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# FERMIONS IN ASYMPTOTICALLY FREE THEORIES*

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## ABSTRACT

It is shown that when the dimension of gauge group exceeds a critical value $N_c$, the only types of asymptotic-freedom-allowed representations for fermions are vector and second rank tensors. A possible connection between this observation and quark-lepton spectrum is discussed.

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A basic idea in dynamical symmetry breaking mechanisms for gauge theories,[1] either the usual ones or the ones with global supersymmetry, is that the effective gauge coupling grows while descending down the mass scale of renormalization. This behavior of gauge coupling, termed asymptotic freedom (or infrared slavery),[2] is unique to non-Abelian gauge theories. It has served as a major justification for the enterprises of quantum chromodynamics[3] and grand unification[4] of strong interactions with electroweak interactions. Equally essential for the generation of spontaneous dynamical symmetry breakdowns is the presence of matter fields -- fermions in the case of usual gauge theories, fermions plus scalar fields in the case of supersymmetric gauge theories. The matter fields, however, have negative effects on the asymptotic freedom.[2]

In this article I will show that, when the dimension of gauge group, say the N of SU(N), exceeds a critical value $N_c$, the only types of asymptotic-freedom-allowed irreducible representations for the matter fields are vector and second rank tensors. The further requirement of triangle-anomaly cancellation[5] has the effect of reducing the value $N_c$ significantly. The following arguments and illustrations will be presented in terms of SU(N) gauge groups; we expect, however, the same constraint also holds for SO(N) and Sp(2N) but with different values of $N_c$.

First we consider SU(N) gauge theories with two-component fermions in representation Γ. The β-function determining the evolution of the gauge coupling (g) is

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\frac{11N}{3} - \frac{2}{3}T(\Gamma)\right] \tag{1}$$

with

$$T(\Gamma) \equiv \frac{C_2(\Gamma)d(\Gamma)}{N^2-1} \tag{2}$$

where $C_2(\Gamma)$ is the value of quadratic Casimir operator in the representation $\Gamma$, and $d(\Gamma)$ is the dimension of the representation $\Gamma$.

If we label the representation $\Gamma$ by the numbers of boxes in each row of the correspondent Young tableau, i.e., $\Gamma = (f_\ell) = (f_1, f_2, \ldots, f_N)$, then[6]

$$C_2(f_\ell) = \frac{1}{2}\left[\sum_\ell f_\ell^2 - 2\sum_\ell \ell f_\ell + f(N+1) - \frac{f^2}{N}\right] \tag{3}$$

where $f \equiv \sum_\ell f_\ell$.

Let $(f'_\ell)$ be a representation obtainable from $(f_\ell)$ by moving a box from the $\ell_1^{th}$-row of the tableau to the $\ell_2^{th}$-row. The change in the value of the Casimir operator due to this relocation is given by

$$\Delta C_2 = C_2(f'_\ell) - C_2(f_\ell)$$

$$= \Delta f_\ell - \Delta\ell + 1 \tag{4}$$

where $\Delta f_\ell \equiv f_{\ell_2} - f_{\ell_1}$ and $\Delta\ell \equiv \ell_2 - \ell_1$. By the nature of Young tableau, we have $\Delta f_\ell < 0$ for $\Delta\ell > 0$ and vice versa. Therefore

$$\Delta C_2 < 0 \quad \text{for} \quad \ell_2 > \ell_1 \quad ,$$
$$\Delta C_2 > 0 \quad \text{for} \quad \ell_2 < \ell_1 \quad . \tag{5}$$

Let us call the transformation: $(f_\ell) \rightarrow (f'_\ell)$ with $\ell_2 > \ell_1$, operation-A, and that with $\ell_2 < \ell_1$, operation-S. It is then easily perceived that given any representation one can always link it to one of the N totally antisymmetric tensors by a chain of operation-A. Thus we have proved the following theorem:

Theorem 1: Let $(f_\ell)$ be any given representation of SU(N), then

$$C_2(f_\ell) \geqslant C_2(\{1^{R(f/N)}\}) \tag{6}$$

where R(f/N) denotes the remainder of f divided by N, and $\{1^k\}$ denotes totally antisymmetric tensor of rank k. The equality holds when $(f_\ell)$ is a totally antisymmetric tensor.

We may now divide the collection of all irreducible representations[7] of SU(N) into 1+[N/2] subsets ([N/2] = n, for N = 2n or 2n+1). The first one consists of the singlet representation and those linked to it by chains of operation-A. The second, the third, etc., are characterized respectively by totally antisymmetric tensors (modulo complex conjugation) of rank one, two, etc.

It follows from Eq. (3) and the formula $d(\{1^k\}) = C_k^N$ that

$$T(\{1^k\}) = \frac{1}{2} C_{k-1}^{N-2} \ . \tag{7}$$

Therefore the number of $\{1^k\}$ allowed by asymptotic freedom is

$$n_k < 11N/C^{N-2}_{k-1} \;. \tag{8}$$

Specifically

$$n_1 < 11N$$

$$n_2 < \frac{11N}{N-2} = \begin{cases} 22 & \text{for} \quad N = 4 \\ \vdots & \qquad\qquad \vdots \\ 12 & \qquad\qquad 24 \\ 11+ & \qquad N \geq 25 \end{cases}$$

$$n_3 < \frac{22N}{(N-2)(N-3)} = \begin{cases} 11 & \text{for} \quad N = 6 \\ \vdots & \qquad\qquad \vdots \\ 2+ & \qquad\qquad 12,\ldots,15 \\ 1+ & \qquad\qquad 16,\ldots,26 \\ 0+ & \qquad N \geq 27 \end{cases}$$

$$n_4 < \frac{66N}{(N-2)(N-3)(N-4)} = \begin{cases} 4+ & \text{for} \quad N = 8 \\ 2+ & \qquad\qquad 9 \\ 1+ & \qquad\qquad 10,11,12 \\ 0+ & \qquad N \geq 13 \end{cases}$$

$$n_5 < \frac{264N}{(N-2)(N-3)(N-4)(N-5)} = \begin{cases} 1+ & \text{for} \quad N = 10 \\ 0+ & \qquad N \geq 11 \end{cases}$$

and $n_k = 0$ for $k \geq 6$ and $N \geq 2k$. Thus amongst the collection of totally antisymmetric tensors, those of rank greater than two are suppressed as $N \to \infty$.

On the other hand, one can show readily that vector and second rank tensors (which consists of second rank antisymmetric tensor, second rank symmetric tensor, and adjoint representation) are the only SU(N) representations with dimensions less than that of $\{1^3\}$. They belong to the first three subsets of our categorization of representations. The remaining subsets of representations have dimensions greater than that of $\{1^3\}$. Furthermore, taking into account also Theorem 1 and Eq. (7), we conclude immediately that these later subsets of representations are suppressed by the requirement of asymptotic freedom at a rate faster than of the case of $\{1^3\}$.

Finally we examine the first three subsets of representations. We find that the ones other than vector and second rank tensors have values of quadratic Casimir operator greater than that of $\{1^3\}$. Consequently they are rapidly suppressed too. The numbers of vectors and second rank antisymmetric tensors allowed by asymptotic freedom are given before. The allowed numbers of adjoint representations and second rank symmetric tensors are respectively

$$n_{adj} < \frac{11N}{2N} = \left\{ \begin{array}{lll} 5+ & \text{for any} & N \geq 2 \end{array} \right.$$

$$n_{2s} < \frac{11N}{N+2} = \left\{ \begin{array}{lll} 5+ & \text{for} & N = 2 \\ \vdots & & \vdots \\ 10 & & 20 \\ 10+ & & N \geq 21 \end{array} \right.$$

Thus we arrive at

Theorem 2: For SU(N) gauge theories, asymptotic freedom alone requires that there exist a critical value for N, namely $N_c = 26$, beyond which the only types of nontrivial representations allowed to fermions are vector and second rank tensors.

Gauge theories with fermions, however, must be subjected to a fundamental constraint, namely triangle-anomaly cancellation, in order to remain renormalizable. We see that $n_3$ can at most equal to one for $16 \leqslant N \leqslant 26$, the anomaly cancellation thus requires that fermions in other representations be added to cancel the anomaly of $\{1^3\}$. The addition of anomaly-cancellating fermions, however, may destroy the asymptotic freedom. Indeed actual enumeration shows that this incompatibility cannot be evaded as long as $N > 17$. Thus our conclusion:

Theorem 3: For SU(N) gauge theories, asymptotic freedom and triangle-anomaly cancellation together demands that there exists a critical value for N, namely $N_c = 17$, beyond which the only types of nontrivial representations allowed to fermions are vector and second rank tensors.

For supersymmetric gauge theories[8] the gauge fields are necessarily accompanied by an adjoint representation of Majorana fermions. The matter fields, in the simplest cases, come as chiral multiplet (of gauge group representation Γ) containing both fermion fields and scalar fields. The β-function for the gauge coupling g is:[9]

$$\beta(g) \;=\; -\frac{g^3}{16\pi^2}\left[3N - T(\Gamma)\right] . \tag{9}$$

Following the identical procedure as before, we obtain

Theorem 4: For supersymmetric SU(N) gauge theories, asymptotic freedom and triangle-anomaly cancellation together requires that there exists a critical value for N, namely $N_c = 12$, beyond which the only types of nontrivial representations allowed to the matter fields (in chiral multiplets) are vector and second rank tensors.

Have the above theorems anything to do with reality? It is encouraging that the known quarks and leptons seem to fit into the $\underline{5}$ and $\underline{10}^*$ representations of SU(5) grand unifying gauge group,[4] which are respectively the vector and conjugate of second rank anti-symmetric tensor representation. Imagine a SU(N) gauge theory with $N > N_c$. The fermions can only belong to the following types of representations: vector (1,0,0,...,0), second rank antisymmetric tensor (1,1,0,...,0), second rank symmetric tensor (2,0,0,...,0), their conjugates, and the adjoint representation (2,1,1,...,1,0). By invoking the concept of naturalness,[10] we neglect the fermions in adjoint representations and any representation of fermions that has a conjugate counter part present to match it. So define

$$m_1 = [\text{number of } (1,0,0,\ldots,0) - \text{number of } (1,0,0,\ldots,0)^*]$$

$$m_2 = [\text{number of } (1,1,0,\ldots,0) - \text{number of } (1,1,0,\ldots,0)^*]$$

and

$$m_{2s} = [\text{number of } (2,0,0,\ldots,0) - \text{number of } (2,0,0,\ldots,0)^*] \,.$$

The triangle-anomaly cancellation requires

$$m_1 + (N-4)m_2 + (N+4)m_{2s} = 0 . \tag{10}$$

Furthermore, by decomposing the representations according to the branching SU(N) $\supset$ SU(5) $\otimes$ SU(N-5) $\otimes$ U(1) [we shall ignore the SU(N-5) $\otimes$ U(1) labelling], we obtain the SU(5) content of the fermions

$$m_1(1,0,0,\ldots,0) \oplus m_2(1,1,0,\ldots,0) \oplus m_{2s}(2,0,0,\ldots,0)$$

$$= \left[m_1 + \left(m_2 + m_{2s}\right)\left(N-5\right)\right] \underline{5} \oplus m_2\, \underline{10} \oplus m_{2s}\, \underline{15} \oplus \text{SU(5) singlets} . \tag{11}$$

In order to reproduce equal numbers of $\underline{5}$ and $\underline{10}^*$, as commonly expected in the SU(5) grand unified theory, we need

$$m_1 + \left(m_2 + m_{2s}\right)\left(N-5\right) = -m_2 = h \tag{12}$$

where h denotes the number of quark-lepton families. Solving Eqs. (10) and (12) simultaneously, we obtain

$$m_1 = (N-4)h \tag{13}$$

and

$$m_{2s} = 0 . \tag{14}$$

In this type of model, u-like quarks, left-handed d-like quarks, and right-handed leptons do not participate in any non-Abelian interactions (such as horizontal gauge interactions) other than the SU(5) gauge interactions. This is because fermions here carry at most two SU(N) gauge group indices.

The repetition of the family structure of quarks and leptons is not a very surprising phenomenon according to our theorems -- there just is not much room for variation if $N \gtrsim N_c$.

Of course quarks and leptons may well turn out to be composite objects, and the theorems may be applied on the preon level.[11]

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