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TRANSVERSE WAKE FORCE IN PERIODICALLY VARYING WAVEGUIDE*

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Introduction

We want to find the wake force[1] due to a perturbed charge distribution moving parallel to the axis of a circular periodic waveguide. We will assume that the perturbed charge and current density may be written as

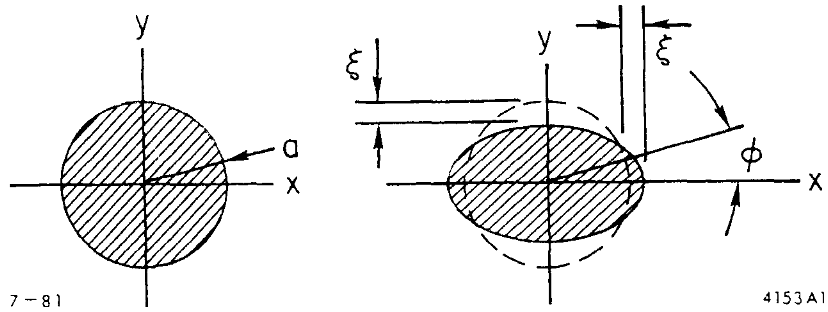
$$\rho = \frac{eN\xi}{\pi a^2 v} \frac{\delta(t - z/v)\delta(r - a) \cos m\phi}{1 + \delta_{m,o}} \tag{1.1}$$

and

$$j_z = \frac{eN\xi}{\pi a^2} \frac{\delta(t - z/v)\delta(r - a)}{1 + \delta_{m,o}} \cos m\phi = v\rho \tag{1.2}$$

where z is the longitudinal coordinate along the axis of the waveguide, t is the time, r and ϕ are the radial and azimuthal coordinates of the guide, $\delta(x)$ is a Dirac delta function, v is the longitudinal velocity of the charge and $\delta_{m,o} = 1$ for $m = 0$ and is zero otherwise.

This distribution corresponds to a perturbation to a cylindrical disk of charge as shown in Fig. 1, for $m = 2$. The total number of electrons in the unperturbed disk of charge is N. We have assumed that $\xi \ll a$ and that the disk is infinitesimally thin. It is interesting to note that for a point



Unperturbed charge distribution   Perturbed charge distribution m=2

Fig. 1

charge of N electrons displaced horizontally by an amount ξ the charge distribution is

$$\rho = \frac{eN}{\xi v}\,\delta(t - z/v)\delta(r - \xi)\delta(\phi) \quad , \tag{1.3}$$

which may be written as

$$\rho = \frac{eN\delta(t - z/v)\delta(r - \xi)}{\pi\xi v}\left[\frac{1}{2} + \sum_{m=1}^{\infty} \cos m\phi\right] \quad . \tag{1.4}$$

The wake force for the m^{th} harmonic of a displaced point charge may be obtained from the wake force derived in this paper by substituting $a = \xi$. We will ignore any effects of the fields produced by the charge upon its own motion and assume that the velocity v in the z direction is constant. This assumption will be valid for the case where both the transverse velocity and the change in the longitudinal velocity in one period are small compared to the longitudinal velocity.

We will assume the waveguide has a period L and a radius b(z) described by

$$b(z) = b_o \left[1 + \sum_{p=-\infty}^{\infty} \tilde{C}_p e^{j\frac{2\pi p}{L} z} \right] \quad , \tag{1.5}$$

with the average radius taken to be b_o so that $\tilde{C}_o = 0$. The reality of b(z) requires that $\tilde{C}_{-p} = \tilde{C}_p^*$. This configuration is shown in Fig. 2.

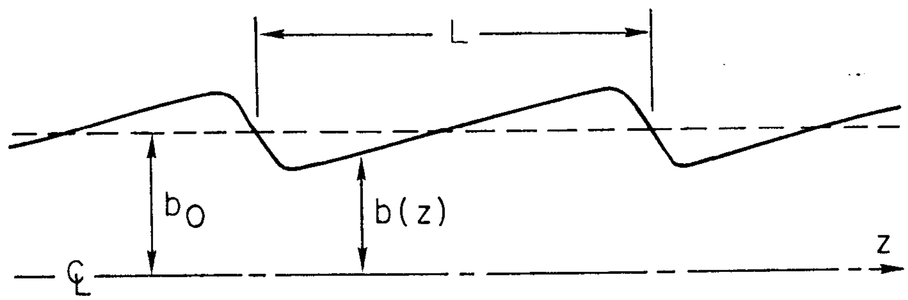


Fig. 2

We will solve for the wake force using a perturbation technique in powers of the parameter $\tilde{C}_p$. This is the same technique used by M. Chatard-Moulin and A. Papiernik in calculating the energy loss of an electron bunch moving along the axis.[2] We not only will follow their technique closely but also utilize a similar notation in order to facilitate comparisons with their results for m=0.

We will mainly be interested in calculating the transverse wake forces, which are present only for the case m≠0, nevertheless the following equations will be valid for the case of a charge eN moving along the axis by setting m=0, dropping the term $(1+\delta_{m,o})$, and setting $\xi = a/2$. The longitudinal wake force for the m=0 case is treated in Appendix 1. Krinsky[3] has also applied the technique of Chatard-Moulin and Papiernik to evaluate the transverse impedance of the periodic waveguide experienced by a coasting beam with a coherent vertical oscillation. Krinsky's result [Eq. (7)] agrees with our result for the transformed forces [Eqs. (6.14) and (6.15)] with m=1 if one replaces Z_0 in his expressions (the impedance of free space) by $4\pi/c$, substitutes for $I_0\Delta$ the value $eN\xi/2\pi$ and uses b_0 as the average pipe radius.

Equations for field components

It will be useful to use Fourier transformations in solving for the electromagnetic fields. The convention we use is that a tilde above a quantity designates the transform as defined by

$$f(r,\phi,z,t) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega\tau} \sum_{p=-\infty}^{\infty} e^{j\frac{2\pi p}{L}z}\, \tilde{f}(r,\omega,p) \begin{Bmatrix} \cos m\phi \\ \text{or} \\ \sin m\phi \end{Bmatrix} \tag{1.6}$$

where $\tau = t - z/v$. Note that $\widetilde{(\partial f/\partial t)} = j\omega\tilde{f}$ and $\widetilde{(\partial f/\partial z)} = jk_p\tilde{f}$ where $k_p = (2\pi p/L - \omega/v)$. The charge and current density are proportional to $\cos m\phi$; the field components E_z, E_r, and B_ϕ are proportional to $\cos m\phi$, while the field components E_ϕ, B_z, and B_r are proportional to $\sin m\phi$.

The transforms of ρ and j_z which correspond to Eqs. (1.1) and (1.2) are given by

$$\tilde{\rho}(r,\omega,p) = \frac{eN\xi}{2\pi^2 a^2 v}\, \delta(r - a)\,\delta_{o,p} \quad , \tag{1.7}$$

and

$$\tilde{j}_z(r,\omega,p) = v\tilde{\rho}(r,\omega,p) \quad . \tag{1.8}$$

We see that only the p = 0 space harmonic for the various field components is synchronous with the charge and current.

From Maxwell's Equations it follows that the transverse transformed field components may be obtained from a knowledge of the longitudinal transformed field components by

$$\alpha_p^2 \tilde{E}_r(r,\omega,p) = j\left\{k_p \frac{\partial \tilde{E}_z}{\partial r}(r,\omega,p) - \frac{m\omega}{rc}\tilde{B}_z(r,\omega,p)\right\} \quad , \tag{1.9}$$

$$\alpha_p^2 \tilde{E}_\phi(r,\omega,p) = j\left\{-\frac{m}{r}k_p\tilde{E}_z(r,\omega,p) + \frac{\omega}{c}\frac{\partial \tilde{B}_z}{\partial r}(r,\omega,p)\right\} \quad , \tag{1.10}$$

$$\alpha_p^2 \tilde{B}_r(r,\omega,p) = j\left\{-\frac{m\omega}{rc}\tilde{E}_z(r,\omega,p) + k_p\frac{\partial \tilde{B}_z}{\partial r}(r,\omega,p)\right\} \quad , \tag{1.11}$$

$$\alpha_p^2 \tilde{B}_\phi(r,\omega,p) = j\left\{-\frac{\omega}{c}\frac{\partial \tilde{E}_z}{\partial r}(r,\omega,p) + \frac{m}{r}k_p\tilde{B}_z(r,\omega,p)\right\} \quad , \tag{1.12}$$

since $\tilde{j}_r = \tilde{j}_\phi = o$ where c is the velocity of light,

$$k_p = \frac{2\pi p}{L} - \frac{\omega}{v} \quad , \tag{1.13}$$

$$\alpha_p^2 = \frac{\omega^2}{c^2} - k_p^2 = -\frac{\omega^2}{\gamma^2 v^2} + \frac{4\pi p\omega}{vL} - \frac{4\pi^2 p^2}{L^2} \quad ,$$

$$\gamma = (1 - v^2/c^2)^{-1/2} \quad , \tag{1.14}$$

and Gaussian units are used throughout the paper. For later work it will be desirable to have the transverse forces on a test particle of charge e:

$$\tilde{F}_r = e\,(\tilde{E}_r - v/c\ \tilde{B}_\phi) \quad \text{and} \quad \tilde{F}_\phi = e\,(\tilde{E}_\phi + v/c\ \tilde{B}_r) \quad ,$$

The synchronous space harmonic, i.e., p = 0, is the only space harmonic which does not vanish upon integration over one period in z. From Eqs. (1.9-1.14) we obtain for the synchronous deflection forces

$$\tilde{F}_r(r,\omega,0) = j\frac{v}{\omega}\, e\, \frac{\partial\tilde{E}_z(r,\omega,0)}{\partial r} \quad , \tag{1.15}$$

$$\tilde{F}_\phi(r,\omega,0) = -j\frac{v}{\omega}\,\frac{m}{r}\, e\tilde{E}_z(r,\omega,0) \quad . \tag{1.16}$$

The longitudinal tranformed field components are obtained from the transformed wave equations

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial\tilde{E}_z}{\partial r}\right) + \left(\alpha_p^2 - \frac{m^2}{r^2}\right)\tilde{E}_z = 4\pi j\left(k_p\tilde{\rho} + \frac{\omega}{c^2}\,\tilde{j}_z\right) \quad , \tag{1.17}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{B}_z}{\partial r}\right) + \left(\alpha_p^2 - \frac{m^2}{r^2}\right)\tilde{B}_z = 0 \quad , \tag{1.18}$$

along with the proper boundary conditions which are discussed in the next section. Using the expressions for $\tilde{\rho}$ and $\tilde{j}_z$ from Eqs. (1.7) and (1.8) we obtain the following equation for $\tilde{E}_z$:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{E}_z}{\partial r}\right) + \left(\alpha_p^2 - \frac{m^2}{r^2}\right)\tilde{E}_z = -j\,\frac{2\omega e N \xi}{\pi a^2 v^2 \gamma^2}\,\delta(r - a)\delta_{o,p} \quad . \tag{1.19}$$

Boundary conditions

The proper boundary conditions for a perfectly conducting wall are that both the parallel component of the electric field, E , and the normal component of the magnetic field, B_n, equal zero at the wall. The technique used in this paper, which is the same as that of Chatard-Moulin and Papiernik,[2] is to replace these boundary conditions at $r = b(z)$ by appropriate boundary conditions on the values of E_z and $\partial B_z/\partial r$ at $r = b_o$. A perturbation technique is used in which we expand the field components in orders of the quantity $\tilde{C}_p$ given in Eq. (1.5). Thus we first solve for E_z and B_z for the case of a perfectly conducting round cylindrical wave guide of radius b_o with the boundary conditions $E_z\big|_{r=b_o} = 0$ and $\partial B_z/\partial r\big|_{r=b_o} = 0$, then using these results obtain new boundary conditions at $r = b_o$ to first order in $\tilde{C}_p$ and solve for the fields which obey these new boundary conditions. This process is repeated to second order in $\tilde{C}_p$ for the synchronous value of E_z from which we can obtain the synchronous transverse deflecting force by Eqs. (1.15) and (1.16).

From Fig. 3 we see that at $r = b(z)$

$$E_{\parallel}\Big|_{b(z)} = E_z\Big|_{b(z)} \cos\alpha + E_r\Big|_{b(z)} \sin\alpha \quad . \tag{2.1}$$

Applying the boundary condition $E_{\parallel} = 0$ we obtain the following relation between E_z and E_r.

$$E_z\Big|_{b(z)} = -E_r\Big|_{b(z)} \tan\alpha = -\frac{db}{dz} E_r\Big|_{b(z)} \quad . \tag{2.2}$$

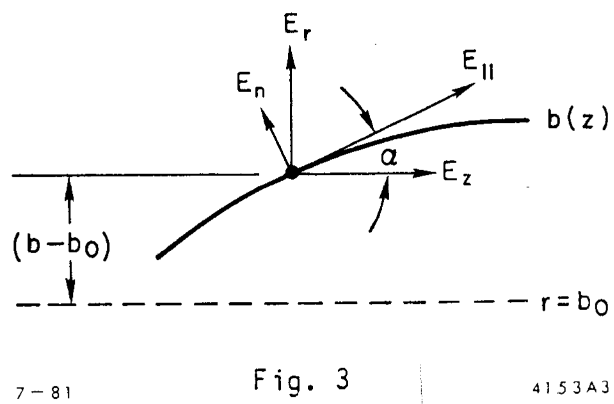


Fig. 3

We can expand the components in terms of a Taylor series such as

$$E_z\Big|_{b(z)} = E_z\Big|_{b_o} + \frac{\partial E_z}{\partial r}\Big|_{b_o} (b - b_o) + \frac{\partial^2 E_z}{\partial r^2}\Big|_{b_o} \frac{(b - b_o)^2}{2} \tag{2.3}$$

By combining Eqs. (2.2) and (2.3) we obtain for the boundary condition at $r = b_o$

$$E_z\Big|_{b_o} = -\frac{\partial E_z}{\partial r}\Big|_{b_o} (b - b_o) - \frac{\partial^2 E_z}{\partial r^2}\Big|_{b_o} \frac{(b - b_o)^2}{2} - E_r\Big|_{b_o} \frac{db}{dz} - \frac{\partial E_r}{\partial r}\Big|_{b_o} (b - b_o)\frac{db}{dz} \quad . \tag{2.4}$$

The above expressions may be written in terms of the transformed field components by using the convolution condition (see Appendix 2)

$$[\widetilde{f \cdot g}](\omega,p) = \int d\omega' \sum_q \tilde{f}(\omega',q)\tilde{g}(\omega - \omega',p-q) \quad ,$$

which for the special case where $\tilde{g}(\omega,p) = \tilde{g}(0,p)\delta(\omega)$ [such as occurs in $\widetilde{(b - b_0)} = b_0\tilde{C}_p\delta(\omega)$ and $\widetilde{(db/dz)} = (jb_0 2\pi p/L)\tilde{C}_p\delta(\omega)$] reduces to

$$[f \cdot g]_p(\omega) = \sum_q \tilde{f}_q(\omega)\tilde{g}_{p-q}(0) \quad , \tag{2.5}$$

where the subscript refers to the spatial harmonic number. Dropping the notation $\big|_{b_0}$ which is to be understood when quantities are evaluated at $r = b_0$, and using the subscript notation for the spatial harmonic number, we can obtain the boundary condition of the transformed field components at $r = b_0$

$$\begin{aligned}(\tilde{E}_z)_p = &- \sum_q \left(\frac{\partial \tilde{E}_z}{\partial r}\right)_q \widetilde{(b - b_o)}_{p-q} - \sum_q (\tilde{E}_r)_q \left(\widetilde{\frac{db}{dz}}\right)_{p-q} \\ &- \frac{1}{2} \sum_{q,n}\left(\frac{\partial^2 \tilde{E}_z}{\partial r^2}\right)_q \widetilde{(b - b_o)}_{p-q-n}\widetilde{(b - b_o)}_n - \sum_{q,n}\left(\frac{\partial \tilde{E}_r}{\partial r}\right)_q \widetilde{(b - b_o)}_{p-q-n}\left(\widetilde{\frac{db}{dz}}\right)_n \quad .\end{aligned} \tag{2.6}$$

The field components are next expanded in orders of $\tilde{C}_p$ such that

$$\tilde{E}_z = \tilde{E}_z^{(0)} + \tilde{E}_z^{(1)} + \tilde{E}_z^{(2)} \qquad \text{etc.} \quad , \tag{2.7}$$

where $\tilde{E}_z^{(0)}$ is the solution for a constant radius waveguide of radius b_o. The term $\tilde{E}_z^{(1)}$ is proportional to $\tilde{C}_p$; the term $\tilde{E}_z^{(2)}$ is proportional to $\tilde{C}_p^2$, etc. The relationship between transformed field component $\tilde{E}_r$ and the longitudinal transformed field components $\tilde{E}_z$ and $\tilde{B}_z$ is given by Eq. (1.9) along with Eq. (2.7) to separate Eq. (2.6) into the following equations. To zero order

$$\tilde{E}_z^{(0)}(b_o,\omega,p) = 0 \quad . \tag{2.8}$$

To first order

$$\tilde{E}_z^{(1)}(b_o,\omega,p) = b_o\tilde{C}_p\left(\frac{2\pi p v\gamma^2}{\omega L} - 1\right)\frac{\partial\tilde{E}_z^{(0)}(b_o,\omega,0)}{\partial r} \quad , \tag{2.9}$$

where we have used the results obtained later [Eq. (3.2) to Eq. (3.4)] which show that

$$\tilde{B}_z^{(0)}(r,\omega,p) = 0 \quad \text{and} \quad \frac{\partial\tilde{E}_z^{(0)}}{\partial r}(r,\omega,p) = \frac{\partial\tilde{E}_z^{(0)}}{\partial r}(r,\omega,0)\delta_{o,p} \quad . \tag{2.10}$$

We will be interested only in the synchronous value of E_z and will stop at second order in $\tilde{C}_p$ to obtain

$$\tilde{E}_z^{(2)}(b_o,\omega,0) = -\sum_q b_o \tilde{C}_{-q} \left(\frac{\alpha_q^2 + \frac{2\pi q}{L} k_q}{\alpha_q^2} \right) \frac{\partial \tilde{E}_z^{(1)}}{\partial r} (b_o,\omega,q)$$

$$+ \sum_q \frac{m\omega}{c} \tilde{C}_{-q} \frac{2\pi q}{\alpha_q^2 L} \tilde{B}_z^{(1)}(b_o,\omega,q) + b_o^2 \sum_q \tilde{C}_q \tilde{C}_{-q} \left(\frac{2\pi q v \gamma^2}{\omega L} - \frac{1}{2} \right) \frac{\partial^2 \tilde{E}_z^{(0)}}{\partial r^2} (b_o,\omega,0) \quad . \tag{2.11}$$

From Fig. 4 we see that at $r = b(z)$

$$B_n \Big|_{b(z)} = B_r \Big|_{b(z)} \cos\alpha - B_z \Big|_{b(z)} \sin\alpha = 0 \quad ,$$

applying the boundary condition $B_n = 0$ we obtain the following relation between B_r and B_z.

$$B_r \Big|_{b(z)} = B_z \Big|_{b(z)} \tan\alpha = \frac{db}{dz} B_z \Big|_{b(z)} \quad . \tag{2.12}$$

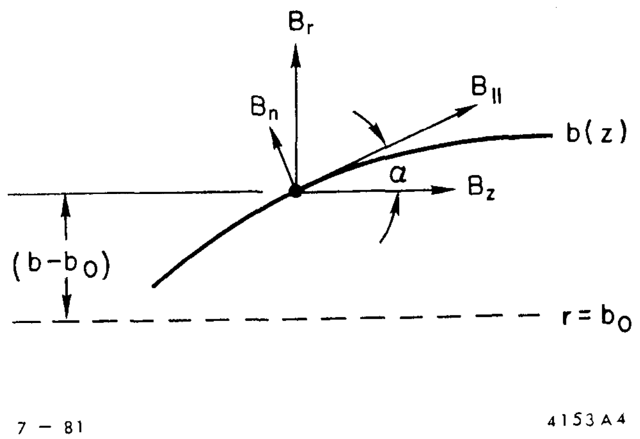


Fig. 4

We use a Taylor series expansion, similar to that used for the electric field, to obtain for the boundary condition at $r = b_o$ through first order in $\tilde{C}_p$

$$B_r\Big|_{b_o} = -\frac{\partial B_r}{\partial r}\Big|_{b_o} (b - b_o) + B_z\Big|_{b_o} \frac{db}{dz} \quad . \tag{2.13}$$

We again drop the notation $\Big|_{b_o}$ use Eq. (1.11) to write $\tilde{B}_r$ in terms of $\tilde{E}_z$ and $\tilde{B}_z$, and expand the field components in orders of $\tilde{C}_p$ to obtain the boundary condition for the transformed longitudinal field components. To zero order

$$\frac{\partial \tilde{B}_z^{(0)}}{\partial r}(b_o,\omega,p) = 0 \quad , \quad \text{and} \quad \frac{\partial^2 \tilde{B}_z^{o}}{\partial r^2}(b_o,\omega,p) = 0 \tag{2.14}$$

where we have used the result from Eq. (2.8) that $\tilde{E}_z^{(0)}(b_o,\omega,p) = 0$. To first order

$$\frac{\partial \tilde{B}_z^{(1)}}{\partial r}(b_o,\omega,p) - \frac{m\omega}{b_o k_p c}\tilde{E}_z^{(1)}(b_o,\omega,p) = \frac{\alpha_p^2 m\omega}{\alpha_o^2 k_p c}\tilde{C}_p \frac{\partial \tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0) \quad . \tag{2.15}$$

We substitute the expression for $\tilde{E}_z^{(1)}(b_o,\omega,p)$ from Eq. (2.9) into Eq. (2.15) to obtain the following expression for $\partial \tilde{B}_z^{(1)}/\partial r$,

$$\frac{\partial \tilde{B}_z^{(1)}}{\partial r}(b_o,\omega,p) = -\frac{2\pi p}{\alpha_o^2 L}\frac{m\omega}{c}\tilde{C}_p \frac{\partial \tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0) \quad . \tag{2.16}$$

Zero order solution

The zero order solutions for $\tilde{E}_z$ and $\tilde{B}_z$ are obtained from solutions to the wave equations Eqs. (1.18) and (1.19) along with the zero order boundary conditions given by Eqs. (2.8) and (2.14). The solutions of Eqs. (1.18) and (1.19) may be written in terms of the Bessel functions[3] $J_m(\alpha_p r)$ and $N_m(\alpha_p r)$. From Eq. (1.19) we see that the radial derivative of $\tilde{E}_z$ is discontinuous at r = a such that

$$\left.\frac{\partial \tilde{E}_z}{\partial r}\right|_{a+} - \left.\frac{\partial \tilde{E}_z}{\partial r}\right|_{a-} = -j\,\frac{2\omega e N \xi}{\pi a^2 v^2 \gamma^2}\,\delta_{o,p} \quad . \tag{3.1}$$

This condition along with the boundary conditions

$$\tilde{E}_z^{(0)}(b_o,\omega,p) = 0 \quad ,$$

$$\frac{\partial \tilde{B}_z^{(0)}}{\partial r}(b_o,\omega,p) = 0 \quad ,$$

and the requirement that $\tilde{E}_z^o$ and $\tilde{B}_z^o$ be finite at r = 0 yields the following zero order solutions:

For $r \leq a$,

$$\tilde{E}_z^{(0)}(r,\omega,p) = -j\delta_{o,p} \frac{\omega e N \xi \left[N_m(\alpha_o a)J_m(\alpha_o b_o) - N_m(\alpha_o b_o)J_m(\alpha_o a)\right]}{a v^2 \gamma^2 J_m(\alpha_o b_o)} J_m(\alpha_o r) \tag{3.2}$$

For $a \leq r \leq b_o$,

$$\tilde{E}_z^{(0)}(r,\omega,p) = -j\delta_{o,p} \frac{\omega e N \xi \left[N_m(\alpha_o r)J_m(\alpha_o b_o) - N_m(\alpha_o b_o)J_m(\alpha_o r)\right] J_m(\alpha_o a)}{a v^2 \gamma^2 J_m(\alpha_o b_o)} \tag{3.3}$$

For $0 \leq r \leq b_o$

$$\tilde{B}_z^{(0)}(r,\omega,p) = 0 \tag{3.4}$$

For future use we also need to have the values of

$$\frac{\partial \tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0) \quad \text{and} \quad \frac{\partial^2 \tilde{E}_r^{(0)}}{\partial r^2}(b_o,\omega,0) \quad .$$

We use the following property of Bessel functions

$$N'(x)J(x) - N(x)J'(x) = \frac{2}{\pi x} \tag{3.5}$$

and

$$N''(x)J(x) - N(x)J''(x) = \frac{-2}{\pi x^2} \tag{3.6}$$

to obtain

$$\frac{\partial \tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0) = -j\,\frac{2\omega e N \xi\, J_m(\alpha_o a)}{\pi a \gamma^2 v^2 b_o\; J_m(\alpha_o b_o)} \quad , \tag{3.8}$$

and

$$\frac{\partial^2 \tilde{E}_z^{(0)}}{\partial r^2}(b_o,\omega,0) = +j\,\frac{2\omega e N \xi\, J_m(\alpha_o a)}{\pi a \gamma^2 v^2 b_o^2\; J_m(a_o b_o)} \quad . \tag{3.9}$$

We note that Eqs. (3.8) and (3.9) yield the simple relationship between the first and second derivatives of the synchronous harmonic of $\tilde{E}_z^{(0)}$ at the boundary,

$$\frac{\partial^2 \tilde{E}_z^{(0)}}{\partial r^2}(b_o,\omega,0) = -\frac{1}{b_o}\frac{\partial \tilde{E}_z^{(0)}}{\partial r} \quad . \tag{3.10}$$

which may also be obtained from the wave equation (1.17).

First order solutions

The first order solutions for $\tilde{E}_z^{(1)}$ and $\tilde{B}_z^{(1)}$ which satisfy the boundary conditions at b_o are

$$\tilde{E}_z^{(1)}(r,\omega,p) = \tilde{E}_z^{(1)}(b_o,\omega,p)\,\frac{J_m(\alpha_p r)}{J_m(\alpha_p b)} \quad , \tag{4.1}$$

$$\tilde{B}_z^{(1)}(r,\omega,p) = \frac{\partial\tilde{B}_z^{(1)}}{\partial r}(b_o,\omega,p)\,\frac{J_m(\alpha_p r)}{\alpha_p J_m'(\alpha_p b)} \quad , \tag{4.2}$$

where the values of $\tilde{E}_z^{(1)}(b_o,\omega,p)$ and $\partial\tilde{B}_z^{(1)}(b_o,\omega,p)/\partial r$ are given by the first order boundary conditions Eqs. (2.9) and (2.16).

The results for $\tilde{E}_z^{(1)}$, $\tilde{B}_z^{(1)}$ and $\partial\tilde{E}_z^{(1)}/\partial r$ may be written as

$$\tilde{E}_z^{(1)}(r,\omega,q) = \tilde{C}_q b_o \frac{\partial\tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0)\left\{\left(\frac{2\pi q v\gamma^2}{\omega L} - 1\right)\frac{J_m(\alpha_q r)}{J_m(\alpha_q b_o)}\right\} \quad , \tag{4.3}$$

$$\tilde{B}_z^{(1)}(r,\omega,q) = \tilde{C}_q \frac{\partial\tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0)\left\{\frac{m2\pi q v^2\gamma^2}{\omega L c}\,\frac{J_m(\alpha_q r)}{\alpha_q J_m'(\alpha_q b_o)}\right\} \quad , \tag{4.4}$$

and

$$\frac{\partial\tilde{E}_z^{(1)}}{\partial r}(r,\omega,q) = \tilde{C}_q b_o \frac{\partial\tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0)\left\{\left(\frac{2\pi q v\gamma^2}{\omega L} - 1\right)\frac{\alpha_q J_m'(\alpha_q r)}{J_m(\alpha_q b_o)}\right\} \tag{4.5}$$

where the value of $\frac{\partial\tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0)$ is given by Eq. (3.8). Note that since $\tilde{C}_o = 0$ there are no synchronous first order fields.

Second order solutions

In order to obtain the expressions for the transverse deflecting force on a test charge following a constant distance $v\tau$ behind the charge exciting the fields it is necessary only to evaluate the synchronous space harmonic of the transformed longitudinal electric field $\tilde{E}_z$ (cf. Eqs. (1.15) and (1.16)). The second order solution for the synchronous space harmonic of the transformed field $\tilde{E}_z^{(2)}$ is given by

$$\tilde{E}_z^{(2)}(r,\omega,0) = \tilde{E}_z^{(2)}(b_o,\omega,0)\,\frac{J_m(\alpha_o r)}{J_m(\alpha_o b_o)} \tag{5.1}$$

The value of $\tilde{E}_z^{(2)}(b_o,\omega,0)$ may be obtained from Eqs. (4.4), (4.5), and (3.10) inserted into Eq. (2.11).

The result for $\tilde{E}_z^{(2)}(r,\omega,0)$ may be written

$$\tilde{E}_z^{(2)}(r,\omega,0) = b_o^2\,\frac{J_m(\alpha_o r)}{J_m(\alpha_o b_o)}\,\frac{\partial \tilde{E}_z^o}{\partial r}(b_o,\omega,0)\sum \tilde{C}_{-q}\tilde{C}_q\left\{\left(\frac{1}{2b_o}\right)\right.$$
$$\left. + \left(1 - \frac{2\pi q v\gamma^2}{\omega L}\right)\left(\frac{\alpha_q^2 + (2\pi q/L)k_q}{\alpha_q}\right)\frac{J_m'(\alpha_q b_o)}{J_m(\alpha_q b_o)} + \left(\frac{4\pi^2 v^2\gamma^2 q^2 m^2}{\alpha_q^3 b_o^2 c^2 L^2}\right)\frac{J_m(\alpha_q b_o)}{J_m'(\alpha_q b_o)}\right\} \tag{5.2}$$

with $\dfrac{\partial \tilde{E}_z^{(0)}}{\partial r}(b_o,\omega,0)$ given by Eq. (3.8).

Inverse Fourier Transform

In this section we show that the transverse forces due to the zero order fields fall off with distance behind the charge at least as fast as

$\exp(-\gamma v\tau/b_0)$. The forces arising from the second-order fields moving with the charge distribution fall off more slowly, and for high energy electrons these longer-range forces are the ones of interest for us.

For reference purposes we include the second order longitudinal force acting on a test particle (charge e) following a distance $v\tau$ behind a centered (m=0) delta function charge with total charge Ne. From Appendix 1 this longitudinal force, averaged over a spatial period is

$$\langle F_z(r,t)\rangle = \frac{-4\pi e^2 N}{cL} \sum_{q=-\infty}^{\infty} q\tilde{C}_q\tilde{C}_{-q} \sum_{s=1}^{\infty} \omega_{os} e^{j\omega_{os}\tau} \quad .$$

This expression diverges as a result of the infinite frequencies generated by the delta-function distribution. When integrated over a reasonable charge distribution this force is a well-behaved function of τ. Such an integration over a Gaussian bunch is performed at the end of this section.

In order to calculate the average transverse deflecting forces, the expressions for $\tilde{E}_z(r,\omega,0)$ may be substituted into Eqs. (1.15) and (1.16) and the inverse transform taken. That is

$$\langle F_r(\tau)\rangle = \frac{1}{L} \int_0^L F_r(z,t = \tau + z/v)dz = \int d\omega\, e^{j\omega\tau}\, \tilde{F}_r(r,\omega,0) \cos m\phi \quad , \tag{6.1}$$

$$\langle F_\phi\rangle = \int d\omega\, e^{j\omega\tau}\, \tilde{F}_\phi(r,\omega,0) \sin m\phi \quad . \tag{6.2}$$

The transformed synchronous deflecting forces may be written as

$$\tilde{F}_r(r,\omega,0) = \tilde{F}_r^{(0)}(r,\omega,0) + \tilde{F}_r^{(2)}(r,\omega,0) \quad , \tag{6.3}$$

$$\tilde{F}_\phi(r,\omega,0) = \tilde{F}_\phi^{(0)}(r,\omega,0) + \tilde{F}_\phi^{(2)}(r,\omega,0) \quad , \tag{6.4}$$

where we have used the fact that the first order solution for the forces do not contain a synchronous space harmonic since we have picked the average radius b_o such that $\tilde{C}_p$ equals zero for $p = 0$. The zero order transformed synchronous deflecting forces are for $r < a$ given by

$$\tilde{F}_r^{(0)}(r,\omega,0) = \frac{e^2 N\xi\alpha_o}{av\gamma^2}\left[N_m(\alpha_o a)J_m(\alpha_o b) - N_m(\alpha_o b)J_m(\alpha_o a)\right]\frac{J_m'(\alpha_o r)}{J_m(\alpha_o b_o)} \quad , \tag{6.5}$$

and

$$\tilde{F}_\phi^{(0)}(r,\omega,0) = \frac{-me^2 N\xi}{av\gamma^2}\left[N_m(\alpha_o a)J_m(\alpha_o b) - N_m(\alpha_o b)J_m(\alpha_o a)\right]\frac{1}{r}\,\frac{J_m(\alpha_o r)}{J_m(\alpha_o b_o)} \quad . \tag{6.6}$$

We see from these equations that the poles of $F_r^{(0)}$ and F_ϕ^0 are for values of ω such that

$$\omega = \omega_{ms}^{(0)} = \pm j\,\frac{\gamma v x_{ms}}{b_o} \quad , \tag{6.7}$$

where x_{ms} are the roots of the equation $J_m(x_{ms}) = 0$.

The inverse transform integral gives the following result for the zero order synchronous deflecting forces

$$\langle \mathcal{F}_r \rangle = -\sum_{s=1}^{\infty} \frac{4e^2 N\xi}{ab_o^2\gamma} \frac{J_m(x_{ms}a/b_o)\ J_m'(x_{ms}r/b_o)}{[J_m'(x_{ms})]^2}\, e^{-\frac{\gamma v x_{ms}}{b_o}|\tau|} \cos m\phi \quad , \tag{6.8}$$

$$\langle \mathcal{F}_\phi \rangle = +\sum_{s=1}^{\infty} \frac{4me^2 N\xi\, b_o}{ab_o^2\, \gamma\, x_{ms} r} \frac{J_m(x_{ms}a/b_o)\ J_m(x_{ms}\, r/b_o)}{[J_m'(x_{ms})]^2}\, e^{-\frac{\gamma v x_{ms}}{b_o}|\tau|} \sin m\phi \quad . \tag{6.9}$$

The forces depend only upon absolute value of τ due to the fact that for $\tau > 0$ we must close the contour integral in the top half of the complex ω plane while for $\tau < 0$ we must close the contour in the lower half plane.

We note that the zero order forces fall off exponentially in a distance $v\tau \sim b_o/\gamma$; this is the well known result that a charge moving in a perfectly smooth conducting vacuum chamber will excite fields that fall off longitudinally from the charge of the order in a distance of the cross section dimension divided by γ.

For high energy electrons we will be interested in the wake fields that fall off more slowly in distance behind the charge. We consider the second order terms in the synchronous deflecting forces. The second order transform of the synchronous deflecting force may be written as

$$\tilde{F}_r^{(2)}(r,\omega,0) = \frac{2e^2N\xi\alpha_o b_o J_m'(\alpha_o r)\ J_m(\alpha_o a)}{\pi a\gamma^2 v\ J_m^2(\alpha_o b_o)} \quad \sum_q \tilde{C}_q\tilde{C}_{-q}\left\{\left(\frac{1}{2b_o}\right)\right. \tag{6.10}$$

$$\left. + \left(1 - \frac{2\pi q v\gamma^2}{\omega L}\right)\left(\frac{\alpha_q^2 + (2\pi q/L)k_q}{\alpha_q}\right)\frac{J_m'(\alpha_q b_o)}{J_m(\alpha_q b_o)} + \left(\frac{4\pi^2 v^2\gamma^2 q^2 m^2}{\alpha_q^3 b_o^2 c^2 L^2}\right)\frac{J_m(\alpha_q b_o)}{J_m'(\alpha_q b_o)}\right\}$$

and

$$\tilde{F}_\phi^{(2)}(r,\omega,0) = \frac{-2me^2N\xi b_o J_m(\alpha_o r)\ J_m(\alpha_o a)}{\pi a r\gamma^2 v\ J_m^2(\alpha_o b_o)} \quad \sum_q \tilde{C}_q\tilde{C}_{-q}\left\{\left(\frac{1}{2b_o}\right)\right. \tag{6.11}$$

$$\left. + \left(1 - \frac{2\pi q v\gamma^2}{\omega L}\right)\left(\frac{\alpha_q^2 + (2\pi q/L)k_q}{\alpha_q}\right)\frac{J_m'(\alpha_q b_o)}{J_m(\alpha_q b_o)} + \left(\frac{4\pi^2 v^2\gamma^2 q^2 m^2}{\alpha_q^3 b_o^2 c^2 L^2}\right)\frac{J_m(\alpha_q b_o)}{J_m'(\alpha_q b_o)}\right\}$$

We see that these transformed forces not only have poles at values of $\omega = \pm jv\gamma x_{ms}/b_o$ but also poles at values of ω such that $\alpha_q b_o = x_{ms}$ and $\alpha_q b_o = x'_{ms}$ where $J_m(x_{ms}) = 0$ and $J_m'(x'_{ms}) = 0$. The first set of poles will contribute to forces that fall off in a distance $c\tau \sim b/\gamma$ and are not of interest to us. The rest of the poles will contribute to the wake fields that have a much slower decay and these are the interesting ones for us to consider. Since we are interested only in the long range wake fields we will approximate the expression for $\tilde{F}_r$ and $\tilde{F}_\phi$ by allowing $\gamma \to \infty$. With these approximations we obtain for $m \neq 0$

$$\tilde{F}_r = \frac{8\pi e^2N\xi m\ r^{m-1}\ a^{m-1}}{cL^2(b_o^{m-1})^2}\sum_q q^2\ \tilde{C}_{-q}\tilde{C}_q\left\{\frac{m^2J_m(\alpha_q b_o)}{\alpha_q^3 b_o^3\ J_m'(\alpha_q b_o)} - \frac{J_m'(\alpha_q b_o)}{\alpha_q b_o J_m(\alpha_q b_o)}\right\} \tag{6.12}$$

and $\quad \tilde{F}_\phi = -\tilde{F}_r$ (6.13)

From Eqs. (6.12) and (6.13) we see that F_r and F_ϕ have poles along the real axis of the ω plane.

For each special harmonic q we find the roots for ω are at

$$\frac{\omega}{c} = \frac{\omega_{ms}}{c} = \left[\frac{\pi q}{L} + \frac{L x_{ms}^2}{4\pi q b_o^2}\right] \tag{6.14}$$

and

$$\frac{\omega}{c} = \frac{\omega'_{ms}}{c} = \left[\frac{\pi q}{L} + \frac{L (x'_{ms})^2}{4\pi q b_o^2}\right] \tag{6.15}$$

Since the wake field in front of the charge (i.e., $\tau < 0$) must be zero, we must close the contour in the lower part of the ω plane for this case; the integration along the real axis must go below the poles as shown in Fig. 5.

It then follows that for $\tau > 0$ the average value of F

$$\langle F \rangle = 2\pi j \sum \text{Residues of } (\tilde{F})\, e^{j\omega_r \tau} \tag{6.16}$$

where ω_r are the roots defined by Eqs. (6.14) and (6.15). For $\tau > 0$ the results for $\langle F_r \rangle$ and $\langle F_\phi \rangle$ can be written as

$$\langle F_r(r,\phi,\tau) \rangle = j\, \frac{8\pi e^2 N \xi m a^{m-1}}{L b_o^{2m}}\, r^{m-1} \sum_{q=-\infty}^{\infty} q\, \tilde{C}_q \tilde{C}_{-q} \sum_s \left\{ \frac{m^2}{m^2 - (x'_{ms})^2}\, e^{j\omega'_{ms}\tau} - e^{j\omega_{ms}\tau} \right\} \cos m\phi \tag{6.17}$$

$$< F_\phi(r,\phi,\tau) > = -j \frac{8\pi e^2 N \xi m a^{m-1}}{L b_o^{2m}} r^{m-1} \sum_q q \, \tilde{C}_q \tilde{C}_{-q} \sum_s \left\{ \frac{m^2}{m^2 - (x'_{ms})^2} e^{j\omega'_{ms}\tau} - e^{j\omega_{ms}\tau} \right\} \sin m\phi \tag{6.18}$$

where ω_{ms} and ω'_{ms} are defined by Eqs. (6.14) and (6.15). We note that ω_{ms} and ω'_{ms} are odd functions of q, and $\tilde{C}_{-q} = \tilde{C}_q^*$ so that we can write Eqs. (6.17) and (6.18) in terms of real variables

$$< F_r(r,\phi,\tau) > = - \frac{4\pi e^2 N \xi m a^{m-1} r^{m-1}}{L \; b_o^{2m}} \sum_{q=1}^{\infty} q |2\tilde{C}_q|^2 \left\{ \sum_s \frac{m^2}{m^2 - (x'_{ms})^2} \sin \omega'_{ms}\tau - \sin \omega_{ms}\tau \right\} \cos m\phi \tag{6.19}$$

and

$$< F_\phi(r,\phi,\tau) > = \frac{4\pi e^2 N \xi m a^{m-1} r^{m-1}}{L \; b_o^{2m}} \sum_{q=1} q |2\tilde{C}_q|^2 \left\{ \sum_s \frac{m^2}{m^2 - (x'_{ms})^2} \sin \omega'_{ms}\tau - \sin \omega_{ms}\tau \right\} \sin m\phi \tag{6.20}$$

where $|2 \tilde{C}_q| b_o$ is the amplitude of the qth harmonic of the waveguide corrugation.

Equations (6.19) and (6.20) represent the synchronous deflection forces that would be exerted on a test particle with charge e following a distance $c\tau$ behind the delta function charge distribution given by Eq. (1.1). As is typical of calculations involving singular distributions, the convergence properties of these expressions are at best poor, and nonexistent without invoking some sort of cutoff frequency. If, however, we use these expressions as Green's functions to treat a reasonable distribution of charge in the longitudinal dimension, then we will obtain well-defined expressions for the transverse deflecting force as a function of position within the distribution as well as behind it. We will perform such a calculation for the case of a Gaussian bunch for which the charge density is

$$\rho = \frac{eN\xi}{\pi a^2}\,\frac{1}{\sigma\sqrt{2\pi}}\,e^{\frac{-(t-z/c)^2 c^2}{2\sigma^2}}\,\delta(r-a)\cos m\phi \quad . \tag{6.21}$$

We must integrate Eqs. (6.19) and (6.20) over the distribution in z. For a test particle of charge e located a distance $c\tau$ after the bunch center (τ may be negative), we obtain

$$\langle F_r(r,\phi,\tau)\rangle = \frac{4\pi e^2 N\xi m a^{m-1} r^{m-1}}{L\ b_o^{2m}}\cos m\phi$$

$$\times \sum_q q\,|2\tilde{C}_q|^2 \sum_s \left\{\frac{m^2}{m^2 - x'^2_{ms}}\ \mathrm{Im}\left[\frac{1}{2}\,e^{\frac{-c^2\tau^2}{2\sigma^2}}\,w\left(\frac{\omega'_{ms}\sigma}{v\sqrt{2}} - \frac{jc\tau}{\sqrt{2}\,\sigma}\right)\right]\right.$$

$$\left. -\ \mathrm{Im}\left[\frac{1}{2}\,e^{\frac{-c^2\tau^2}{2\sigma^2}}\,w\left(\frac{\omega_{ms}\sigma}{v\sqrt{2}} - \frac{jc\tau}{\sqrt{2}\,\sigma}\right)\right]\right\} \quad , \tag{6.22}$$

and

$$<F_{\phi}(r,\phi,\tau)> = \frac{-4\pi e^2 N\xi m a^{m-1} r^{m-1}}{L\ b_0^{2m}} \sin m\phi$$

$$\times \sum_q q|2\tilde{C}_q|^2 \sum_s \left\{ \frac{m^2}{m^2 - x'^{\,2}_{ms}} \operatorname{Im}\left[\frac{1}{2} e^{\frac{-c^2\tau^2}{2\sigma^2}} w\left(\frac{\omega'_{ms}\sigma}{v\sqrt{2}} - \frac{jc\tau}{\sqrt{2}\,\sigma}\right)\right]\right.$$

$$\left. - \operatorname{Im}\left[\frac{1}{2} e^{\frac{-c^2\tau^2}{2\sigma^2}} w\left(\frac{\omega_{ms}\sigma}{v\sqrt{2}} - \frac{jc\tau}{\sqrt{2}\,\sigma}\right)\right]\right\} , \quad (6.23)$$

where we have used the relationship

$$\int_{-\infty}^{\tau} \sin\omega(\tau - t)\, e^{\frac{-t^2c^2}{2\sigma^2}}\, dt = \sqrt{\frac{\pi}{2}}\,\frac{\sigma}{v} \operatorname{Im}\left[e^{\frac{-c^2\tau^2}{2\sigma^2}} w\left(\frac{\omega\sigma}{v\sqrt{2}} - \frac{jc\tau}{\sqrt{2}\,\sigma}\right)\right] \quad (6.24)$$

where $w(z)$ is the complex error function.[4]

A similar calculation for the average longitudinal force resulting from a centered ($m = 0$) charge yields the result

$$<F_z(r,\tau)> = \frac{-2\pi e^2 N}{cL} \sum_{q=1}^{\infty} q|2\tilde{C}_q|^2 \sum_{s=1}^{\infty} \omega_{os} \operatorname{Re}\left[\frac{1}{2} e^{\frac{-c^2\tau^2}{2\sigma^2}} w\left(\frac{\omega_{ms}\sigma}{\sqrt{2}\,v} - j\frac{c\tau}{\sqrt{2}\,\sigma}\right)\right] . \quad (6.25)$$

Observations and A Numerical Example

In integrating the results 6.19 and 6.20 over a charge distribution, one is led to calculate

$$\int_{-\infty}^{\tau} \sin \omega(\tau-\tau')\rho(\tau')d\tau' = \mathrm{Im}\left\{e^{j\omega\tau}\int_{-\infty}^{\tau} e^{-j\omega\tau'}\rho(\tau')d\tau'\right\} \quad .$$

If $c\tau$ represents a distance within the bunch, this last integral represents the Fourier transform of a distribution truncated at the point of observation. This transform of a discontinuous (because of the truncation) function will have an amplitude which falls off as ω^{-1}. On the other hand, if $c\tau$ falls outside (and behind) the bunch, this integral is the Fourier transform of the total bunch shape, which for a continuous density distribution falls off in amplitude at least as rapidly as ω^{-2}, and for a Gaussian falls off as $\exp(-\alpha\omega^2)$.

These qualitative observations can be seen in Fig. 5, which is a three-dimensional plot of $|\exp(-y^2)w(x+jy)|$, the function resulting from integrating over a Gaussian distribution [cf. Eq. (6.24)]. In the summation over s, the frequencies ω_{ms} and ω'_{ms} correspond to distances along the x axis, while the length $c\tau$ (=0 in the center of the bunch, and positive behind the bunch) corresponds to a distance along the y axis. Thus it is clear that within the bunch many frequencies will contribute to the wake force, while outside the bunch the wake force will be determined principally by those frequencies for which the Fourier transform of the bunch distribution has appreciable amplitude.

As an example of the previous remarks, Figs. 6a through 6d show wake force calculations for m=1 with only $\tilde{C}_1 \neq 0$ and taking L=b=5 cm, with varying values of σ. Clearly, as one proceeds from small σ to large σ the number of significant frequency components in the wake force diminishes. In fact, for $c\tau > 2\sigma$ the wake force is well represented by including, in the sums occuring in Eqs. (6.22) and (6.23), only terms for which $\omega\sigma/\sqrt{2c} < 1.5$.

Further studies of the implications of Eqs. (6.22), (6.23), and (6.25) for beam dynamics are planned.

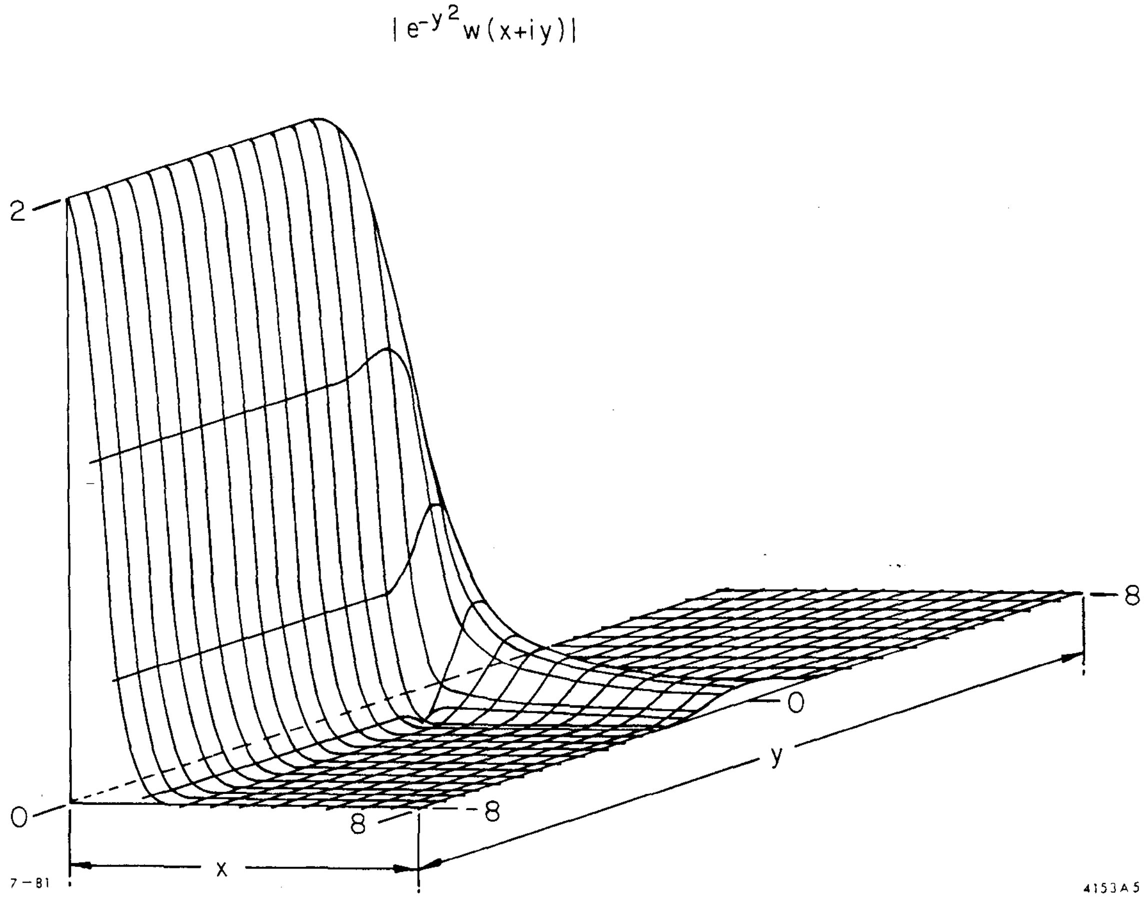


Fig. 5. Three-dimensional plot of the function $\exp(-y^2)|w(x+jy)|$.

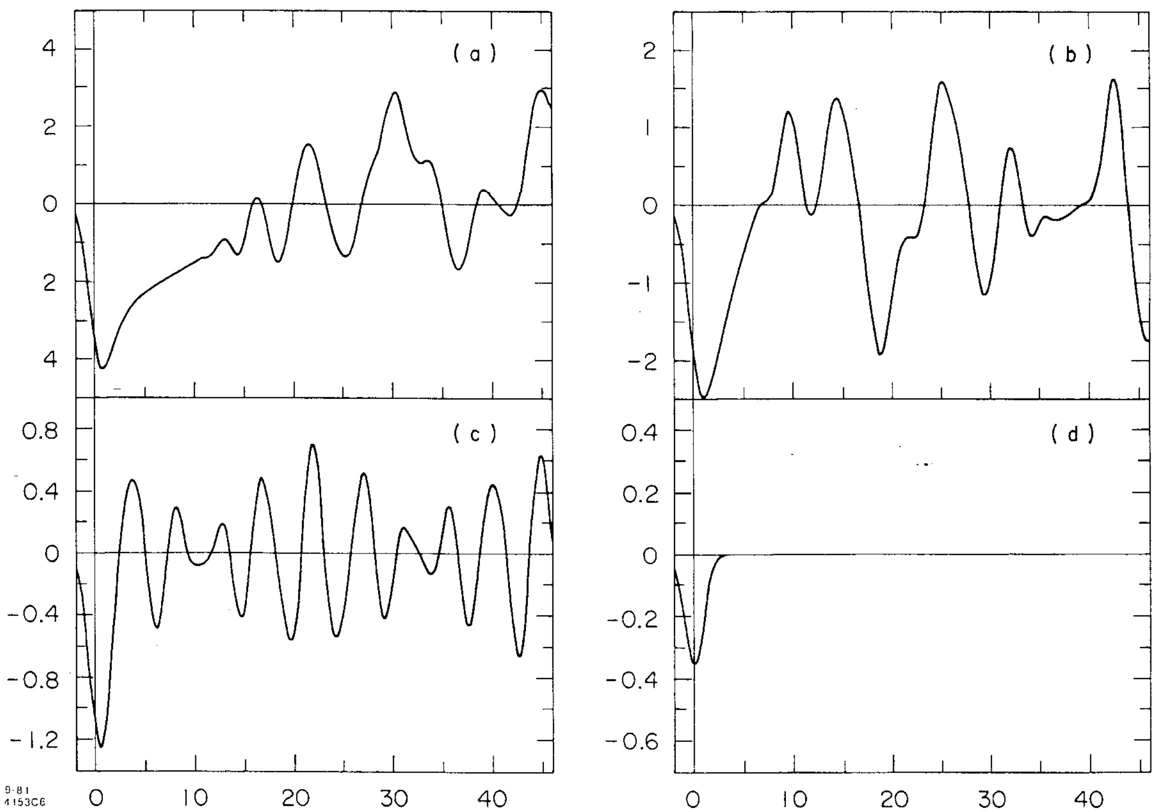


Fig. 6. Plots of the wake force for L=5 cm, b=5 cm, and varying values of σ. In Fig. 6a, $\sigma=(0.5/\sqrt{10})$ cm; in Fig. 6b $\sigma=0.5$ cm; in Fig. 6c $\sigma=(0.5\ \sqrt{10})$ cm; in Fig. 6d $\sigma=5$ cm. The force is indicated in the same arbitrary units for each plot; but the scales differ. The abscissa in each plot is $v\tau/\sigma$, ranging from -2 to 46.

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APPENDIX 1

The synchronous longitudinal wake field, as $\gamma \to \infty$, for a charge equal to eN traveling down the center of the periodic wave guide may be obtained from the transformed longitudinal electric field component given by Eq. (5.2) and (3.8) if we set m=0 and replace ξ by a/2. For $\gamma \to \infty$ Eq. (5.2) becomes

$$\tilde{E}_z^{(2)}(r,\omega,0) = j \frac{4\pi e N \omega b_o^2}{c^2 L^2} \sum_{q=-\infty}^{\infty} q^2 \tilde{C}_q \tilde{C}_{-q} \frac{J_o'(\alpha_q b_o)}{\alpha_q b_o J_o(\alpha_q b_o)}$$

Expanding $J_o(\alpha_q b_o)$ about the poles at $\alpha_q b_o = x_{os}$ we have near the pole

$$\alpha_q b_o \, J_o(\alpha_q b_o) \approx \frac{2\pi q b_o^2}{cL} J_o'(x_{os})(\omega - \omega_{os})$$

where x_{os} are the roots of $J_o(x_{os}) = 0$

and $$\frac{\omega_{os}}{c} = \frac{\pi q}{L} + \frac{L x_{os}^2}{4\pi q b_o^2}$$

The inverse transform of $\tilde{E}_z^{(2)}(r,\omega,0)$ will give the synchronous component of $E_z(r,t)$:

$$E_z^{(2)}(r,\tau) = -\frac{4\pi e N}{cL} \sum_q q \tilde{C}_q \tilde{C}_{-q} \sum_s \omega_{os} \, e^{j\omega_{os}\tau}$$

APPENDIX 2

We consider the product of the functions $f(r,z,t)$ and $g(r,z,t)$ each of which may be written in the following manner.

$$f(r,z,t) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega\tau} \sum_{p=-\infty}^{\infty} e^{j\frac{2\pi p}{L}z}\, \tilde{f}(r,\omega,p) \quad , \qquad \text{(A.1)}$$

$$\text{with } \tau = (t - \frac{z}{v}) \quad . \qquad \text{(A.2)}$$

Clearly the product is given by

$$f \cdot g = \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega'' \sum_{p=-\infty}^{\infty} \sum_{p'=-\infty}^{\infty} \tilde{f}(r,\omega',p)\tilde{g}(r,\omega'',p')\, e^{j(\omega'+\omega'')\tau}\, e^{j\frac{2\pi}{L}(p+p')z} \qquad \text{(A.3)}$$

If we define $\omega = \omega' + \omega''$ and $q = p + p'$ we obtain

$$f \cdot g = \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \sum_{q=-\infty}^{\infty} e^{j\frac{2\pi q}{L}z} \int_{-\infty}^{\infty} d\omega' \sum_{p=-\infty}^{\infty} \tilde{f}(r,\omega',p)\tilde{g}(r,\omega-\omega',q-p) \qquad \text{(A.4)}$$

Using the definition of

$$[fg](r,z,t) = \int_{-\infty}^{\infty} d\omega\, e^{j\omega t} \sum_{q=-\infty}^{\infty} e^{j\frac{2\pi q}{L}z}\, \widetilde{[fg]}(r,\omega,q) \tag{A.5}$$

we obtain the following expression

$$\widetilde{[fg]}(r,\omega,p) = \int_{-\infty}^{\infty} d\omega' \sum_{q=-\infty}^{\infty} \tilde{f}(r,\omega',q)\tilde{g}(r,\omega-\omega',p-q) \tag{A.6}$$