frac{1}{\gamma} \right) \hat{y} \times \vec{S} \quad (17)$$

With the precession axis along  $\hat{y}$ , the  $y$ -component of spin  $S_y$  is preserved. If we adopt the coordinate system  $(\hat{x}, \hat{y}, \hat{z})$  that rotates with the ideal electron with  $\hat{z}$  along the electron's velocity and  $\hat{x}$  the horizontal direction, the other two spin components  $S_x$  and  $S_z$  rotates with the angular speed  $a\gamma eB_0/E$  which is  $a\gamma$  times the speed that the coordinate system rotates. As the electron completes one revolution, the coordinate system rotates by  $2\pi$  and the spin has precessed around  $\hat{y}$  by an angle  $2\pi a\gamma$ . In analogy to the definitions of tunes  $\nu_x$ ,  $\nu_y$  and  $\nu_s$  for the horizontal and the vertical betatron motions and the longitudinal synchrotron motion, we define

$$\text{spin tune} = a\gamma, \quad (18)$$

which can be easily shown to be identically equal to  $E/0.44065$  GeV.

Consider an electron beam polarized initially in a certain direction. As the beam circulates around, only the polarization projection along  $\hat{y}$  is preserved; components perpendicular to  $\hat{y}$  precess around  $\hat{y}$  and since different particles precess with somewhat different rates, rapidly smear out. As a result, if the beam is polarized at all, the equilibrium polarization can only be in the  $\hat{y}$  direction. It is also interesting to note the fact that the spin tune involves not the gyromagnetic ratio  $g$  but only the anomalous part of  $g$ , i.e.,  $g-2$  is a consequence of the Thomas precession.

## 2.2 The Hamiltonian

For a non-relativistic electron in a magnetic field  $\vec{B}$ , the Hamiltonian is

$$H = \frac{1}{2m} \left( \vec{p} - \frac{e}{c} \vec{A} \right)^2 - \vec{\mu} \cdot \vec{B} \quad (19)$$

where  $\vec{\mu}$  is the magnetic moment defined in Eq. (11) and  $\vec{A}$  is the vector potential associated with  $\vec{B}$ . In the semiclassical calculation of electromagnetic radiation, one needs the part of Hamiltonian that describes the interaction between the electron and the field  $\vec{B}$ :

$$H_{\text{int}} = -e\vec{\beta} \cdot \vec{A} - \vec{\mu} \cdot \vec{B} \quad (20)$$

We have dropped from (19) the term  $\vec{p}^2/2m$  that describes a free electron and the term  $e^2 \vec{A}^2/2mc^2$  that describes the negligible two-photon processes.

Eq. (20) is the Hamiltonian in the non-relativistic limit. To describe the radiation by a relativistic electron, we need the relativistic generalization by Eq. (20). A rigorous derivation of the relativistic semi-classical Hamiltonian should be obtained by making canonical transformations on the Dirac Hamiltonian,<sup>13</sup> but since this is not a course on relativistic quantum mechanics, we shall content ourselves with something less glorious. The first term in (20) does not require extra work;

it remains the same relativistically. To see that, we note that the Hamiltonian  $(\vec{p} - \frac{e}{c} \vec{A})^2/2m$  should be replaced by the relativistic counterpart  $[m^2 c^4 + c^2 (\vec{p} - \frac{e}{c} \vec{A})^2]^{1/2}$  which, up to the linear order in  $e\vec{A}$ , can be written as a free particle term  $[m^2 c^4 + c^2 \vec{p}^2]^{1/2}$  plus an interaction term  $-e \vec{\beta} \cdot \vec{A}$ . To generalize the second term of (20), we first rewrite it as

$$\hbar \vec{S} \cdot \vec{\Omega} \quad (21)$$

where  $\vec{\Omega}$  is given by the non-relativistic expression (13) and use has been made of Eq. (11). Generalization is then obtained simply by inserting the relativistic expression (16) to replace (13) into  $\vec{\Omega}$ . Adding the two terms together, the Hamiltonian reads<sup>6,12</sup>

$$H_{\text{int}} = -e \vec{\beta} \cdot \vec{A} - \frac{e\hbar}{mc} \vec{S} \cdot \left[ \left( a + \frac{1}{\gamma} \right) \vec{B} - \frac{a\gamma}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \vec{B}) - \left( a + \frac{1}{\gamma+1} \right) \vec{\beta} \times \vec{E} \right] \quad (22)$$

In the non-relativistic limit, (22) reduces to (20) as it should.

### 2.3 Power and Transition Rate of Synchrotron Radiation

To describe synchrotron radiation, we let  $\vec{A}$ ,  $\vec{E}$  and  $\vec{B}$  in the interaction Hamiltonian to contain, in addition to an external applied field, the field due to radiation. The interaction Hamiltonian then contains two terms: a time-independent term due to external fields and a time-varying term due to radiation field. The external-field term is grouped with the free particle term to form an "unperturbed" Hamiltonian (unperturbed by radiation field),  $H_0$ . The total Hamiltonian is then written as  $H_0 + H_{\text{int}}$ , where  $H_{\text{int}}$  is given by (22) with the understanding that  $\vec{A}$ ,  $\vec{E}$  and  $\vec{B}$  only contain the radiation field:

$$\vec{A} = \hat{\epsilon} \left( \frac{2\pi\hbar c}{k} \right)^{1/2} e^{-i\vec{k} \cdot \vec{r} + i\omega t} \quad (23)$$

where  $\hat{\epsilon}$ ,  $\omega$  and  $\vec{k}$  are the polarization, the frequency and wave vector of the emitted synchrotron photon, respectively. The normalization constant of  $\vec{A}$  is chosen so that there is one such photon per unit volume. Complex conjugate of  $\vec{A}$  is not included in (23) since it contributes to a photon absorption process that does not concern us here. From the Maxwell's equations, we have

$$\begin{aligned}\vec{E} &= -ik\vec{A} \\ \vec{B} &= \hat{k} \times \vec{E}\end{aligned}\tag{24}$$

To find the synchrotron radiation power and transition rate, we use the standard technique of quantum mechanics used to deal with time-dependent perturbations.<sup>13</sup> Let  $|n(t)\rangle$  be the  $n$ -th eigenstate of the unperturbed Hamiltonian that evolves in time according to  $\exp(-iE_n t/\hbar)$ . Let the electron be initially in the state  $|i(t)\rangle$ . The time-dependent perturbation theory says that the probability amplitude that the electron is found in the state  $|f(t)\rangle$  after perturbation, to first order of the perturbation strength, is given by

$$C_{fi} = \frac{1}{i\hbar} \int_{-\infty}^{\infty} dt \langle f(t) | H_{int}(t) | i(t) \rangle \tag{25}$$

$H_{int}$  in Eq. (25) is obtained by inserting (23) and (24) into (22):

$$H_{int} = (-e\hat{\epsilon} \cdot \vec{\beta} + i \frac{e\hbar k}{mc} \vec{S} \cdot \vec{V}) \left(\frac{2\pi\hbar c}{k}\right)^{1/2} e^{-i\vec{k} \cdot \vec{r} + i\omega t} \tag{26}$$

where we have followed Jackson<sup>12</sup> to define

$$\vec{V} = \left(a + \frac{1}{\gamma}\right) \hat{k} \times \hat{\epsilon} - \frac{a\gamma}{\gamma+1} \vec{\beta} (\vec{\beta} \cdot \hat{k} \times \hat{\epsilon}) - \left(a + \frac{1}{\gamma+1}\right) \vec{\beta} \times \hat{\epsilon} \tag{27}$$

The first term of the interaction Hamiltonian (26) describes a spinless point charge and is independent of  $\hbar$  aside from a normalization constant. The second term involves spin and is linear in  $\hbar$ .

In expressions (25), (26) and (27), we understand  $\vec{\beta}$ ,  $\vec{r}$ ,  $\vec{S}$  and  $H_{int}$  are quantum mechanical operators. In our semi-classical calculations, however, they will be substituted by their classical values. Consequently, we avoid most of the troubles in taking expectation values between  $|i\rangle$  and  $|f\rangle$  and in keeping record of how variables are arranged in order. The only exception will be for the spin  $\vec{S}$  when spin-flips are involved, of which we take care by using the 2x2 Pauli matrices.

The differential probability that a photon of polarization  $\hat{\epsilon}$  is emitted with wave vector between  $\vec{k}$  and  $\vec{k} + d\vec{k}$  is

$$dp = |C_{fi}|^2 \frac{d^3\vec{k}}{(2\pi)^3} \quad (28)$$

where the factor  $d^3\vec{k}/(2\pi)^3$  is the number of photon states per unit volume. The power  $d\mathcal{P}$  is given by the probability  $dp$  times the photon energy  $\hbar\omega$ , times the instantaneous frequency of revolution  $c\beta/2\pi\rho$ . This gives the instantaneous power radiated per unit solid angle, per unit frequency interval, and summed over the two possibilities of photon polarizations:

$$\frac{d^2\mathcal{P}}{d\omega d\Omega} = \frac{\hbar\omega^3}{(2\pi)^4 c^2 \rho} \sum_{\hat{\epsilon}} |C_{fi}|^2 \quad (29)$$

Substituting explicitly Eqs. (25) and (26) into the above expression, we find

$$\frac{d^2\mathcal{P}}{d\omega d\Omega} = \frac{\omega^2}{(2\pi)^3 \rho} \sum_{\hat{\epsilon}} |I_1 + I_2|^2 \quad (30)$$

where we have defined a spin-independent integral

$$I_1 = \int_{-\infty}^{\infty} dt \langle f(t) | -e \hat{\epsilon} \cdot \vec{\beta} e^{-i\vec{k} \cdot \vec{r} + i\omega t} | i(t) \rangle \quad (31)$$

and a spin-dependent integral proportional to  $\hbar$ :

$$I_2 = \int_{-\infty}^{\infty} dt \langle f(t) | \frac{i e \hbar \vec{k}}{m c} \vec{S} \cdot \vec{V} e^{-i\vec{k} \cdot \vec{r} + i\omega t} | i(t) \rangle \quad (32)$$

Transition rate is, of course, obtained by taking away a factor of  $\hbar\omega$  from the power:

$$\frac{d^2W}{d\omega d\Omega} = \frac{1}{\hbar\omega} \frac{d^2\mathcal{P}}{d\omega d\Omega} \quad (33)$$

#### 2.4 The Classical Limit

The classical result of synchrotron radiation is obtained by ignoring terms involving  $\hbar$ 's. The integral  $I_2$  is therefore dropped from

Eq. (30), and if we do not care about the electron motion after photon emission, the integral  $I_1$  can be replaced by

$$I_1 = -e \hat{\epsilon} \cdot \int_{-\infty}^{\infty} dt \vec{\beta}(t) e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t} \quad (34)$$

with  $\vec{\beta}(t)$  and  $\vec{r}(t)$  now given by their classical values. Calculation of  $I_1$  using Eq. (34) can be found in textbooks. We first note that there is the identity

$$\sum_{\hat{\epsilon}} (\hat{\epsilon} \cdot \vec{Z}_1)(\hat{\epsilon} \cdot \vec{Z}_2) = (\hat{k} \times \vec{Z}_1) \cdot (\hat{k} \times \vec{Z}_2) \quad (35)$$

for any complex vectors  $\vec{Z}_1$  and  $\vec{Z}_2$ . If we consider both  $\vec{Z}_1$  and  $\vec{Z}_2$  to be the integral that appears in (34), we realize that the quantity to be evaluated in the classical limit is

$$\hat{k} \times \int_{-\infty}^{\infty} dt \vec{\beta}(t) e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t} \quad (36)$$

The coordinate system is shown in Figure 1. The bending field is along  $\hat{y}$ ;  $\theta$  and  $\phi$  define the direction of photon emission:

$$\hat{k} = \hat{z} \cos\theta + \hat{x} \sin\theta \cos\phi + \hat{y} \sin\theta \sin\phi \quad (37)$$

In the classical limit, the electron motion is unperturbed by radiation and follows a circular path:

$$\begin{aligned} \vec{\beta}(t) &= \beta \left( \hat{z} \cos \frac{\beta ct}{\rho} + \hat{x} \sin \frac{\beta ct}{\rho} \right) \\ \vec{r}(t) &= \rho \left[ \hat{z} \sin \frac{\beta ct}{\rho} + \hat{x} \left( 1 - \cos \frac{\beta ct}{\rho} \right) \right] \end{aligned} \quad (38)$$

We recall that synchrotron radiation by a relativistic electron almost always is emitted in the direction of electron motion. The angle between the directions of motion of the electron and the photons is of the order of  $1/\gamma$ . We therefore expect  $\theta \lesssim 1/\gamma$ . Also, for a given  $\hat{k}$ , the time interval that takes an electron to emit a photon in the  $\hat{k}$  direction can

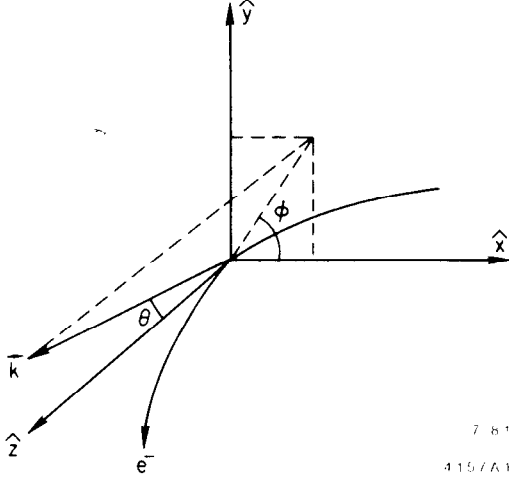


Fig. 1 Relative orientations of the coordinate system, the electron trajectory and the wave vector  $\vec{k}$  of an emitted synchrotron photon. The bending magnetic field is in the  $\hat{y}$  direction. The polar angle  $\theta$  is defined relative to  $\hat{z}$ .

only last for a short time:  $|\beta ct/\rho| \lesssim 1/\gamma$ . What we do is now straightforward; substitute Eqs. (37) and (38) into (36), keeping only leading terms in  $1/\gamma$ . The quantity (36) is found to be (a phase factor has been dropped):

$$-\frac{2\rho}{\sqrt{3}c\gamma^2} (1+t^2)^{1/2} [\hat{x} t K_{1/3}(\eta) + i\hat{y} (1+t^2)^{1/2} K_{2/3}(\eta)] \quad (39)$$

It follows that the classical differential radiation power is

$$\frac{d^2\mathcal{P}_{\text{class}}}{d\omega d\Omega} = \frac{e^2 \rho \omega^2}{6\pi^3 c^2 \gamma^4} (1+t^2) [t^2 K_{1/3}^2(\eta) + (1+t^2) K_{2/3}^2(\eta)] \quad (40)$$

where we have, again following Jackson,<sup>12</sup> defined

$$t = \theta \gamma \sin\phi \quad (41)$$

$$\eta = \frac{\omega}{\omega_c} (1+t^2)^{3/2}$$

with  $\omega_c$  the critical frequency given by Eq. (2). Geometrically,  $t/\gamma$  is the angle between  $\hat{k}$  and the orbital plane of the electron. The modified Bessel functions  $K_{1/3}$  and  $K_{2/3}$ , together with some useful integrals involving them, are given in Table 2. Integrating (40) over  $\omega$  gives the angular distribution of instantaneous power:

$$\frac{d\mathcal{P}_{\text{class}}}{d\Omega} = \frac{e^2 c \gamma^5}{32\pi\rho^2} \frac{7 + 12 t^2}{(1+t^2)^{7/2}} \quad (42)$$



Table 2

Definition and Some Integrals of the  
Modified Bessel Functions  $K_{1/3}$  and  $K_{2/3}$

$$\int_{-\infty}^{\infty} du e^{i z_0 u + \frac{i}{3} u^3} = 2 \left( \frac{z_0}{3} \right)^{\frac{1}{2}} K_{1/3} \left( \frac{2}{3} z_0^{3/2} \right)$$

$$\int_{-\infty}^{\infty} du u e^{i z_0 u + \frac{i}{3} u^3} = \frac{2i}{\sqrt{3}} z_0 K_{2/3} \left( \frac{2}{3} z_0^{3/2} \right)$$

$$\int_0^{\infty} x^2 K_{1/3}^2(x) dx = \frac{5\pi^2}{144}$$

$$\int_0^{\infty} x^2 K_{2/3}^2(x) dx = \frac{7\pi^2}{144}$$

$$\int_0^{\infty} x^3 K_{1/3}^2(x) dx = \frac{16\pi}{81\sqrt{3}}$$

$$\int_0^{\infty} x^3 K_{1/3}(x) K_{2/3}(x) dx = \frac{35\pi^2}{864}$$

$$\int_0^{\infty} x^3 K_{2/3}^2(x) dx = \frac{20\pi}{81\sqrt{3}}$$

Making a change of variable

$$\int d\Omega = 2\pi \int_{-\infty}^{\infty} \frac{dt}{\gamma}$$

One can integrate (42) over solid angles. The result is, of course, just Eq. (1).

## 2.5 Quantum Correction for a Spinless Charge

By quantum correction here we mean correction to the classical results up to first order in  $\hbar$ . When we wrote down the integrals  $I_1$  and  $I_2$  for the synchrotron radiation power, we were not too careful about the order in which the various operators appeared. Since non-commutability of operators are of the order of  $\hbar$ , this carelessness is acceptable for  $I_2$ , which is already first order in  $\hbar$ . It is, in fact, also acceptable for  $I_1$  because  $I_1$  is independent of spin and it is only the spin-dependent  $\hbar$  correction that we are interested in for later calculations. Nevertheless, one can insist on doing the job right and obtain the quantum correction for a spinless charge (for which  $I_2$  vanishes). This has been done by Schwinger,<sup>3</sup> who showed that the first order correction can be obtained by simply making a replacement

$$\omega \longrightarrow \omega \left( 1 + \frac{\hbar\omega}{E} \right) \quad (43)$$

in the classical result of (40). One can then integrate the result over frequency to obtain the angular distribution

$$\frac{d\mathcal{P}_{\text{class}}}{d\Omega} \left[ 1 - \frac{64}{3\sqrt{3}\pi} \frac{\hbar\omega_c}{E} \frac{5 + 9t^2}{(1+t^2)^{3/2}(7+12t^2)} \right] \quad (44)$$

Integrating again over solid angles gives Eq. (3).

## 2.6 Radiation Power without Spin-flip

Although the fact that  $g$  is not exactly equal to 2 plays an important role in how spin precesses in a storage ring, it is not so essential for the synchrotron radiation of the electron. In the rest of this section, we choose to ignore the difference between  $g$  and 2.

Just like  $I_1$  defined by Eq. (31) can be approximated by a classical integral, Eq. (34), similar approximation can be made on  $I_2$ :

$$I_2 = \frac{ie\hbar k}{mc} \int_{-\infty}^{\infty} dt \langle f | \vec{S}(t) | i \rangle \cdot \vec{V}(t) e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t} \quad (45)$$

where  $\vec{V}$ ,  $\vec{r}$  are now classical quantities,  $|i\rangle$  and  $|f\rangle$  refer to the initial and final spin states of the electron. To find  $I_2$ , we need to evaluate  $\langle f | \vec{S}(t) | i \rangle$ .

Let the electron spin be instantaneously ( $t = 0$ ) in the direction  $\hat{n}$  and define angles  $\theta_0$  and  $\phi_0$  as shown in Figure 2. Note that  $\theta_0$  is defined with respect to  $\hat{y}$ , while  $\theta$  of Figure 1 is defined with respect to  $\hat{z}$ . We distinguish between two cases according to whether there is a spin flip or not after photon emission. In case of no spin-flip,  $\langle f | \vec{S}(t) | i \rangle$  is easy to find. Knowing  $\hat{n}$  at time  $t = 0$  and knowing that, for  $g = 2$ , spin precesses with the same angular frequency  $\omega_0 = |e|B_0/mc\gamma$  as the electron circulates in the field  $\vec{B}_0$ , we find

$$\langle f | \vec{S}(t) | i \rangle = \frac{\hat{x}}{2} \sin\theta_0 \sin(\omega_0 t + \phi_0) + \frac{\hat{y}}{2} \cos\theta_0 + \frac{\hat{z}}{2} \sin\theta_0 \cos(\omega_0 t + \phi_0) \quad (46)$$

where a factor of  $1/2$  is included on the right-hand side because the electron spin is  $\hbar/2$ .

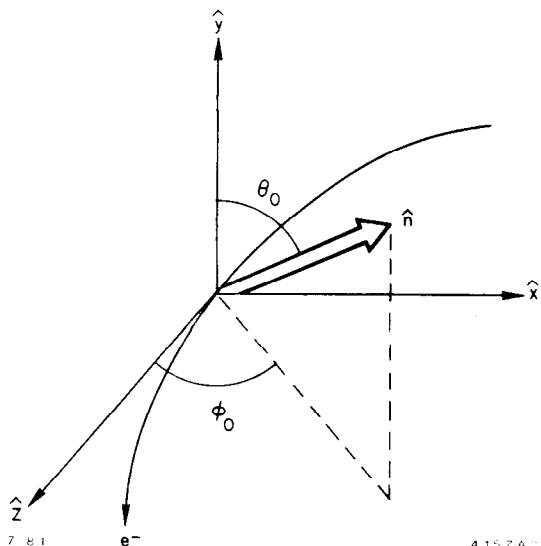


Fig. 2 Relative orientations of the coordinate system, the electron trajectory and the instantaneous direction  $\hat{n}$  of electron spin. The bending magnetic field is in the  $\hat{y}$  direction. The polar angle  $\theta_0$  is defined relative to  $\hat{y}$ .

We insert Eq. (46) and the expression (27) for  $\vec{V}$  (remembering we have set  $g = 2$  or equivalently  $a = 0$ ) into  $I_2$  to obtain

$$I_2 = \frac{ie\hbar k}{2mc} \hat{\varepsilon} \cdot \left( \cos \theta_o \vec{u}_1 + \frac{1}{2} \sin \theta_o e^{i\phi_o} \vec{u}_2 + \frac{1}{2} \sin \theta_o e^{-i\phi_o} \vec{u}_3 \right) \quad (47)$$

where we have defined three more new symbols:

$$\vec{u}_1 = \hat{y} \times \int_{-\infty}^{\infty} dt \left[ \frac{\hat{k}}{\gamma} - \frac{\vec{\beta}(t)}{\gamma+1} \right] e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t} \quad (48)$$

$$\vec{u}_{2,3} = (\hat{z} \mp i\hat{x}) \times \int_{-\infty}^{\infty} dt \left[ \frac{\hat{k}}{\gamma} - \frac{\vec{\beta}(t)}{\gamma+1} \right] e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t \pm i\omega_o t}$$

In the expression for  $\vec{u}_{2,3}$ , the upper signs are for  $\vec{u}_2$  and the lower signs are for  $\vec{u}_3$ . The reason we factor  $\hat{\varepsilon}$  outside of the parentheses in Eq. (47) is so that we can make use of the identity (35).

The  $\hbar$  correction to synchrotron radiation power involves, from Eq. (30), the interference between the spin-independent amplitude and the spin-dependent amplitude. Explicitly, it involves the real part of  $\sum_{\hat{\varepsilon}} (I_1 I_2^*)$ . Making use of Eq. (35), one finds

$$\sum_{\hat{\varepsilon}} (I_1 I_2^*) = \frac{e^2 \hbar k}{2mc} \left[ \hat{k} \times \int_{-\infty}^{\infty} dt \vec{\beta}(t) e^{-i\vec{k} \cdot \vec{r}(t) + i\omega t} \right] \cdot \left[ \hat{k} \times \left( \cos \theta_o \vec{u}_1^* + \frac{1}{2} \sin \theta_o e^{-i\phi_o} \vec{u}_2^* + \frac{1}{2} \sin \theta_o e^{i\phi_o} \vec{u}_3^* \right) \right] \quad (49)$$

The quantity in the first pair of square brackets has been evaluated before; it is given by Eq. (39). Similar steps that led to (39) also lead to

$$\hat{\mathbf{k}} \times \vec{\mathbf{u}}_1^* = \frac{2}{\sqrt{3}} \frac{\rho \hat{\mathbf{z}}}{\gamma^3 c} (1 + t^2)^{\frac{1}{2}} K_{1/3}(\eta) \quad (50)$$

$$\hat{\mathbf{k}} \times \vec{\mathbf{u}}_{2,3}^* = -\frac{2}{\sqrt{3}} \frac{\rho}{\gamma^3 c} (1 + t^2)^{\frac{1}{2}} \left[ (t\hat{\mathbf{y}} \mp i\hat{\mathbf{x}}) K_{1/3}(\eta) + i\hat{\mathbf{x}} (1 + t^2)^{\frac{1}{2}} K_{2/3}(\eta) \right]$$

which can be readily substituted into (49) to find the spin-dependent correction to synchrotron radiation power to first order in  $\hbar$ . If we put this result together with our previous spin-independent results, we obtain

$$\frac{d^2 \mathcal{P}}{d\omega d\Omega} = \left( \frac{d^2 \mathcal{P}_{\text{class}}}{d\omega d\Omega} \right)_{\omega \rightarrow \omega + \frac{\hbar\omega}{E}} \quad (51)$$

$$+ \frac{e^2 \rho \hbar \omega^3}{6\pi^3 m c^4 \gamma^5} (1 + t^2)^{3/2} (-\hat{\mathbf{n}} \cdot \hat{\mathbf{y}} + 2t \hat{\mathbf{n}} \cdot \hat{\mathbf{z}}) K_{1/3}(\eta) K_{2/3}(\eta)$$

From (51), we have

$$\frac{d\mathcal{P}}{d\Omega} = \frac{d\mathcal{P}_{\text{class}}}{d\Omega} \left[ 1 - \frac{64}{3\sqrt{3}\pi} \frac{\hbar\omega_c}{E} \frac{5 + 9t^2}{(1+t^2)^{3/2} (7+12t^2)} \right. \quad (52)$$

$$\left. + \frac{35}{6} \frac{\hbar\omega_c}{E} \frac{-\hat{\mathbf{n}} \cdot \hat{\mathbf{y}} + 2t \hat{\mathbf{n}} \cdot \hat{\mathbf{z}}}{(1+t^2)(7+12t^2)} \right]$$

in which the spin-independent terms are those that appeared in Eq. (44). For a longitudinally polarized electron, the term proportional to  $\hat{\mathbf{n}} \cdot \hat{\mathbf{z}}$  gives rise to an up-down asymmetry of synchrotron radiation. With positive helicity ( $\hat{\mathbf{n}} = \hat{\mathbf{z}}$ ), there is more radiation in the upper plane, while with negative helicity ( $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$ ), more radiation is found in the lower plane. Integrating over solid angles averages out the up-down asymmetry and we get Eq. (5). We now see an asymmetry with respect to whether the spin is up ( $\hat{\mathbf{n}} = \hat{\mathbf{y}}$ ) or down ( $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$ ); more energy is radiated if the electron spin points against the bending magnetic field. As we will see in section 3.5, the  $\hat{\mathbf{n}} \cdot \hat{\mathbf{y}}$  term in Eq. (5) plays a role in determining the beam polarization in an electron storage ring.

## 2.7 Transition Rate with Spin-flip

The spin-independent integral  $I_1$  does not contribute to spin-flip radiation. To evaluate  $I_2$  using Eq. (45), we need first to find the spin transition amplitude  $\langle f | \vec{S}(t) | i \rangle$ . Unlike the case without spin-flip, this can not be done in two lines.

Let us choose the spin operator at time  $t = 0$  to be  $\vec{S}(0) = \vec{\sigma}/2$  with  $\vec{\sigma}$  the Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (53)$$

(We have made a cyclic permutation upon the more familiar definition of Pauli matrices. This choice is more convenient because our magnetic field  $\vec{B}_0$  is in the  $\hat{y}$  direction.) The spin operator at other times can be obtained from  $\vec{S}(0)$  by considering the precession:

$$\begin{aligned} S_x(t) &= \frac{1}{2} (\sigma_x \cos \omega_0 t + \sigma_z \sin \omega_0 t) \\ S_y(t) &= \frac{1}{2} \sigma_y \\ S_z(t) &= \frac{1}{2} (-\sigma_x \sin \omega_0 t + \sigma_z \cos \omega_0 t) \end{aligned} \quad (54)$$

Again let  $\hat{n}$  be the direction of the electron spin before radiation. The initial and final states  $|i\rangle$  and  $|f\rangle$  in the matrix representation are

$$|i\rangle = \begin{bmatrix} \cos \frac{\theta_0}{2} e^{-i\phi_0/2} \\ \sin \frac{\theta_0}{2} e^{i\phi_0/2} \end{bmatrix}, \quad |f\rangle = \begin{bmatrix} -\sin \frac{\theta_0}{2} e^{-i\phi_0/2} \\ \cos \frac{\theta_0}{2} e^{i\phi_0/2} \end{bmatrix} \quad (55)$$

which are eigenstates of the operator  $\hat{n} \cdot \vec{\sigma}$  with eigenvalues  $+1$  and  $-1$ , respectively. Angles  $\theta_0$  and  $\phi_0$  are the same as before (see Figure 2). The matrix representation of  $|f\rangle$  can be obtained from that of  $|i\rangle$  by replacing  $(\theta_0, \phi_0) \rightarrow (\pi - \theta_0, \pi + \phi_0)$ . Having obtained (54) and (55), it is straightforward by matrix multiplication to find

$$\begin{aligned} \langle f | \vec{S}(t) | i \rangle = & -\frac{\hat{y}}{2} \sin\theta_o + \cos^2 \frac{\theta_o}{2} \left( \frac{\hat{z}}{2} + i \frac{\hat{x}}{2} \right) e^{-i\phi_o - i\omega_o t} \\ & - \sin^2 \frac{\theta_o}{2} \left( \frac{\hat{z}}{2} - i \frac{\hat{x}}{2} \right) e^{i\phi_o + i\omega_o t} \end{aligned} \quad (56)$$

We are now in the position to calculate  $I_2$ . Eq. (45) gives

$$I_2 = \frac{ie\hbar k}{2mc} \hat{\epsilon} \cdot \left( -\sin\theta_o \vec{u}_1 - \sin^2 \frac{\theta_o}{2} e^{i\phi_o} \vec{u}_2 + \cos^2 \frac{\theta_o}{2} e^{-i\phi_o} \vec{u}_3 \right) \quad (57)$$

using  $\vec{u}_{1,2,3}$  already defined in Eq. (48). The next step, by now familiar, is to sum over the photon polarizations  $\hat{\epsilon}$  using the identity (35). After doing so, we get an expression that contains  $\hat{k} \times \vec{u}_{1,2,3}$ . Substituting from (50) then yields

$$\begin{aligned} \frac{d^2 W}{d\omega d\Omega} = & \frac{e^2 \hbar \omega^3 \rho}{24\pi^3 m^2 c^6 \gamma^6} (1+t^2) \left\{ \sin^2 \theta_o K_{1/3}^2 \right. \\ & + (1+t^2) \left( \frac{1+\cos^2 \theta_o}{2} \right) [K_{1/3}^2 + K_{2/3}^2] \\ & + 2 \cos \theta_o (1+t^2)^{\frac{1}{2}} K_{1/3} K_{2/3} + t \cos \phi_o \sin 2\theta_o K_{1/3}^2 \\ & \left. - \frac{1}{2} \cos 2\phi_o \sin^2 \theta_o \left[ (1+t^2) K_{2/3}^2 - (1-t^2) K_{1/3}^2 \right] \right\} \end{aligned} \quad (58)$$

We have given the transition rate rather than the power. The reason has been explained when we discussed Eq. (7). Integrating over frequency gives

$$\begin{aligned} \frac{dW}{d\Omega} = & \frac{2}{3\sqrt{3}\pi^2} \frac{e^2 \hbar \gamma^6}{m^2 c^2 \rho^3} \frac{1}{(1+t^2)^5} \left\{ \sin^2 \theta_o + \frac{9}{8} (1+t^2)(1+\cos^2 \theta_o) \right. \\ & \left. + \frac{105\sqrt{3}\pi}{256} \cos \theta_o (1+t^2)^{\frac{1}{2}} + t \cos \phi_o \sin 2\theta_o - \frac{1}{8} \cos 2\phi_o \sin^2 \theta_o (1+9t^2) \right\} \end{aligned} \quad (59)$$

We have kept the five terms in the curly brackets in the same order as we had them in Eq. (58). The fourth term, being proportional to  $t$ , gives an up-down asymmetry to spin-flip radiation. This asymmetry disappears if the spin direction is in the  $xz$ -plane or in the  $xy$ -plane.

For example, one does not observe up-down asymmetry if  $\hat{n}$  is along  $\hat{x}$ , or  $\hat{y}$  or  $\hat{z}$ . The total spin-flip transition rate is obtained by integrating (59) over solid angles. Using the fact that  $\cos \theta_0 = \hat{n} \cdot \hat{y}$  and  $\sin \theta_0 \cos \phi_0 = \hat{n} \cdot \hat{z}$ , we discover Eq. (6).

## 2.8 Radiative Polarization

We briefly mentioned the mechanism for the beam to polarize itself naturally in a storage ring when we discussed Eqs. (7), (8) and (9). We will now do it more systematically.

In the above semi-classical treatments, we have been considering the radiation by a single electron. Polarization, of course, is the net spin of a group of many electrons. Let  $\vec{\zeta}$  be the polarization vector. Its direction is along the direction of the net spin and its magnitude  $\zeta$  ( $\leq 100\%$ ) is the beam polarization. The equation of motion of  $\vec{\zeta}$  contains, of course, precession described by the BMT terms, Eq. (16). In addition, it must also take into account the polarizing effect of spin-flip synchrotron radiation. In fact, it even has to include the various depolarization effects so far not yet described. Here, let us consider an idealistic situation in which the electrons form a point bunch of zero emittances and no energy spread; all electrons follow the circular designed trajectory and see only a guiding magnetic field in the vertical direction  $\hat{y}$ . The only relevant terms are then the BMT precession and the spin-flip transition rate, Eq. (6):

$$\frac{d\vec{\zeta}}{dt} = \frac{(a\gamma + 1)c}{\rho} \hat{y} \times \vec{\zeta} - \frac{1}{\tau_0} \left[ \vec{\zeta} - \frac{2}{9} \hat{z} (\vec{\zeta} \cdot \hat{z}) + \frac{8}{5\sqrt{3}} \hat{y} \right] \quad (60)$$

where the factor  $a\gamma$ , we recognize, is the spin tune;  $c/\rho$  is the revolution frequency of the electron and  $\tau_0$  has been defined in Eq. (9). An additional factor of 2 appears in the transition rate term because in one spin-flip event, polarization changes by 2 units of electron spin.

Admittedly Eq. (60) is somewhat awkward since in the first precession term, we have included, and indeed we must include, the fact that  $g \neq 2$ . In the second spin-flip term, however, we have insisted to set  $g = 2$ . The justification is that taking into account of  $g \neq 2$  in



the second term does not change our final result much (since, after all,  $g$  is very close to 2), while the mathematics becomes much more complicated. Those who are interested in the general case for arbitrary  $g$  should refer to the literature.<sup>6,12</sup> [See also Eq. (65).]

Let us rewrite (60) in terms of the three components  $\zeta_x$ ,  $\zeta_y$ , and  $\zeta_z$  of the polarization in a coordinate system that rotates with the beam.

$$\begin{aligned}\dot{\zeta}_x &= a\gamma \frac{c}{\rho} \zeta_z - \frac{1}{\tau_0} \zeta_x \\ \dot{\zeta}_y &= -\frac{1}{\tau_0} \left( \zeta_y + \frac{8}{5\sqrt{3}} \right) \\ \dot{\zeta}_z &= -a\gamma \frac{c}{\rho} \zeta_x - \frac{7}{9\tau_0} \zeta_z\end{aligned}\tag{61}$$

Note that  $\zeta_x$  and  $\zeta_z$  are coupled while  $\zeta_y$  is independent. From Eq. (61), we observe that at equilibrium when  $\dot{\zeta}_x = \dot{\zeta}_y = \dot{\zeta}_z = 0$ , we must have  $\zeta_x = \zeta_z = 0$  and  $\zeta_y = -8/5\sqrt{3} = -92.38\%$ .

In order to get a feeling about how this polarization is reached in time, let us simplify the problem by considering a uniform magnetic field;  $\rho$  and  $\tau_0$  are then constants. We readily solve  $\zeta_y$ :

$$\zeta_y(t) = \left[ \zeta_y(0) + \frac{8}{5\sqrt{3}} \right] e^{-t/\tau_0} - \frac{8}{5\sqrt{3}}\tag{62}$$

The vertical component of polarization thus approaches its equilibrium with time constant  $\tau_0$ . To find the time evolution for  $\zeta_x$  and  $\zeta_z$  we first note that if we ignore spin precession,  $\zeta_x$  will approach 0 with a rate  $\tau_0^{-1}$  while  $\zeta_z$  will take a slightly lower rate,  $7\tau_0^{-1}/9$ , to reach its 0. Both rates are very slow compared with the rate  $a\gamma c/\rho$  at which  $\zeta_x$  and  $\zeta_z$  rotate and mix into each other. It is therefore a good approximation if we replace  $1/\tau_0$  in the  $\dot{\zeta}_x$  equation and  $7/9\tau_0$  in the  $\dot{\zeta}_z$  equation by their average value  $8/9\tau_0$ . After doing so, we can solve  $\zeta_x$  and  $\zeta_z$ :<sup>8</sup>

$$\begin{aligned}\zeta_x(t) &= \left[ \zeta_x(0) \cos \frac{a\gamma ct}{\rho} + \zeta_z(0) \sin \frac{a\gamma ct}{\rho} \right] e^{-8t/9\tau_0} \\ \zeta_z(t) &= \left[ -\zeta_x(0) \sin \frac{a\gamma ct}{\rho} + \zeta_z(0) \cos \frac{a\gamma ct}{\rho} \right] e^{-8t/9\tau_0}\end{aligned}\tag{63}$$

Eqs. (62) and (63) describe the time evolution of polarization if we inject into a storage ring a beam with initial polarization  $\vec{\zeta}(0)$ . In particular, if the injected beam is unpolarized, the spin-flip synchrotron radiation will cause the beam to build up its polarization against the field:

$$\vec{\zeta}(t) = -\frac{8}{5\sqrt{3}} \hat{y} (1 - e^{-t/\tau_0})\tag{64}$$

Up to now, we have been considering electrons. For positrons, the equilibrium polarization will be parallel to the magnetic field. One may try to draw a more intuitive picture of the effect of polarization build-up. For that, one imagines a magnetic moment  $\vec{\mu}$  in a magnetic field  $\vec{B}$ . Two energy states are generated, one with  $\vec{\mu}$  parallel to  $\vec{B}$ , another with  $\vec{\mu}$  anti-parallel to  $\vec{B}$ . Particles, of course, prefer to stay in the lower energy state, namely the one with  $\vec{\mu}$  parallel to  $\vec{B}$ . One concludes then that electrons must polarize against  $\vec{B}$  while positrons are polarized along  $\vec{B}$ . The difficulty with such a picture has been discussed by Jackson.<sup>12</sup> The two states cannot be regarded as isolated states; orbital motion of the electron or the positron must be considered together with the spin as one coupled system. During the time interval it takes an electron to complete the process of emitting a photon, the electron has rotated by an angle  $\sim 1/\gamma$ . In the mean time, the electron spin has precessed by an angle  $a\gamma$  times as much, i.e., it has precessed by an angle  $\sim a$ . In order for the two energy states to be regarded as being isolated, the spin must complete at least one turn of precession during the photon emission process. This is true only if  $|a| \gtrsim 2\pi$ , or equivalently,  $|g| \gtrsim 4\pi$ . For electrons and positrons, this is far from being valid. The above intuitive picture remains not too much more than a quick way to memorize the direction of polarization correctly for both electrons and positrons. In fact, even for this limited purpose, the fact that it does work is only accidental. According to this picture, electron polarization will be in the  $-\hat{y}$  direction if  $g > 0$  and  $+\hat{y}$

direction if  $g < 0$ . The general calculation, which is not only valid for  $g = 2$  as we have done, but also valid for arbitrary values of  $g$ , shows differently: the electron polarization switches direction between  $-\hat{y}$  and  $+\hat{y}$  not at  $g = 0$  but at  $g = 1.198$ . More explicitly, let us copy the result for the case of arbitrary  $g$ :<sup>6,12</sup>

$$\begin{aligned} \frac{\tau_o(a)}{\tau_o} = & \left[ \left( 1 + \frac{41}{45} a - \frac{23}{18} a^2 - \frac{8}{15} a^3 + \frac{14}{15} a^4 \right) e^{-\sqrt{12}|a|} \right. \\ & - \frac{8}{5\sqrt{3}} \frac{a}{|a|} \left( 1 + \frac{11}{12} a - \frac{17}{12} a^2 - \frac{13}{24} a^3 + a^4 \right) e^{-\sqrt{12}|a|} \\ & \left. + \frac{8}{5\sqrt{3}} \frac{a}{|a|} \left( 1 + \frac{14}{3} a + 8a^2 + \frac{23}{3} a^3 + \frac{10}{3} a^4 + \frac{2}{3} a^5 \right) \right]^{-1} \quad (65) \\ P_o(a) = & - \frac{8}{5\sqrt{3}} \frac{\tau_o(a)}{\tau_o} \left( 1 + \frac{14}{3} a + 8a^2 + \frac{23}{3} a^3 + \frac{10}{3} a^4 + \frac{2}{3} a^5 \right) \end{aligned}$$

Plotted in Figures 3(A) and 3(B) are the values of  $\tau_o(a)/\tau_o$  and  $P_o(a)$  versus  $a$ . For large  $|a|$ , the magnetic moment is large; the polarization time constant becomes short and the level of polarization approaches 100% as we would expect. Our results, however, correspond only to the values at  $a = 0$ . If we insist on using the right value of  $a = 0.00116$ , the equilibrium polarization would have been 92.44%, somewhat higher than the value 92.38% we have been talking about; and the polarization time constant would have been shorter by about half a percent.

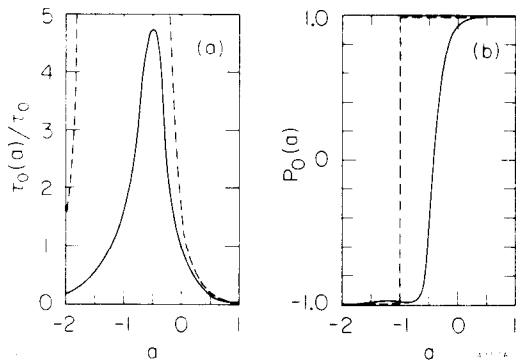


Fig. 3 (A) The characteristic time  $\tau_o(a)$  for radiative polarization build-up, normalized by  $\tau_o$  of Eq. (9), versus the magnetic anomaly parameter  $a$ . (B) The equilibrium beam polarization  $P_o(a)$  versus the parameter  $a$ . Our results of  $\tau_o$  and  $P_o$  correspond to the values at  $a = 0$ . The dotted curves indicate what one would expect from an intuitive picture that is valid for large  $|a|$ .

Since protons have  $a = 1.793$ , a proton beam will be fully radiative polarized. The problem is its polarization time: with  $\tau_0(1.793)/\tau_0 = 116.5$  and  $\tau_0$  from Eq. (9), the polarization time is about  $10^{19}$  minutes for a 500 GeV ring of 1 Km radius.

### 3. Beam Polarization

When we discussed spin precession in the previous section, we mentioned that if an electron follows the designed trajectory exactly, its spin will precess around the vertical direction  $\hat{y}$ ; and if all electrons do so, the net beam polarization direction  $\hat{n}$  will have to be along  $\hat{y}$ . We defined a spin tune as the rate of spin precession and found it is equal to  $a\gamma$ . Then in section 2.8, we concluded that under this same condition the radiative beam polarization will be 92%. In other words, we showed:

$$\hat{n} = \hat{y} \tag{66a}$$

$$\text{spin tune} = a\gamma \tag{66b}$$

and

$$P_0 = 92\% \tag{66c}$$

We know the assumption that all electrons follow the designed trajectory is never fulfilled because the beam distribution has a finite size. Even if we build a storage ring for which all electric and magnetic devices are constructed and installed perfectly, the designed trajectory is followed only by the center of beam distribution and not by all electrons. One urgent question to be answered is what happens to the polarization properties (66) if we take into account the finite size of the beam.\*

#### 3.1 Polarization for a Perfect Storage Ring

Finite beam sizes in an electron storage ring come from the recoil perturbations that electrons receive as they radiate synchrotron photons. Let us define the orbital state of an electron by a vector

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\*For this discussion, we assume that the perfect storage ring does not have skew quadrupole and sextupole fields. Those fields will be later discussed as error fields.

$$X = \begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \Delta E/E \end{bmatrix} \quad (67)$$

where  $x$ ,  $y$  and  $z$  are the displacements of the electron relative to the center of particle distribution;  $x'$  and  $y'$  are the corresponding conjugate momenta defined to be the slopes of the electron's trajectory;  $\Delta E/E$  is the relative energy deviation. Immediately after radiation, only the  $\Delta E/E$ -coordinate of the electron is perturbed. As the electron keeps on circulating, this perturbation in  $\Delta E/E$  in general propagates into the other five orbital coordinates, giving the beam finite sizes in all six dimensions. In a perfect storage ring, however, the perturbation on  $\Delta E/E$  propagates only into the  $x$ -,  $x'$ - and  $z$ -coordinates, leaving  $y$ - and  $y'$ -coordinates free from being perturbed. As a result, the beam distribution is an infinitely thin ribbon with finite width and length but zero height. A particle in such a ribbon distribution sees, in addition to the bending magnetic field and the rf accelerating electric field seen along the designed trajectory, the perturbing magnetic fields in the quadrupoles. The nice thing is, with  $y = 0$ , these quadrupole fields are all along  $\hat{y}$ . If we look at the BMT equation (16), we find that the spin precession angular velocity  $\vec{\Omega}$  is also along  $\hat{y}$ . [The second term in Eq. (16) vanishes, the other two terms are along  $\hat{y}$ .] This establishes (66a) since any polarization components perpendicular to  $\hat{y}$  will disappear rapidly due to the different precession phases and rates of different particles. We also find from Eq. (16) that the contributions from quadrupoles and rf cavities oscillate between positive and negative values as  $x$  and  $x'$  executes betatron and synchrotron oscillations.\* As a result, the average rate of spin precession is determined by the bending magnets alone. This establishes (66b). (It is in fact a general result that the spin tune is always determined from the EM field seen along the closed orbit.) But we shall keep in

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\*In general,  $x$  contains a betatron part and a synchrotron part.

mind that the actual spin precession angle per turn deviates slightly from the average value  $2\pi a\gamma$  by an amount that depends on the betatron and synchrotron motions of the electron. This effect of "frequency modulation," as we will see later, is in fact one of the mechanisms that depolarize the beam. As to the level of radiative polarization, Eq. (66c), it is also unaffected by the finite size of the beam. This is because the rate of spin-flip transition, which we recall is the mechanism responsible for polarization build-up, is proportional to the magnetic field to the cubic power [see Eq. (60)]. The magnetic field in quadrupoles is too weak to have an appreciable effect on radiative polarization build-up. We thus conclude that in a perfectly constructed storage ring, beam polarization satisfies the nice properties listed in (66).

### 3.2 The Case of a Ribbon Beam

A real storage ring is never perfect. So we want to know what happens to beam polarization if the storage ring contains error fields.

Let us first consider two types of error fields: those due to sextupole magnets and those accidental dipole fields that cause a horizontal closed orbit distortion. An example of the later type is when a quadrupole magnet is horizontally misaligned. These error fields are special because they do not cause particles to execute vertical motions and the beam keeps its ribbon distribution. The perturbing magnetic fields seen by particles always point in the  $\pm\hat{y}$  directions. Most of the previous discussions for a perfect machine still apply. In particular, (66a) follows from the fact that  $\vec{\Omega}$  is always along  $\pm\hat{y}$  directions. Remembering that the beam distribution center always follows the closed orbit to rotate  $2\pi$  radians per turn--no matter how distorted the closed orbit may be--and that spin precesses  $a\gamma$  times faster than the coordinate does, we find the spin tune is always equal to  $a\gamma$ , i.e., (66b) is assured. As to (66c), it again follows if the perturbing fields are weaker than the main bending fields, which is satisfied for almost all

practical cases.\* We thus conclude that as long as particles do not execute  $y$ -motions, the beam (a ribbon beam!) will happily polarize itself according to (66).

### 3.3 Integer Resonances

Problems occur as soon as we include error fields that cause  $y$  excursions in particle motion. One might think these fields are weak and question why should they do any harm. For example, if an electron passes through a quadrupole of strength  $G\ell = 50$  kilogauss off-centered by 1 mm, the spin precesses by an angle of 0.3 mrad, which looks harmless. The answer to this question lies in two facts: (1) the electron passes through this quadrupole not just once but again and again as it circulates around. The innocent-looking 0.3 mrad may add up every time the electron passes through the quadrupole. The conditions for those small spin rotations to add up are referred to as the depolarization resonance conditions. (2) Even more importantly, the strengths of some of those depolarization resonances are greatly enhanced due to the presence of a noise source--the synchrotron radiation. The enhancement factor involved is typically as large as  $10^6$ .

Before we go on to discuss depolarization effects, let me make a comment here. In storage rings, typical error fields coming from, for example, magnet misalignments are proportional to beam energy  $E$ . (This is because strengths of all magnets scale with  $E$ .) This means as a particle passes through the error field, its angular deflection  $\theta$  is independent of  $E$ . The same thing does not happen for spin; it is perturbed by an angle  $a\gamma\theta$  which is proportional to  $E$ . In other words, the higher the beam energy is, the more sensitive are the particle spins to magnet misalignments and therefore the more vulnerable is the beam polarization to the depolarization resonances.

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\*That is all except one. If we insert a wiggler device--a series of bending magnets with alternating positive and negative polarities--in the storage ring, the beam remains ribbon-shaped but the associated "error" fields are strong enough to have an appreciable effect on polarization level. In fact, it always makes the polarization lower. See problem 6.

One type of depolarization resonances occurs when the spin tune  $a\gamma$  is close to an integer. To see that, let us start with Eq. (66a), i.e., the polarization direction  $\hat{n}$  is along the vertical direction  $\hat{y}$ . The reason Eq. (66a) is important is that the radiative polarization built up painstakingly by the spin-flip synchrotron radiation is along  $\hat{y}$ . If  $\hat{n} \neq \hat{y}$ , the beam will keep only the polarization component along  $\hat{n}$  and the net beam polarization will be reduced by a cosine factor  $\hat{n} \cdot \hat{y}$ . Clearly, one loses polarization if  $\hat{n}$  deviates appreciably from  $\hat{y}$ .\*

Consider a particle at the center of beam distribution. It follows the closed orbit and sees an external EM field that is periodic in  $s$  with period  $2\pi R$ . Its spin therefore precesses with a periodic angular velocity. Starting with  $s$ , we can integrate this angular velocity through one turn to obtain a net rotation on spin. If the spin is represented as a 3-dimensional vector:

$$\vec{S} = \begin{bmatrix} S_x \\ S_y \\ S_z \end{bmatrix}, \quad (68)$$

the net rotation can be written as a  $3 \times 3$  matrix  $R(s)$ . The beam polarization direction  $\hat{n}(s)$  is then given by the rotational axis of  $R(s)$  (use right-hand rule):

$$R(s) \hat{n}(s) = \hat{n}(s). \quad (69)$$

In case particles do not execute  $y$ -motions (the ribbon beam case),  $R(s)$  is simply a rotation about  $\hat{y}$ :

$$R(s) = \begin{bmatrix} \cos 2\pi a\gamma & 0 & \sin 2\pi a\gamma \\ 0 & 1 & 0 \\ -\sin 2\pi a\gamma & 0 & \cos 2\pi a\gamma \end{bmatrix} \quad (70)$$

The rotational axis of (70) is  $\hat{n}(s) = \hat{y}$ , which of course is just (66a). The calculation of  $R(s)$  and  $\hat{n}(s)$  for the general case will be described

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\*Strictly speaking, such a loss of polarization is not a "depolarization" mechanism. It is rather a "lack of polarization."



in detail later. Here let us consider a somewhat idealized case in which the spin precession from  $s = 0$  to  $s = 2\pi R$  is given by Eq. (70) but at  $s = 0$  there is a perturbing magnetic field  $B_x$  along the horizontal direction. Such a field may come from a vertical closed orbit distortion at a quadrupole. The spin rotation across the perturbing field is described by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \quad (71)$$

with  $\theta = (1 + a\gamma) B_x \ell / B\rho$ . The total rotation matrix for one revolution is given by the product of (70) and (71). It can easily be shown that the corresponding rotational axis  $\hat{n}(s)$  everywhere outside the perturbing field region has the cosine factor  $\hat{n} \cdot \hat{y} = 1/\sqrt{1 + \tan^2 \frac{\theta}{2} \csc^2 \pi a\gamma}$ . It follows that the beam polarization vanishes ( $\hat{n} \cdot \hat{y} = 0$ ) on an "integer resonance," i.e., when the spin tune  $a\gamma$  is equal to an integer  $k$ . One can also calculate the width in  $a\gamma - k$  within which beam polarization is significantly reduced. The width is found to be of the order of  $\theta/2\pi$ . Taking again  $\theta = 0.3$  mrad as a typical value from our numerical example mentioned before, the resonance width is about  $0.5 \times 10^{-4}$ , which is much narrower than the spacing between the integer resonances. This means integer depolarization resonances are easy to avoid. Furthermore, the integer resonances depolarize the beam through the cosine factor  $\hat{n} \cdot \hat{y}$ . Unlike other depolarization resonances to be mentioned later, they are not enhanced by the noise due to synchrotron radiation. We thus expect that integer resonances are not a serious problem in storage rings.

### 3.4 Sideband Resonances

In the previous idealized example, we have followed a particle at the beam center to obtain the polarization direction  $\hat{n}$ . Consider now instead a particle that executes a horizontal betatron oscillation  $x_\beta$ . The part of precession described by Eq. (70) needs to be modified; the angle  $2\pi a\gamma$  now contains an additional term that is proportional to the betatron amplitude  $\hat{x}_\beta$  and is oscillatory with the betatron tune  $\nu_x$ . One might say that the spin precession motion is "frequency modulated"

by the  $x_\beta$ -motion. A result of such a frequency modulation is the occurrence of frequency sidebands. In other words, to first order in  $\hat{x}_\beta$ , the system now contains, in addition to the natural frequency  $a\gamma$ , two more frequencies  $a\gamma \pm \nu_x$ . If we now introduce the perturbation (71), the spin motion will be seriously influenced if  $a\gamma \pm \nu_x$  is equal to an integer  $k$ .

A similar thing happens if the electron executes a synchrotron oscillation. The spin precession motion described by Eq. (70), is frequency modulated by the synchrotron motion at the synchrotron tune  $\nu_s$ . Two sidebands at  $a\gamma \pm \nu_s$  occur and the spin motion is seriously influenced by the perturbation (71) if  $a\gamma \pm \nu_s = k$ .

The spin motion is also seriously perturbed at the vertical betatron sidebands  $a\gamma \pm \nu_y = k$ . The mechanism, however, is different from the frequency modulation mechanism for the previous cases. In the idealized example, the source of the problem is now not Eq. (70) but Eq. (71). As the particle executes a  $y_\beta$ -oscillation, the magnetic field it experiences at the quadrupole contains two terms: the static  $B_x$  that causes the spin to precess according to Eq. (71) and an additional  $B_x$  that oscillates with  $y_\beta$ . One might now say that the simple harmonic spin precession is "driven" by an oscillatory driving force every time the electron passes through the quadrupole. If the frequency  $\nu_y$  of the driving force and the natural simple harmonic frequency  $a\gamma$  satisfy the resonance condition  $a\gamma \pm \nu_y = k$ , we expect a strong response of spin to the driving.

Once we deviate from our idealized case, the situation rapidly becomes complicated. For example, if there is a skew quadrupole field somewhere, it produces a perturbation (71) when the electron has an  $x$ -displacement. The resonance driving mechanism now also applies to the  $a\gamma \pm \nu_x = k$  and the  $a\gamma \pm \nu_s = k$  sidebands. One can also imagine that the simple harmonic precession will be frequency modulated by  $y_\beta$ -motion if there are vertical bending dipoles in the storage ring. It is clear that studying these effects case by case is cumbersome, if not impossible. What is needed is a general, more formal description, which we will offer in section 3.7.

I have not yet explained the role of synchrotron radiation in enhancing the depolarization resonances. Imagine an electron following the closed orbit with its spin  $\vec{S}$  happily polarized along  $\hat{n}$ . Now suddenly it emits a photon of energy  $\delta E$  at time  $t = 0$ . After the emission, the electron starts to execute orbital oscillations around the closed orbit. The oscillations can be decomposed into three modes, which we somewhat loosely refer to as the horizontal and vertical betatron modes and the synchrotron mode. We know that these excited orbital oscillations are damped by radiation damping. The damping times  $\tau_{\text{rad}}$  for the three modes are somewhat different but they are all comparable, typically about several miniseconds.

A few  $\tau_{\text{rad}}$  after the radiation, the electron damps to the closed orbit and quiets down again. Meanwhile,  $\vec{S}$  starts to precess away from  $\hat{n}$  due to the perturbing EM fields seen away from the closed orbit. Similar to the orbital motion, this excited spin motion will also quiet down. The time constant, however, is not  $\tau_{\text{rad}}$  but the polarization time constant  $\tau_0$  given by Eq. (9), which typically reads at least several minutes. We have illustrated in Figures 4(A)-(D) the spin motion during this whole process.

The perturbing EM field that acts on the spin from  $t = 0$  to  $t = \text{a few } \tau_{\text{rad}}$  is oscillatory with frequencies  $\nu_x$ ,  $\nu_y$  and  $\nu_s$ . This field perturbs the spin through both the frequency modulation and the driving mechanisms mentioned before. In case the spin tune  $a\gamma$  is such that one of the sideband conditions is fulfilled or nearly fulfilled, this photon emission event will destroy the polarization of this electron. (In Figure 4, this means  $\theta \rightarrow \infty$ .) One can imagine doing a calculation of the widths of the sideband resonances just like we did for the integer resonances. Within the widths, the angle  $\theta$  of Figure 4 is of the order of 1 radian. One then probably finds that the widths are very narrow and concludes that sideband resonances are not a serious problem for beam polarization. What happens, however, is that photons are constantly being emitted. Staying outside of such a resonance width not necessarily guarantee a good polarization. For example, if each photon emission causes the spin to deviate from  $\hat{n}$  by an angle  $\theta$  of, say,  $10^{-6}$  rad, then the spin will random-walk away from  $\hat{n}$  in about  $10^{12}$  emissions. For a

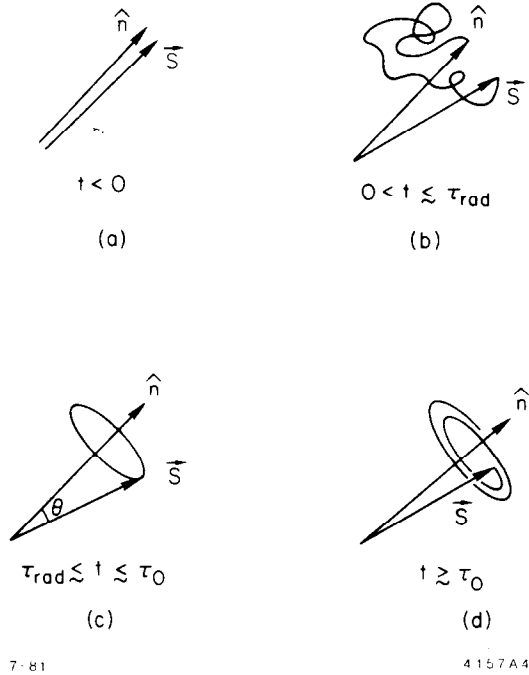


Fig. 4 The motion of an electron spin  $\vec{S}$  following the sudden emission of a synchrotron photon of energy  $\delta E$ . (A) Before emission ( $t < 0$ ), the electron is polarized with  $\vec{S}$  along the direction  $\hat{n}$  of the net beam polarization. (B) Photon emission excites the orbital motions of the electron, which cause the electron to see some perturbing EM fields. After the emission ( $t > 0$ ) and before the orbital motions are radiation damped ( $t \lesssim \tau_{\text{rad}}$ ),  $\vec{S}$  precesses according to the perturbing fields in some complicated manner. The radiation damping time  $\tau_{\text{rad}}$  is typically several miniseconds. (C) After the orbital motions are damped ( $t \gtrsim \tau_{\text{rad}}$ ),  $\vec{S}$  sees no perturbing fields and starts to execute a simple precession motion around  $\hat{n}$ . The angle  $\theta$  is an important parameter that determines the strength of depolarization due to synchrotron radiation.

(D) The precessing  $\vec{S}$  slowly spirals in toward  $\hat{n}$  due to the polarizing effect of synchrotron radiation. Significant spiralling occurs after a time  $\tau_0$  given by the polarization time, typically at least several minutes. A few  $\tau_0$  later,  $\vec{S}$  damps to  $\hat{n}$ . The excitation-damping process (A) to (D) is repeated every time a synchrotron photon is emitted.

5 GeV storage ring of 25m radius, this means a depolarization time of about  $10^9$  revolutions (there will be  $10^3$  emissions per revolution) or about 10 minutes. To guarantee good polarization, the depolarization time must be much larger than the polarization time. This means one must stay away from the sideband resonances far enough so that  $\theta$  is less than something like  $10^{-6}$  rad rather than 1 rad. Synchrotron radiation thus greatly enhances the sideband depolarization resonances.

Since  $\tau_{\text{rad}}$  is so much shorter than  $\tau_0$ , one can ignore the time period  $0 < t \lesssim \tau_{\text{rad}}$  [Figure 4(B)] as far as spin polarization is concerned. For  $t < 0$ , we have  $\vec{S} - \hat{n} = 0$ . For  $t > 0$ , the deviation of  $\vec{S}$  from  $\hat{n}$  is proportional to the perturbation  $\delta E/E$ . If we extrapolate the spin precession motion of Figure 4(C) backwards in time to the moment of emission,  $t = 0$ , we can write

$$\vec{S} - \hat{n} = \frac{\delta E}{E} \cdot \gamma \frac{\partial \hat{n}}{\partial \gamma} \quad \text{at } t = 0, \quad (72)$$

where we have defined a proportionality vector  $\gamma \partial \hat{n} / \partial \gamma$ .

The vector  $\gamma \hat{n} / \partial \gamma$  is a crucial quantity in determining the radiative polarization of the beam. It is a 2-dimensional vector perpendicular to  $\hat{n}$ . For a ribbon beam, perturbations due to synchrotron radiation is decoupled from the spin motion and we have  $\gamma \partial \hat{n} / \partial \gamma = 0$ . In general, it is a vector completely determined by the storage ring lattice depending only on the location  $s$  where the photon is emitted, independently of synchrotron radiation and spin. Following Buon,<sup>15</sup> we shall call  $\gamma \partial \hat{n} / \partial \gamma$  the "spin chromaticity." The notation used here follows that of Derbenev, Kondratenko and Skrinsky.<sup>9</sup> It should be mentioned that although this notation suggests more or less its physical meaning, it is not to be taken too literally to mean the partial derivative of  $\hat{n}$  relative to  $\gamma$ . Note also that the angle  $\theta$  shown in Figure 4 is equal to  $|(\delta E/E) \gamma \partial \hat{n} / \partial \gamma|$ . It specifies the random walk step-size of quantum diffusion on spin motion.

### 3.5 Determining the Beam Polarization

We assume that the storage ring fields, including the error fields, and the associated closed-orbit distortion are known. From this information, one can obtain the polarization direction  $\hat{n}(s)$  and the spin chromaticity  $\gamma \partial \hat{n} / \partial \gamma(s)$  around the storage ring (see Figure 5). A matrix formulation will be described in sections 3.6 and 3.7 for this purpose. Here let us assume  $\hat{n}$  and  $\gamma \partial \hat{n} / \partial \gamma$  are already known and we will look for an expression of beam polarization in terms of these quantities.

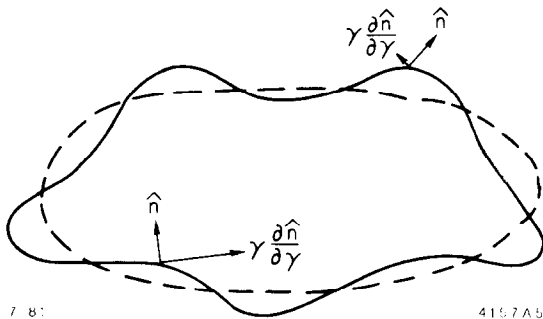


Fig. 5 A schematic drawing of the direction of polarization  $\hat{n}$  and the spin chromaticity  $\gamma \partial \hat{n} / \partial \gamma$ . The dotted line indicates the designed trajectory. The solid line is the distorted closed orbit. Note that  $\hat{n}(s)$  is a unit vector but the magnitude of  $\gamma \partial \hat{n} / \partial \gamma(s)$  varies with  $s$ .

Consider an unpolarized electron beam stored at time  $t = 0$ . Due to synchrotron radiation, with all its polarizing as well as depolarizing effects, the beam slowly acquires a polarization  $\zeta(t) \hat{n}$  along  $\hat{n}$ . We expect  $\zeta(t)$  to approach an equilibrium value  $P$  with a time constant  $\tau$ .

The ideal case has been worked out in section 2.8. Here we want to find the general expressions for P and  $\tau$ .

We start with Eq. (60). The first term in (60) describes the precession motion. For a polarization along  $\hat{n}$ , it can be dropped since  $\hat{n}$  is the precession axis. The second term comes from spin-flip radiation. It of course must be kept and we have

$$\dot{\zeta}(t) = -\frac{1}{\tau_0} \left[ \zeta(t) - \frac{2}{9} \zeta(t) (\hat{n} \cdot \hat{z})^2 + \frac{8}{5\sqrt{3}} \hat{n} \cdot \hat{y} \right] \quad (73)$$

where  $\hat{z}$  is along the beam motion,  $\hat{y}$  is along the bending magnetic field.

Since we expect the polarization to be very slowly changing, it is a good approximation to average the right-hand side of Eq. (73) over the circumference of the ring. Inserting  $\tau_0$  from Eq. (9), this gives

$$\begin{aligned} \dot{\zeta}(t) = & -\frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{m^2 c^2} \frac{1}{2\pi R} \left\{ \zeta(t) \oint ds \frac{\left[1 - \frac{2}{9} (\hat{n} \cdot \hat{z})^2\right]}{|\rho|^3} \right. \\ & \left. + \frac{8}{5\sqrt{3}} \oint ds \frac{\hat{n} \cdot \hat{y}}{|\rho|^3} \right\} \end{aligned} \quad (74)$$

Eq. (74) is incomplete; it contains only the polarizing effect. We will have to include more terms coming from the spin chromatic effects due to a nonzero  $\gamma \partial \hat{n} / \partial \gamma$ . Let us, however, ignore  $\gamma \partial \hat{n} / \partial \gamma$  for a short moment. The equilibrium level of polarization would then be given by

$$-\frac{8}{5\sqrt{3}} \frac{\oint ds \hat{n} \cdot \hat{y} / |\rho|^3}{\oint ds \left[1 - \frac{2}{9} (\hat{n} \cdot \hat{z})^2\right] / |\rho|^3} \quad (75)$$

The factor  $\hat{n} \cdot \hat{y}$  is the cosine reduction factor mentioned when we discussed integer depolarization resonances. The (less important) factor  $\left[1 - \frac{2}{9} (\hat{n} \cdot \hat{z})^2\right]$  in the denominator comes from the slight dependence of spin-flip radiation on the  $\hat{z}$ -component of electron spin.

The effects of spin chromaticity  $\gamma \partial \hat{n} / \partial \gamma$  are associated with synchrotron radiation without spin-flips. Consider an electron polarized along  $\hat{n}$ . As a photon of energy  $\delta E$  is emitted, its spin starts to precess around  $\hat{n}$  with a small rotating deviation  $\vec{d}$ . After emission, the electron has lost a polarization

$$\frac{\Delta P}{P} = -\frac{1}{2} |\vec{d}|^2 = -\frac{1}{2} \left| \frac{\delta E}{E} \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 . \quad (76)$$

Let  $\dot{N}$  be the number of photon emissions per unit time, we obtain the quantum diffusion rate on polarization:

$$\dot{\zeta}(t) = \frac{\zeta(t)}{2\pi R} \oint ds \left(-\frac{1}{2}\right) \left\langle \dot{N} \frac{\delta E^2}{E^2} \right\rangle_s \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|_s^2 \quad (77)$$

where we have averaged over  $s$  and  $\langle \dot{N} \delta E^2 / E^2 \rangle$  is given by integrating  $\hbar \omega d^2 \mathcal{P} / d\omega d\Omega$  [see Eq. (40)] over  $\omega$  and  $\Omega$ :

$$\left\langle \dot{N} \frac{\delta E^2}{E^2} \right\rangle_s = \frac{55}{24\sqrt{3}} \frac{\hbar e^2 \gamma^5}{m^2 c^2 |\rho^3(s)|} \quad (78)$$

There is another effect on polarization due to the spin chromaticity. Consider now an electron that is not perfectly polarized before radiation. Let its spin be  $\hat{n} + \vec{D}$  with  $\vec{D}$  a small rotating vector orthogonal to  $\hat{n}$ . Now the electron emits a photon of energy  $\delta E$ . After emission, the spin acquires another rotating deviation  $\vec{d}$ . Let  $\vec{D}_0$  and  $\vec{d}_0$  be the values of  $\vec{D}$  and  $\vec{d}$  extrapolated to the moment of emission. If  $\delta E$  does not depend on  $\vec{D}_0$ , i.e., if the synchrotron radiation does not depend on the instantaneous spin,  $\vec{d}_0$  is uncorrelated with  $\vec{D}_0$  and we simply have observed a random walk in spin motion. The story is quite different if  $\delta E$  does depend on  $\vec{D}_0$ . Then  $\vec{d}_0$  correlates with  $\vec{D}_0$  and the original amount of depolarization will decrease or increase according to how  $\vec{d}_0$  and  $\vec{D}_0$  are correlated. In the former case the correlation is polarizing, while in the latter case, depolarizing.

More quantitatively, the polarization of the electron before and after the radiation are equal to  $1 - \frac{1}{2} |\vec{D}_0|^2$  and  $1 - \frac{1}{2} |\vec{D}_0 + \vec{d}_0|^2$ , respectively. Change of polarization due to the radiation is therefore

$$\Delta P = -\vec{D}_0 \cdot \vec{d}_0 - \frac{1}{2} |\vec{d}_0|^2 . \quad (79)$$

The second term in (79) is the random walk term already discussed. Summing up on photon emission events, the contribution of the first term in (79) to the polarization process is found to be

$$\begin{aligned}\dot{\zeta}(t) &= \langle -\dot{\mathbf{N}} \cdot \vec{\mathbf{D}}_0 \cdot \vec{\mathbf{d}}_0 \rangle \\ &= -\frac{\langle \dot{\mathbf{N}} \delta E \rangle}{E} \vec{\mathbf{D}}_0 \cdot \gamma \frac{\partial \hat{\mathbf{n}}}{\partial \gamma}\end{aligned}\quad (80)$$

where, as before,  $\langle \rangle$  means averaging over the radiation spectrum. In addition, an averaging over  $s$  is understood. Expression for  $\langle \dot{\mathbf{N}} \delta E \rangle$  has been obtained before; it is given by Eq. (5). The spin-independent terms in Eq. (5) do not concern us here. Keeping only the spin-dependent term gives

$$\langle \dot{\mathbf{N}} \delta E \rangle = -\frac{1}{2} \mathcal{P}_{\text{class}} \vec{\mathbf{S}} \cdot \hat{\mathbf{y}} \frac{\hbar \omega}{E}, \quad (81)$$

where the spin direction in Eq. (5) has been replaced by the instantaneous value  $\vec{\mathbf{S}} = \hat{\mathbf{n}} + \vec{\mathbf{D}}_0$ . Also since  $\hat{\mathbf{n}}$  is perpendicular to  $\gamma \partial \hat{\mathbf{n}} / \partial \gamma$ , the vector  $\vec{\mathbf{D}}_0$  in Eq. (80) can be replaced by  $\vec{\mathbf{S}}$ . Substituting Eq. (81) into Eq. (80) yields

$$\dot{\zeta}(t) = \frac{\hbar \omega}{2E^2} \mathcal{P}_{\text{class}} (\vec{\mathbf{S}} \cdot \hat{\mathbf{y}}) (\vec{\mathbf{S}} \cdot \gamma \frac{\partial \hat{\mathbf{n}}}{\partial \gamma}) \quad (82)$$

Since  $\vec{\mathbf{S}}$  can in principle point in any arbitrary direction, the next step is to average over its solid angles, keeping its magnitude constant. When this is done, the factor  $(\vec{\mathbf{S}} \cdot \hat{\mathbf{y}}) (\vec{\mathbf{S}} \cdot \gamma \partial \hat{\mathbf{n}} / \partial \gamma)$  becomes  $\frac{1}{3} (\hat{\mathbf{y}} \cdot \gamma \partial \hat{\mathbf{n}} / \partial \gamma) |\vec{\mathbf{S}}|^2$ . Now the question is what to use for  $|\vec{\mathbf{S}}|^2$ . One may argue that since  $\vec{\mathbf{S}}$  is the unit spin direction, it obviously has  $|\vec{\mathbf{S}}|^2 = 1$ . The correct answer, however, is  $|\vec{\mathbf{S}}|^2 = 3$ , which comes from the fact that the magnitude of the electron spin must be determined from the quantum mechanical relation  $|\frac{1}{2} \hbar \vec{\mathbf{S}}|^2 = \frac{1}{2} (\frac{1}{2} + 1) \hbar^2$ . Thus, Eq. (82) becomes, after averaging over  $s$ :

$$\dot{\zeta}(t) = \frac{1}{2\pi R} \oint ds \frac{\hbar \omega}{2E^2} \mathcal{P}_{\text{class}} \hat{\mathbf{y}} \cdot \gamma \frac{\partial \hat{\mathbf{n}}}{\partial \gamma} \quad (83)$$

We have now obtained three separate contributions to  $\dot{\zeta}(t)$ ; they are given by Eqs. (74), (77) and (83). Adding them up gives the final expression obtained by Derbenev and Kondratenko:<sup>6</sup>



$$\dot{\zeta}(t) = -\frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{m^2 c^2} \left[ \alpha_+ \zeta(t) - \frac{8}{5\sqrt{3}} \alpha_- \right] \quad (84)$$

where

$$\begin{aligned} \alpha_+ &= \frac{1}{2\pi R} \oint \frac{ds}{|\rho(s)|^3} \left[ 1 - \frac{2}{9} (\hat{n} \cdot \hat{v}) + \frac{11}{18} \left| \gamma \frac{\partial \hat{n}}{\partial \gamma} \right|^2 \right]_s \\ \alpha_- &= \frac{1}{2\pi R} \oint \frac{ds}{|\rho(s)|^3} \left[ \frac{\hat{v} \times \dot{\hat{v}}}{|\dot{\hat{v}}|} \cdot (\hat{n} - \gamma \frac{\partial \hat{n}}{\partial \gamma}) \right]_s \end{aligned} \quad (85)$$

In Eq. (84), the symbol for the instantaneous direction of beam motion has been changed (hopefully for clarity) from  $\hat{z}$  to  $\hat{v}$ ; the symbol for the magnetic field direction has been changed from  $\hat{y}$  to  $-\hat{v} \times \dot{\hat{v}} / |\dot{\hat{v}}|$  with  $\dot{\hat{v}}$  along the direction of acceleration. The later change of symbol has the advantage that it also takes care of positrons. From Eq. (84), it follows that the equilibrium beam polarization is equal to

$$P = \frac{8}{5\sqrt{3}} \frac{\alpha_-}{\alpha_+}, \quad (86)$$

and the time constant to reach the equilibrium is

$$\tau = \left( \frac{5\sqrt{3}}{8} \frac{e^2 \gamma^5 \hbar}{m^2 c^2} \alpha_+ \right)^{-1}. \quad (87)$$

In the case of a perfect planar storage ring, Eqs. (86) and (87) reduce to the results of section 2.8. An indication of why the symbols  $\alpha_+$  and  $\alpha_-$  were chosen can be found by comparing the expressions (86) and (87) with (8) and (9).

We have thus obtained an expression for the equilibrium level of beam polarization. The integer resonances  $a\gamma = k$  show their effect in causing  $\hat{n}$  to deviate from  $\hat{y}$ . They are not enhanced by synchrotron radiation noise. The sideband resonances  $a\gamma \pm \nu_{x,y,s} = k$ , on the other hand, cause the spin chromaticity to become large. They are enhanced by synchrotron radiation noise and are responsible for most of the loss of beam polarization in electron storage rings.

An inspection of Eqs. (85) and (86) shows that the spin chromaticity appears as a quadratic term in the denominator of P and only linearly in the numerator. Loss of polarization occurs if  $|\gamma \partial \hat{n} / \partial \gamma| \gtrsim 1$ . This means the angle  $\theta$  of Figure 4(c) will be bigger than  $\delta E/E$ , which is of the order of  $\hbar \omega_c / E$ . The spin chromaticity term  $\hat{y} \cdot \gamma \partial \hat{n} / \partial \gamma$  in  $\alpha_-$  is small for most practical cases. This follows from the fact that  $\hat{n}$  is nearly equal to  $\hat{y}$  and that  $\gamma \partial \hat{n} / \partial \gamma$  is perpendicular to  $\hat{n}$ . Finally, skeptical readers who wonder if P, as given by Eq. (86), could be larger than unity (then something is obviously wrong!) should work out problem 9.

### 3.6 The Polarization Direction $\hat{n}(s)$

We assume that the 6-dimensional closed-orbit vector  $X_0 = (x_0, x'_0, y_0, y'_0, z_0, \delta_0)$  in the presence of various error fields has been obtained around the storage ring. From the electric and magnetic field along the closed orbit, one obtains from Eq. (16) the angular velocity  $\vec{\Omega}(X_0)$ . Adopting the thin-lens approximation, we let  $\vec{\Omega}(X_0)$  to be uniform in a given lattice element. The matrix which transforms the spin components (68) as the particle travels through a distance  $s$  in a uniform EM field is given by

$$\begin{bmatrix} \alpha^2(1-C) + C & \alpha\beta(1-C) - \gamma S & \alpha\gamma(1-C) + \beta S \\ \alpha\beta(1-C) + \gamma S & \beta^2(1-C) + C & \beta\gamma(1-C) - \alpha S \\ \alpha\gamma(1-C) - \beta S & \beta\gamma(1-C) + \alpha S & \gamma^2(1-C) + C \end{bmatrix} \quad (88)$$

where  $\alpha$ ,  $\beta$  and  $\gamma$  are the direction cosines  $\hat{\Omega} \cdot \hat{x}$ ,  $\hat{\Omega} \cdot \hat{y}$  and  $\hat{\Omega} \cdot \hat{z}$ ; and  $C = \cos(\Omega s)$   $S = \sin(\Omega s)$ . Knowing  $\vec{\Omega}(X_0)$ , one obtains the  $3 \times 3$  matrix which transforms the spin components through a given lattice element.

One then multiplies all  $3 \times 3$  matrices successively to obtain the total spin precession transformation  $R_{\text{tot}}$  for one revolution around  $s = 0$ . A right-handed orthonormal base  $(\hat{n}, \hat{m}, \hat{\ell})$  with  $\hat{n}$  rotation axis of  $R_{\text{tot}}$  is then chosen. Successive transformations bring this base to other positions. In one revolution,  $\hat{n}$  comes back to its starting value; but  $\hat{m}$  and  $\hat{\ell}$  have rotated around  $\hat{n}$  by an angle  $2\pi\nu$ , where  $\exp(\pm i2\pi\nu)$  are the two nontrivial eigenvalues of  $R_{\text{tot}}$ . The quantity  $\nu$  gives the spin precession

tune and  $\hat{n}$  gives the direction of beam polarization. For a storage ring with planar geometry and without error fields,  $\nu$  is equal to  $a\gamma$  and  $\hat{n}$  is along  $\hat{y}$ . For rings with error fields,  $\nu \approx a\gamma$  and  $\hat{n} \approx \hat{y}$  to a high degree of accuracy provided  $a\gamma$  is a distance  $\gtrsim 10^{-3}$  away from integers.

### 3.7 Spin Chromaticity $\gamma \partial \hat{n} / \partial \gamma$

We assume that the closed orbit  $X_0$  and the spin base vectors  $(\hat{n}, \hat{m}, \hat{\ell})$  are now obtained. The spin of a nearly polarized electron is written as

$$\vec{S} \cong \hat{n} + \alpha \hat{m} + \beta \hat{\ell}, \quad |\alpha, \beta| \ll 1 \quad (89)$$

The quantities  $\alpha$  and  $\beta$  thus describe the spin to a linear approximation and  $\frac{1}{2}(\alpha^2 + \beta^2)$  specifies the degree of depolarization of this electron. The assumption  $|\alpha, \beta| \ll 1$  is acceptable since, as explained before, we are interested in cases down to the  $|\alpha, \beta| \lesssim 10^{-6}$  level.

For an electron that deviates from the closed orbit by the state vector  $X$ , given by Eq. (67), the angular velocity is given by  $\vec{\Omega}(X_0 + X)$ . In a linear approximation,  $\vec{\Omega}$  can be decomposed into  $\vec{\Omega}(X_0) + \vec{\omega}(X)$ , where the perturbation  $\omega$  is small compared with  $\Omega$ .

We need now to know how the orbital coordinates  $X$  and the spin coordinates  $\alpha, \beta$  evolve in time. We know that<sup>4</sup> the orbital motion of a particle in an accelerator is most conveniently described by the transport matrices. In the absence of coupling, transport matrices of a small dimension ( $2 \times 2$  for y-motion,  $3 \times 3$  for x-motion, etc.) will be sufficient. With x-y coupling, one uses  $4 \times 4$  matrices and in case x-, y- and z-motions are all coupled together, one must deal with  $6 \times 6$  matrices. It does not require too much imagination to realize that the next step is to construct an 8-dimensional state vector

$$\begin{bmatrix} x \\ x' \\ y \\ y' \\ z \\ \Delta E/E \\ \alpha \\ \beta \end{bmatrix} \quad (90)$$

The corresponding transport matrices are then  $8 \times 8$ .

To appreciate the need of dealing with such a generality, we remember that the spin motion of an electron depends on the electric and magnetic fields it experiences according to the BMT equation; and that those fields, in turn, depend on its orbital coordinates. This means coupling effects between spin and orbital coordinates play an important role as far as spin motion is concerned.

Concerning the spin-orbit coupling, we mentioned that the spin motion is influenced by the orbital motion. In fact, orbital motion of an electron is also influenced by its spin. The influence is expected to be extremely weak (of the order of  $\hbar$ ) and will be ignored. To see how small these effects are, let us consider a vertically polarized electron with magnetic moment  $\vec{\mu} = \mu\hat{y}$ . A quadrupole magnet, which is a focusing element for an electric charge, acts on  $\vec{\mu}$  as a bending element. The bending is done in the horizontal plane and the bending angle is  $\Delta x' = \mu G\ell/E$  with  $G$  the field gradient and  $\ell$  the magnet length. Taking a typical quadrupole magnet in an electron storage ring, we might have  $G\ell = 50$  kilogauss, the kicking angle  $\Delta x'$  is found to be about  $10^{-13}$  rad. Similarly, a sextupole magnet which produces a nonlinear field for an electric charge acts on  $\vec{\mu}$  as a linear focusing element. The focal length is given by  $f^{-1} = \mu G'\ell/E$  with  $G' = \partial^2 B_y / \partial x^2$  the sextupole strength. Again taking a typical sextupole strength  $G'\ell = 500$  kilogauss/m, we find the focal length is about  $10^{12}$  meters. Both the bending by quadrupoles and focusing by sextupoles are indeed exceedingly weak.

Noting that  $\hat{n}$ ,  $\hat{m}$  and  $\hat{\ell}$  satisfy

$$\frac{d}{ds} \vec{S} = \vec{\Omega}(X_0) \times \vec{S}$$

one obtains by substituting Eq. (89) into the precession equation that

$$\begin{aligned} \frac{d}{ds} \alpha &\simeq \vec{\omega}(X) \cdot \hat{\ell} \\ \frac{d}{ds} \beta &\simeq -\vec{\omega}(X) \cdot \hat{m} \end{aligned} \tag{91}$$

The  $8 \times 8$  transport matrix looks like

$$\left[ \begin{array}{c|cc} T_{6 \times 6} & & 0 \\ \hline D & & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{array} \right] \quad (92)$$

where  $T_{6 \times 6}$  means the  $6 \times 6$  transport matrices describing the transformation among the orbital coordinates; the upper right corner is a  $6 \times 2$  matrix filled by 0's because we are ignoring the effects of spin on orbital motions; the spin-orbit coupling coefficients in the  $2 \times 6$  matrix  $D$  are obtained from Eq. (91). Explicit expressions of the  $8 \times 8$  matrices are given in the Appendix.

One must not forget that, due to the discontinuous transition in the definition of the base vectors as the electron travels across  $s = 2\pi R$ , an extra transformation for the spin components is required:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{s=2\pi R^+} = \begin{bmatrix} \cos 2\pi\nu & -\sin 2\pi\nu \\ \sin 2\pi\nu & \cos 2\pi\nu \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}_{s=2\pi R^-} \quad (93)$$

where  $\nu$  is the spin tune found in section 3.6. Starting from  $s$  we multiply matrices to obtain a transformation matrix  $T(s)$  for one revolution. Let the eigenvalues and eigenvectors of  $T(s)$  be  $\lambda_k$  and  $E_k(s)$ , respectively, with

$$\begin{aligned} T(s) E_k(s) &= \lambda_k E_k(s) \\ \lambda_k^* &= \lambda_{-k} \\ E_k^* &= E_{-k}, \quad k = \pm I, \pm II, \pm III, \pm IV. \end{aligned} \quad (94)$$

Eigenvectors at other positions,  $E_k(s')$ , are obtained from  $E_k(s)$  by successive transformations from  $s$  to  $s'$ . The first three sets of eigenmodes are the orbital modes. The three eigenvalues give the orbital tunes. The corresponding eigenvectors in general carry nonzero spin components. The fourth pair of eigenvectors,  $E_{\pm IV}$ , on the other hand, contains only spin components and no orbital components. The corresponding eigenvalues are given by  $\exp(\pm i2\pi\nu)$  with  $\nu$  the spin tune.

Consider an electron that follows the closed orbit with spin along  $\hat{n}$ . A photon of energy  $\delta E$  is emitted at  $s_0$ . Immediately after the emission, the electron is left in a state

$$X(S_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\delta E/E \\ 0 \\ 0 \end{bmatrix} \quad (95)$$

Decomposed into eigenstates, it can be written as

$$\begin{aligned} X(s_0) &= \sum_k A_k E_k(s_0) \\ &= \sum_{k=\pm I, \pm II, \pm III} A_k E_k(s_0) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \alpha \\ \beta \end{bmatrix}_{s_0} \end{aligned} \quad (96)$$

where we have used the fact that  $E_{\pm IV}$  contain no orbital coordinates.

Eq. (96) contains 8 unknowns  $A_{\pm I, \pm II, \pm III}$  and  $\bar{\alpha}, \bar{\beta}$ , and 8 equations to determine them. The first six equations obtained by equating the orbital components of Eq. (96) can be written as

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\delta E/E \end{bmatrix} = \sum_k A_k e_k(s_0) \quad (97)$$

where  $e_k$  is a 6-dimensional vector whose components are the orbital coordinates of  $E_k$ . To find  $A_k$ , some orthogonality condition on the eigenvectors  $e_k$  is needed. This condition is provided by the symplecticity property of the transport matrices  $T_{6 \times 6}(s)$ , i.e.,<sup>4</sup>

$$\tilde{T}_{6 \times 6}^T S T_{6 \times 6} = S \quad , \quad (98)$$

where a tilde means taking the transpose of a matrix and

$$S = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (99)$$

From Eq. (98), one can find the orthogonality condition

$$\tilde{e}_j S e_i = 0 \quad \text{unless} \quad j = -i. \quad (100)$$

When  $j = -i$ , we choose the normalization so that

$$\tilde{e}_{-k} S e_k = i \quad , \quad k = I, II, III \quad . \quad (101)$$

The reason we normalize it to  $i$  rather than  $1$  is, as one can easily show, this quantity must be purely imaginary. For  $k = -I, -II$  and  $-III$ , they are normalized to  $-i$ . Conditions (100) and (101) are preserved as a function of  $s$  due to the symplecticity of  $T(s)$ . Using Eqs. (100) and (101), Eq. (97) yields

$$A_k = -i \frac{\delta E}{E} E_{k5}^* (s_o) \quad (102)$$

where  $E_{ki}$  means the  $i^{\text{th}}$  component of the vector  $E_k$ .

Having found  $A_k$ , one then solves the remaining 2 equations corresponding to the spin part of Eq. (96):

$$\begin{bmatrix} \bar{\alpha} \\ \bar{\beta} \end{bmatrix}_{s_o} = -2 \frac{\delta E}{E} \sum_{k=I, II, III} \begin{bmatrix} \text{Im}(E_{k5}^* E_{k7}) \\ \text{Im}(E_{k5}^* E_{k8}) \end{bmatrix}_{s_o} \quad (103)$$

After the photon emission, the orbital components of Eq. (96) are rapidly damped by the radiation damping, leaving the spin to precess around  $\hat{n}$  as if it started at  $s_o$  with an initial deviation  $\vec{S} - \hat{n} = \bar{\alpha}\hat{m} + \bar{\beta}\hat{l}$ . Associating with the definition of the spin chromaticity, we find

$$\left( \gamma \frac{\partial \hat{n}}{\partial \gamma} \right)_s = -2 \sum_{k=I, II, III} \left[ \text{Im} (E_{k5}^* E_{k7}) \hat{m} + \text{Im} (E_{k5}^* E_{k8}) \hat{l} \right]_s \quad (104)$$

in which we have dropped the subscript on  $s_0$  since spin chromaticity is defined for all  $s$ .

### 3.8 Numerical Examples

To calculate the polarization numerically, we have to know the arrangement of all lattice elements around the storage ring, including elements coming from imperfections. These computational steps are then followed:

- (1) the closed orbit  $X_0$
- (2) polarization direction  $\hat{n}$
- (3)  $8 \times 8$  transport matrices
- (4) spin chromaticity  $\gamma \partial \hat{n} / \partial \gamma$
- (5)  $\tau$  and  $P$

A computer code is developed following these steps. It is applied to estimate the beam polarizations for the storage rings SPEAR and PEP. Lattice elements for the ideal rings include horizontal bending magnets, quadrupole magnets, sextupole magnets, rf cavities and drift spaces. Without field imperfections, the ideal lattices produce an equilibrium polarization of 92%. To simulate field imperfections, we introduce a random distribution of vertical orbit kickers. The resulting vertical closed orbit distortion makes sextupoles behave like skew quadrupoles and quadrupoles behave like additional vertical kickers. In the presence of these field imperfections, the degree of polarization  $P$  can be plotted as a function of the beam energy  $E$ . Results will be different for different simulations of field imperfections. Typical results for SPEAR and PEP are shown in Figures 6 and 7, respectively.



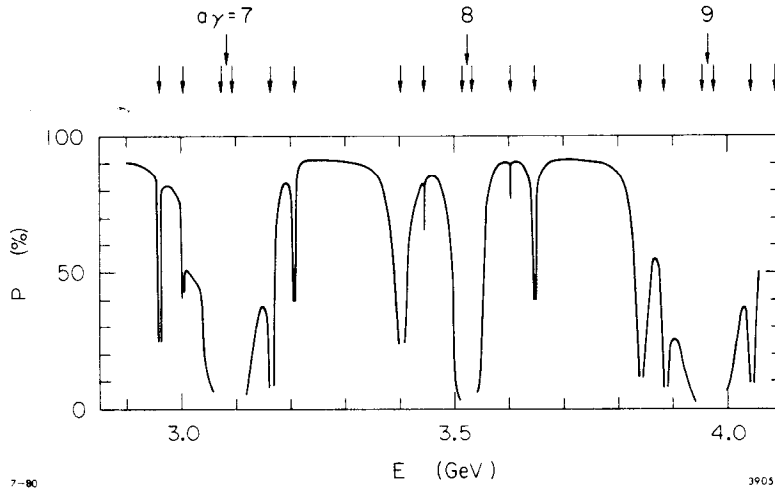


Fig. 6 Expected beam polarization P versus beam energy E around 3-4 GeV for a typical lattice configuration of the SPEAR storage ring. The simulated field imperfections contribute to an rms vertical closed orbit distortion of 1.2 mm. Locations of depolarization resonances are indicated by arrows.

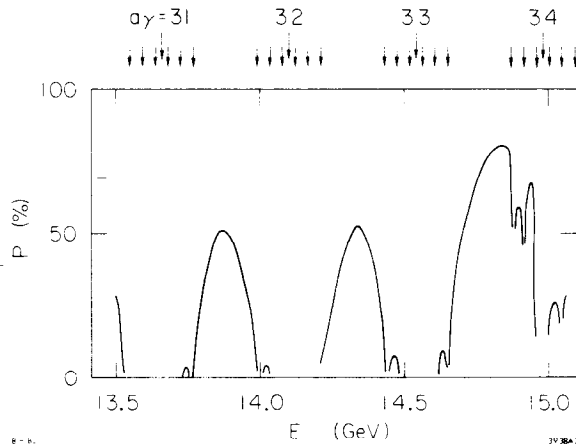


Fig. 7 Expected beam polarization P versus beam energy E around 13.5-15 GeV for a typical lattice configuration of the PEP storage ring. The simulated field imperfections contribute to an rms vertical closed orbit distortion of 0.6 mm.

The SPEAR lattice used is specified by the lattice parameters:  
 $\nu_x = 5.28$ ,  $\nu_y = 5.18$ ,  $\nu_s = 0.022$ ,  $\beta_x^* = 1.2 \text{ m}$ ,  $\beta_y^* = 0.10 \text{ m}$  and  $\eta_x^* = 0$ ,  
 where  $\beta_x^*$ ,  $\beta_y^*$  and  $\eta_x^*$  are the horizontal beta-function, vertical beta-function and the energy dispersion function at the points where positron and electron beams collide. The strengths of the vertical kickers are normalized such that the rms closed orbit distortion after orbit correction is  $\Delta y_{\text{rms}} = 1.2 \text{ mm}$ , which is typical for SPEAR operation. Locations of the depolarization resonances are indicated by arrows at the top of Figure 6. Each integer resonance is surrounded by six sideband resonances. The integer resonances and the two nearby synchrotron sideband resonances overlap and are shown as single depolarization dips.

We have expanded the energy scale of Figure 6 and plotted the result again in Figure 8. Superimposed are polarization measurements performed by the SPEAR polarization team.<sup>16</sup> The agreement between calculation and measurements is acceptable except that the calculation has missed the depolarization resonance located at  $a\gamma - \nu_x + \nu_s = 3$ . In general, there are depolarization resonances located at

$$a\gamma + n_x \nu_x + n_y \nu_y + n_s \nu_s = k \quad (105)$$

for all integers  $n_{x,y,s}$  and  $k$ . The matrix formalism we developed takes care of only the integer and the linear sideband resonances.

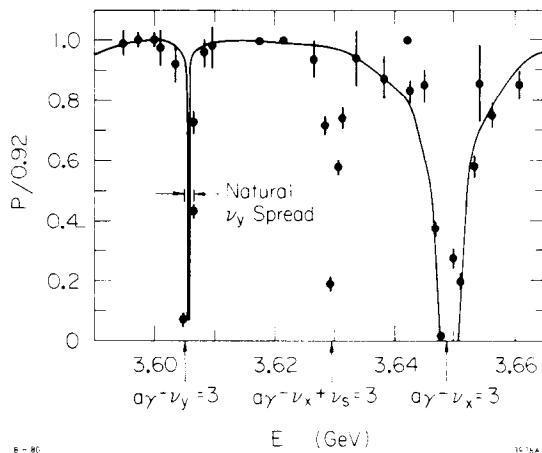


Fig. 8 Comparison of calculation and measurements for SPEAR. Agreement near the two linear sideband resonances  $a\gamma - \nu_y = 3$  and  $a\gamma - \nu_x = 3$  is acceptable. The third resonance  $a\gamma - \nu_x + \nu_s = 3$ , however, is missed by the calculation.

The PEP configuration used has  $\nu_x = 21.15$ ,  $\nu_y = 18.75$  and  $\nu_s = 0.05$ . The corresponding rms closed orbit distortion is set to be 0.6 mm, which is half of what we use for SPEAR because PEP has a more sophisticated orbit correction scheme. Nevertheless, the expected PEP polarization is lower than for SPEAR due to its higher beam energies.

Appendix  
GENERALIZED TRANSPORT MATRICES

The generalized transport matrices for the state vector  $(x, x', y, y', z, \delta, \alpha, \beta)$  are listed below for various lattice elements. Thin-lens approximation has been used. For the rf cavity,  $\phi_s$  is synchronous phase and  $\hat{V}$  is the peak voltage.

Drift Space

$$\begin{bmatrix} 1 & \ell & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \ell & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Horizontal Bend Magnet or Kicker:  $q = B_y \ell / B\rho$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & ayq\ell_z & 0 & ql_y & 1 & 0 \\ 0 & 0 & 0 & -ayqm_z & 0 & -qm_y & 0 & 1 \end{bmatrix}$$

Vertical Bend Magnet or Kicker:  $q = B_x \ell / B\rho$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -q & 0 & 0 \\ 0 & 0 & q & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & ayq\ell_z & 0 & 0 & 0 & ql_x & 1 & 0 \\ 0 & -ayqm_z & 0 & 0 & 0 & -qm_x & 0 & 1 \end{bmatrix}$$

Appendix (Cont.)

Quadrupole:  $q = \frac{\ell}{\beta\rho} \frac{\partial}{\partial x} B_y$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q & 1 & 0 & 0 & 0 & qx_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 1 & 0 & -qy_0 & 0 & 0 \\ -qx_0 & 0 & qy_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -(1+a\gamma)q\ell_y & 0 & -(1+a\gamma)q\ell_x & 0 & 0 & 0 & 1 & 0 \\ (1+a\gamma)qm_y & 0 & (1+a\gamma)qm_x & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Skew Quadrupole:  $q = \frac{\ell}{B\rho} \frac{\partial}{\partial y} B_y$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -q & 0 & 0 & qy_0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -q & 0 & 0 & 1 & 0 & qx_0 & 0 & 0 \\ -qy_0 & 0 & -qx_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ (1+a\gamma)q\ell_x & 0 & -(1+a\gamma)q\ell_y & 0 & 0 & 0 & 1 & 0 \\ -(1+a\gamma)qm_x & 0 & (1+a\gamma)qm_y & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

RF Cavity:  $q = e \hat{V} \cos\phi_s / RE$  ;  $r = (1+a\gamma) e \hat{V} \sin\phi_s / E$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 1 & 0 & 0 \\ 0 & -r\ell_y & 0 & r\ell_x & 0 & 0 & 1 & 0 \\ 0 & rm_y & 0 & -rm_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

Appendix (Cont.)

Sextupole:  $q = \frac{\ell}{B\rho} \frac{\partial^2}{\partial x^2} B_y$  ;  $r = (1 + a\gamma)q$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -qx_0 & 1 & qy_0 & 0 & 0 & \frac{q}{2}(x_0^2 - y_0^2) & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ qy_0 & 0 & qx_0 & 1 & 0 & -qx_0y_0 & 0 & 0 \\ -\frac{q}{2}(x_0^2 - y_0^2) & 0 & qx_0y_0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -r(y_0 \ell_x + x_0 \ell_y) & 0 & -r(x_0 \ell_x - y_0 \ell_y) & 0 & 0 & 0 & 1 & 0 \\ r(y_0 m_x + x_0 m_y) & 0 & r(x_0 m_x - y_0 m_y) & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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Problems

1. Spin precesses ahead of the coordinate system by a factor of  $\alpha\gamma$  if we bend the trajectory by a magnetic field. What if we bend by an electric field? What will be the spin tune? Along which direction is the beam polarized?
2. Show that the Hamiltonian (21), together with  $\dot{\vec{S}} = \frac{i}{\hbar} [H, \vec{S}]$  and  $[S_i, S_j] = i \sum_k \epsilon_{ijk} S_k$  gives the precession equation (13). This is true whether  $\vec{\Omega}$  is nonrelativistic, Eq. (14), or has been generalized, Eq. (16).
3. Show Eq. (35). Make use of the fact that the two polarization vectors  $\hat{e}_1$  and  $\hat{e}_2$  are orthogonal to  $\hat{k}$ .
4. Follow the calculation of  $\langle f | \vec{S}(t) | i \rangle$  of section 2.7 to evaluate the case without spin flip. One should get Eq. (46).
5. If the gyromagnetic ratio  $|g| \gg 1$ , Eq. (65) becomes

$$\tau_o^{-1} = \frac{|g|^5 e^2 \hbar \gamma^5}{48 m^2 c^2 \rho^3} \quad \text{and} \quad P_o = -\frac{g}{|g|} .$$

Show that these results agree with the intuitive picture discussed before Eq. (65). Refer to Ref. 12.

6. If there are wiggler devices (see the footnote of section 3.2) in the storage ring, show that the equilibrium beam polarization is still along  $\hat{y}$  but with a reduced magnitude:

$$P_o = - \frac{8}{5\sqrt{3}} \frac{\oint ds / \rho^3(s)}{\oint ds / |\rho^3(s)|}$$

where  $\rho < 0$  for a reversed bending. Refer to section 2.8. Remember that the "up" and the "down" states switch roles in a reversed bending magnet. See also Eq. (75).

7. Let  $\vec{S}_n$  be the spin of an electron as it circulates the n-th revolution. Transform the spin according to

$$\vec{S}_{n+1} = M_1(a\gamma + \epsilon \cos n\nu) M_2(\theta) \vec{S}_n$$

where  $M_1(a\gamma)$  is the precession matrix (70);  $M_2(\theta)$  is the perturbation matrix (71);  $|\epsilon| \ll 1$  is a small parameter;  $\nu$  is the frequency at which the spin precession frequency  $a\gamma$  is modulated. Starting with  $\vec{S}_0$  along the rotational axis of the transformation  $M_1(a\gamma) M_2(\theta)$ , show that  $\vec{S}_n$  deviates from  $\hat{y}$  significantly if  $a\gamma \pm \nu$  is close to an integer and then estimate the resonance width. Do the analysis to first order in  $\epsilon$ . Repeat the problem for the case when the spin is driven by an oscillating perturbation:

$$\vec{S}_{n+1} = M_1(a\gamma) M_2(\epsilon \cos n\nu) \vec{S}_n .$$

8. To obtain Eq. (83) from Eq. (82), we have used a semi-classical argument to replace the quantity  $(\vec{S} \cdot \vec{A})(\vec{S} \cdot \vec{B})$  by  $\vec{A} \cdot \vec{B}$  by averaging over the solid angles of spin  $\vec{S}$ , where  $\vec{A}$  and  $\vec{B}$  are arbitrary vectors. Prove the quantum mechanical counterpart of this argument:

$$\left\{ (\vec{\sigma} \cdot \vec{A}) , (\vec{\sigma} \cdot \vec{B}) \right\} = \vec{A} \cdot \vec{B} ,$$

where  $\vec{\sigma}$  is a vector whose components are the Pauli matrices,  $\left\{ , \right\}$  is the symmetric anti-commutator of two operators.

9. If one can design a storage ring with arbitrary  $\hat{n}$  and  $\gamma \partial \hat{n} / \partial \gamma$ , how should he choose these quantities so that the beam polarization given by Eqs. (85) and (86) is maximized? The answer is given in Ref. 6:

$$\hat{n} = \sqrt{\frac{7}{11}} \frac{\dot{\hat{v}} \times \hat{v}}{|\dot{\hat{v}}|} \pm \sqrt{\frac{4}{11}} \hat{v}$$

and

$$\gamma \frac{\partial \hat{n}}{\partial \gamma} = \frac{2\sqrt{7}}{11} \left( -\sqrt{\frac{4}{11}} \frac{\dot{\hat{v}} \times \hat{v}}{|\dot{\hat{v}}|} \pm \sqrt{\frac{7}{11}} \hat{v} \right) .$$

The maximum value is  $P_{\max} = 72/5 \sqrt{231} = 94.7\%$ .