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SCATTERING THEORY WITHOUT INFRARED PROBLEMS BASED ON THE LOCAL OBSERVABLES OF QUANTUM ELECTRODYNAMICS*

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ABSTRACT

A conjecture, motivated by plausible physical arguments, is presented, from which is derived a scattering theory for QED based entirely on gauge invariant local fields. The conjecture is shown to be a natural extrapolation of results of Araki and Haag, and models are considered which indicate that the resulting scattering theory is free of the infrared problems usually associated with QED scattering theory. The unsolved problems of proving the conjecture and of obtaining a practical calculation scheme are discussed.

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1. Introduction

A field theory has more predictive power than an S-matrix theory. QED, for example, can be applied at finite temperatures or, to some extent, at the level of classical correspondence, yielding in either case predictions outside of scattering theory. Thus, scattering theory is just one of field theory's many interesting consequences.

The above viewpoint is blemished if the field theory of interest is less than satisfactory. Such is the case for QED as typically formulated, which lacks some of the following desirable properties: gauge invariance, Lorentz invariance, locality, and positive-definite metric. These desirable properties may be retained if we restrict our attention to the local gauge-invariant fields, which presumably yield a perfectly acceptable Wightman field theory. This rescues the above viewpoint, but leaves us with the problem of constructing scattering theory without the fields ψ and A^{μ} at our disposal. To solve this problem we present a conjecture from which follows a derivation of scattering theory from the theory of local observables associated with QED. Our results have both advantages and disadvantages compared to the usual (LSZ) approach of constructing scattering states from asymptotic limits of unobservable fields. The advantages are:

1. Conceptual superiority: scattering probabilities are obtained from a dynamical theory of observables rather than from gauge non-invariants in a conceptually unjustified manner. Gauge invariance is manifest from the outset and does not require subsequent verification.

2. Simplicity: The various constructs presently used to deal with the infrared problem -- e.g., infrared cutoffs, Coulomb phase factors, soft photon radiation damping factors, and non-Fock phase space

summations -- are entirely avoided. Furthermore, it is not necessary to consider different gauges, nonlocal fields, indefinite metrics, or the difficult to construct and Lorentz noninvariant[15,8] charged sectors. The disadvantages are:

1. Although we argue that the conjecture is plausible, we do not prove it.

2. Although our results are in principle as widely applicable as the LSZ approach, they are in practice cumbersome to a degree which depends mainly on the initial state. To make our results competitive in a practical sense requires the solution of a deep problem which is discussed in Section IV. However, the benefits of solving this problem could extend far beyond scattering theory.[16]

We proceed as follows. In Section II we present the conjecture and show how it may be used to obtain scattering probabilities. In Section III we argue that the conjecture is plausible. [This will mostly involve the consideration of the various infrared difficulties which arise in the LSZ approach.] In Section IV we consider the disadvantages of our approach and discuss how they might be overcome.

Notation: Our metric is -+++. Covariant normalizations are used. In particular, instead of $d\vec{p}$ we use $\overline{dp} \equiv d\vec{p}/(2\pi)^3 2p^o$, where $p^o = \sqrt{\vec{p}^2 + m^2}$, m being the mass of whatever particle is relevant. Given a function f of momentum and a region Ω in momentum space, $f(\Omega) \equiv \int_\Omega \overline{dp}\, f(p)$. Finally, if A_j is a collection of commuting self-adjoint operators and λ_j is a collection of real numbers, $E(A_j = \lambda_j)$ denotes the projection operator onto the space of states ψ satisfying $A_j\, \psi = \lambda_j \psi$.

2. The Conjecture and QED Scattering Theory

[As will soon be clear, I have in what follows frequently chosen to sacrifice correctness for brevity. In particular, distributions are sometimes smeared with inappropriate functions and the domains of unbounded operators are never considered. I do not feel that these defects are crucial.]

Our conjecture consists of two parts. The first is that there exist number operators which we think of as counting outgoing leptons (electrons and positrons). Specifically, we assume the existence of operators $n_i^{out}(p)$, $i = 1,\ldots,4$, satisfying

(a) $n_i^{out\,\dagger} = n_i^{out}$

(b) $n_i^{out}|0\rangle = 0$

(c) for any region Ω in momentum space, the spectrum of $n_i^{out}(\Omega)$ is the nonnegative integers

(d) $[n_i^{out}(p), n_j^{out}(q)] = 0$

(e) a regularity condition (described below).

The second part of our conjecture gives meaning to the number operators by relating them to local observables. Let $T^{\mu\nu}$, J^{ν}, and $M^{i\nu}$ denote respectively the energy-momentum tensor, the electric current, and the angular-momentum current. Let $h: \mathbb{R}^3 \to \mathbb{R}^1$ denote an arbitrary smooth function with support <u>inside</u> the unit ball. Then we conjecture that

$$\lim_{t\to\infty} \int d\vec{x}\, h\left(\frac{\vec{x}}{t}\right) T^{\mu o}(t,\vec{x})$$

$$= \int \overline{dp}\, h\left(\frac{\vec{p}}{p^o}\right) p^\mu \left[n_1^{out}(p) + n_2^{out}(p) + n_3^{out}(p) + n_4^{out}(p)\right] \quad (1.a)$$

$$\lim_{t\to\infty}\int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right) J^{o}(t,\vec{x})$$

$$= \int \overline{dp}\; h\left(\frac{\vec{p}}{p^{o}}\right) e\left[n_1^{out}(p) + n_2^{out}(p) - n_3^{out}(p) - n_4^{out}(p)\right] \quad (1.b)$$

$$\lim_{t\to\infty}\int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right) \sum_{i=1}^{3} \frac{x^{i}}{|\vec{x}|}\; M^{io}(t,\vec{x})$$

$$= \int \overline{dp}\; h\left(\frac{\vec{p}}{p^{o}}\right) \frac{1}{2}\left[n_1^{out}(p) - n_2^{out}(p) + n_3^{out}(p) - n_4^{out}(p)\right] \quad (1.c)$$

(We assume that the limits hold weakly on some suitable domain. Also, $T^{\mu o}$, J^{o}, and M^{io} need to be smeared in time; we suppress the necessary modifications.)

We think of the $n_i^{out}(p)$, $i = 1,\ldots,4$, as respectively counting helicity $+\frac{1}{2}$ electrons, helicity $-\frac{1}{2}$ electrons, helicity $+\frac{1}{2}$ positrons, and helicity $-\frac{1}{2}$ positrons of momentum p. Equation (1) is then easily motivated: Consider first Eq. (1.b) sandwiched between a state whose leptonic content consists of a single outgoing electron of nearly well-defined momentum p'. Then for large t, $\langle J^{o}(t,\vec{x})\rangle$ is localized near $\vec{x} = (\vec{p}\,'/p'^{o})t$ modulo wave-packet spreading and other, less important, effects. Consequently, $h(\vec{x}/t)$, which varies slowly with $\vec{x}$ for large t, may be replaced with $h(\vec{p}\,'/p'^{o})$ in the left-hand side of the equation. The left-hand side then reduces to $h(\vec{p}\,'/p'^{o})e$, which clearly equals the right-hand side. Suitable modifications extend this argument to the case of several outgoing leptons.

Analogous arguments apply to Eq. (1.a) except that $\langle T^{\mu o}(t,\vec{x})\rangle$ receives contributions from both the outgoing electron and whatever outgoing photons are present. However, the photon contributions at large times are localized "near" $|\vec{x}|/t = 1$ where, by definition, $h(\vec{x}/t)$

vanishes. Thus, only the electron contributes to the limit and the argument proceeds as before. And the same argument applies to Eq. (1.c) once it is noticed that $\vec{x}/|\vec{x}| = (\vec{x}/t)/(|\vec{x}/t|)$ may be replaced with $(\vec{p}'/p'^{o})/(|\vec{p}'/p'^{o}|) = \vec{p}'/|\vec{p}'|$.

Equation (1) is closely related to -- and was, in fact, originally motivated by -- results of Araki and Haag (described in Section III). Consequently, we refer to our results as Araki-Haag scattering theory applied to QED.

We now proceed to obtain QED scattering theory from the theory of local observables, using our conjecture. The first step is to construct the number operators in terms of local observables. To do this we shall need the regularity condition (e) of the conjecture which we now motivate.

Consider a region Ω of momentum space divided into a large number of exceedingly tiny bins Ω_k. For a given state, it is highly unlikely that any two outgoing leptons will be found in a single bin. Therefore, except for a small error, to find $n_i^{out}(\Omega)$, it suffices to count how many bins contain precisely one lepton of type i and no other leptons. The operator that counts this is $\sum_k E(n_j^{out}(\Omega_k) = \delta_{ij})$. (See notation.) Therefore we assume

$$\text{(e)} \quad n_i^{out}(\Omega) = \lim \sum_k E(n_j^{out}(\Omega_k) = \delta_{ij}),$$

where, in the limit, the size of the bins is taken to zero.

We now construct the number operators in terms of local observables. Define

$$n^{out} \equiv n_1^{out} + n_2^{out} + n_3^{out} + n_4^{out}$$

$$q^{out} \equiv n_1^{out} + n_2^{out} - n_3^{out} - n_4^{out}$$

$$\ell^{out} \equiv n_1^{out} - n_2^{out} + n_3^{out} - n_4^{out} .$$

As a consequence of the spectral properties of $n_i^{out}(\Omega)$ assumed in the conjecture, it follows that

$$E(n^{out}(\Omega) = 1,\ q^{out}(\Omega) = \pm 1,\ \ell^{out}(\Omega) = \pm 1) = E(n_j^{out}(\Omega) = \delta_{ij}) ,$$

where the values of $\pm,\pm$ assumed are $(+,+)$, $(+,-)$, $(-,+)$, and $(-,-)$ respectively for $i = 1,\ldots,4$. Using (e) above we obtain

$$n_i^{out}(\Omega) = \lim \sum_k E(n^{out}(\Omega_k) = 1,\ q^{out}(\Omega_k) = \pm 1,\ \ell^{out}(\Omega_k) = \pm 1), \quad (2)$$

where the meaning of lim and the dependence of $\pm,\pm$ on i are the same as above. Equation (2) is the desired construction of the lepton number operators in terms of local observables, since n^{out}, q^{out}, and ℓ^{out} are expressed in terms of the local observables $T^{\mu o}$, J^o, and M^{io} in Eq. (1).

Next we need number operators which count hard photons. Hard particles are particles with enough energy to be detected. (Electrons, we assume, are always hard.) Photons with too little energy to be detected are called soft. The meaning of "hard" and "soft" depends of course on the efficiency of one's detectors, but the set of hard photon momenta k always excludes a neighborhood of k = 0.

It is tempting to try to get a handle on the photon number operators by reconsidering Eq. (1) without the restriction that the support of h lie inside the unit ball. For example, if we consider a state consisting entirely of outgoing photons and let $n_{ph}^{out}(k)$ count the total number of photons, we expect to find

$$\lim_{t\to\infty} \int d\vec{x}\ h\left(\frac{\vec{x}}{t}\right) T^{\mu o}(t,\vec{x}) = \int \overline{dk}\ h\left(\frac{\vec{k}}{k^o}\right) k^\mu\ n_{ph}^{out}(k) \quad .$$

However, it is not possible by varying h to extract $n_{ph}^{out}(k)$ from the right hand side, because the mapping $\vec{k} \to \vec{k}/k^o$ is many to one. At best, we can extract $\int_0^\infty d\lambda\, \lambda^2\, n_{ph}^{out}(\lambda k)$, which is insufficient for our purposes. We must therefore take a different approach.

Fortunately, the work of constructing hard photon number operators has already been done: the photon annihilation operator $a_\lambda^{out}(k)$ has been constructed[3] from the electromagnetic field $F^{\mu\nu}$ and it has been shown that $n_\lambda^{out}(k) \equiv a_\lambda^{out}(k)^\dagger a_\lambda^{out}(k)$ makes sense away from $k = 0$.[8] Setting $n_5^{out} \equiv n_{\lambda=1}^{out}$ and $n_6^{out} \equiv n_{\lambda=-1}^{out}$, we have number operators $n_i^{out}(k)$, $i = 1,\ldots,6$, for all <u>hard</u> particles.

Finally, we shall express the scattering probabilities of interest in terms of the number operators. For a given state $\rangle$, let $P_m(k_1,\ldots,k_m)$ denote the probability density for finding precisely m outgoing hard particles of momenta $k_1,\ldots,k_m$ accompanied by an arbitrary collection of soft photons. (For simplicity we suppress the indices that label particle type.) Let Ω_i denote m regions of hard phase space and define $\chi_i(k) = \begin{cases} 1, & k\in\Omega_i \\ 0, & k\notin\Omega_i \end{cases}$. Then clearly,

$$\langle n^{out}(\Omega_1) \ldots n^{out}(\Omega_m)\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{hard} \overline{d\ell_1} \ldots \overline{d\ell_n}\; P_n(\ell_1,\ldots,\ell_n) \prod_{i=1}^{m} \left(\sum_{j=1}^{n} \chi_i(\ell_j)\right),$$

where the $\overline{d\ell}$ integrations run over hard phase space. A little work then yields, for <u>$k_i \neq k_j$</u>,

$$\langle n^{out}(k_1) \ldots n^{out}(k_m)\rangle$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{hard} \overline{d\ell_1} \ldots \overline{d\ell_n}\; P_{m+n}(k_1,\ldots,k_m, \ell_1,\ldots,\ell_n) . \quad (3)$$

If we extend the left-hand side of Eq. (3) <u>by continuity</u> to coinciding values of k_i, then Eq. (3) should hold for all values of k_i. Labeling the extension with a prime, it is easy to check that Eq. (3) is equivalent to

$$P_m(k_1, \ldots, k_m) = \sum_{n=o}^{\infty} \frac{(-1)^n}{n!} \int_{hard} \overline{d\ell_1} \ldots \overline{d\ell_n} \langle n^{out}(k_1) \ldots n^{out}(k_m)\, n^{out}(\ell_1) \ldots n^{out}(\ell_n)\rangle' \quad . \tag{4}$$

Equation (4) completes our derivation of QED scattering theory from local observables. All quantities of experimental interest may be expressed in terms of the P_m. Equation (4) expresses P_m in terms of asymptotic number operators which are themselves defined in terms of local observables by Eqs. (1) and (2) and in Refs. 3 and 8. We shall consider the practical applicability of Eq. (4) in Section IV.

3. Plausibility of the Conjecture

Our argument for the plausibility of the conjecture runs as follows: As shall be shown below, our conjecture is a natural extrapolation to QED of the results of Araki and Haag. Now, Araki-Haag scattering theory[1] and LSZ scattering theory[11] have both been justified for Wightman field theories with a mass gap. It is therefore reasonable to expect these scattering theories to apply to QED provided that the various (infrared) phenomena associated with the absence of a mass gap do not cause trouble. These phenomena include long range (Coulomb) interactions, emission and absorption of infinite numbers of soft photons, and the infraparticle nature of the electron. The crucial result of this paper is that while each of these three phenomena presents a serious difficulty for the LSZ approach, none of them seems to present a problem for the Araki-Haag approach. To show this we shall consider three models, each displaying one of the three phenomena. In each case it will be seen that the usual scattering theory breaks down while the Araki-Haag approach applies without modification.

That's the argument. It remains for us to fill in the gaps: we first obtain our conjecture as a natural extrapolation of the Araki-Haag scattering theory, and then consider the three models which display infrared phenomena.

In obtaining our conjecture, we shall focus our attention on Theorem 4 of Ref. 1 which we present in slightly altered form. Given a Wightman field theory with mass gap, one can define scattering states (which we henceforward assume to be dense) and annihilation operators $b_i^{out}(p)$. Let C(x) be a Geiger-counter field, defined by $C(x) = e^{-ix\cdot P} C\, e^{ix\cdot P}$, where

i) C is a bounded, self-adjoint operator

ii) C is quasilocal (the meaning of which is not important to us)

iii) $C\,|0\rangle = 0$

iv) $\langle pi\,|\,C\,|\,p'j\rangle$ is differentiable in $\vec{p}$ and $\vec{p}\,'$ ($|pi\rangle \equiv b_i^{out}(p)^{\dagger}|0\rangle$.)

Set $\Gamma_{ij}(p) = (2p^o)^{-1} \langle pi|C|pj\rangle$. Then Theorem 4 states that

$$\lim_{t\to\infty} \int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right) C(t,\vec{x}) = \sum_{i,j} \int \widetilde{dp}\; h\left(\frac{\vec{p}}{p^o}\right) \Gamma_{ij}(p)\; b_i^{out}(p)^{\dagger} b_j^{out}(p) \quad , \qquad (5)$$

where the limit is weak and $\sum_{i,j}$ runs over all pairs i,j of particle types with $m_i = m_j$.

The one-particle states $|pi\rangle$ are crucial to the statement of the theorem. As will be clear from the models considered below, there is in

QED no sensible way to define a state $|pi\rangle$ consisting of a single electron and no other particles. Consequently, Theorem 4 has no immediate extension to QED. Nevertheless, by massaging Eq. (5) a bit as follows, we can arrive at Eq. (1).

First, we loosen the technical requirement that C(x) be a bounded-operator-valued function and allow it to be an unbounded-operator-valued distribution, i.e., a field. This is a natural assumption since practical LSZ calculations are inevitably done with fields and no difficulties are known to arise from this.

Second, the requirement $C|0\rangle = 0$ seems to be used in the proof of Theorem 4 only to conclude that $C|0\rangle$ is orthogonal to both the vacuum and to the single-"lepton" states. The latter requirement is automatic in QED since there are no local observables which create charged states from the vacuum.

Our loosened requirements on C are satisfied if we take C(x) to be the zero component of a conserved current $j^\mu(x)$ whose associated "charge" $\int d\vec{x}\, j^o(x)$ defines quantum numbers $\Gamma_i(p)$ which label the various particles. In this case it is easy to check that Eq. (5) reduces to

$$\lim_{t\to\infty}\int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right) j^o(t,\vec{x}) = \sum_i \int \overline{dp}\; h\left(\frac{\vec{p}}{p^o}\right) \Gamma_i\,(p)\; b_i^{out}(p)^\dagger b_i^{out}(p) \quad . \tag{6}$$

Equation (6) generalizes naturally to QED and yields Eq. (1) for the special cases $j^\nu = T^{\mu\nu}$, $j^\nu = J^\nu$, and, with a little extra finagling, $j^\nu = M^{i\nu}$. This completes our argument that our conjecture is a natural extrapolation to QED of the results of Araki and Haag.

We now turn to the arguments central to this paper -- those indicating that Araki-Haag scattering theory is untroubled by infrared phenomena which present significant difficulties for the LSZ approach. Specifically we present three models, each manifesting some infrared phenomenon, for which the usual scattering theory breaks down but to which Araki-Haag scattering theory applied in a straightforward manner.

A. Couloumb Scattering

Our first model is the nonrelativistic quantum theory of a particle moving in a Coulomb potential. The Hamiltonian is $H = -\frac{\vec{\nabla}^2}{2m} + \frac{k}{|\vec{x}|} = H_0 + V$. If V were short range, we could proceed in the usual manner -- that is, by defining $\Omega_{out} = \text{s-lim}_{t\to\infty}\, e^{iHt}e^{-iH_0t}$ and $|\vec{p}\ \text{out}\rangle = \Omega_{out}|\vec{p}\rangle$, with similar definitions for in-states. The S-matrix would then be $S_{\vec{p}\vec{p}\,'} = \langle\vec{p}\ \text{out}|\vec{p}\,'\ \text{in}\rangle$. For the Coulomb potential, the above limit defining Ω_{out} does not converge. Instead, one defines[6]

$$\Omega_{out} = \text{s-lim}_{t\to\infty}\, e^{iHt}\, e^{-iH_0t}\, e^{-ikm(-\vec{\nabla}^2)^{-\frac{1}{2}}\log(-2t\vec{\nabla}^2/m)} \quad ,$$

which solves the problem. This solution has two drawbacks. First the extra factor (known as the Coulomb phase factor) needed to obtain convergence is unsightly and becomes more so when generalized to QED. Second and more important, the factor contains an arbitrariness: as noted in Ref. 6, the factor may be multiplied by an arbitrary unitary function of $-i\vec{\nabla}$ without affecting any significant results. Thus, $|\vec{p}\ \text{out}\rangle$ has a $\vec{p}$ dependent phase arbitrariness.

To better understand this arbitrariness, we reconsider nonrelativistic potential scattering from a more field theoretical viewpoint. Let $\vec{X}(t)$ and $\vec{P}(t)$ be the Heisenberg picture position and momentum operators. For suitable potentials V, the usual scattering theory holds and implies the existence of the limits

$$\text{w-lim}_{t\to\infty}\, \vec{P}(t) = \vec{P}_{out}(0) \tag{7.a}$$

$$\text{w-lim}_{t\to\infty} \left[\vec{X}(t) - \frac{\vec{P}(t)}{m}t\right] = \vec{X}_{out}(0) \quad . \tag{7.b}$$

$\vec{P}_{out}(0)$ and $\vec{X}_{out}(0)$ comprise an irreducible set of operators and may be used to define a representation. In particular, $|\vec{p}\ out\rangle$ may be uniquely defined (modulo an irrelevant $\vec{p}$-independent phase) in a natural way.[5] In the case of Coulomb scattering, Eq. (7.a) holds (this follows from results in Ref. 6), but Eq. (7.b) does not. Consequently, one can make sense out of expressions like $f(\vec{P}_{out}) = $ "$\int d\vec{p}\ f(\vec{p})|\vec{p}\ out\rangle\langle\vec{p}\ out|$", but $|\vec{p}\ out\rangle$ standing by itself has a phase arbitrariness which cannot be fixed in a natural way.

How does Araki-Haag scattering theory fit in? In the nonrelativistic analog of Eq. (6) we choose for j^{ν} the probability current density,[12] whose zero component is just $j^{o}(t,\vec{x}) = e^{iHt}|\vec{x}\rangle\langle\vec{x}|e^{-iHt}$. Since $\int d\vec{x}\ j^{o}(t,\vec{x}) = 1$, $\Gamma(\vec{p}) = 1$. Also, the nonrelativistic analog of $b^{out}(p)^{\dagger}b^{out}(p)$ is $|\vec{p}\ out\rangle\langle\vec{p}\ out|$. Thus, Araki-Haag scattering theory applied to our model predicts

$$\lim_{t\to\infty}\int d\vec{x}\ h\left(\frac{\vec{x}}{t}\right)\ e^{iHt}\ |\vec{x}\rangle\langle\vec{x}|e^{-iHt} = \int d\vec{p}\ h\left(\frac{\vec{p}}{m}\right)\ |\vec{p}\ out\rangle\langle\vec{p}\ out| \quad . \tag{8}$$

And, using the results of Ref. 6, Eq. (8) is easily verified. (The Coulomb phase factor which arises in the course of the proof cancels against its complex conjugate.)

Extrapolating to QED, we expect attempts to define lepton creation operators $b_i^{out}(p)^{\dagger}$ to require a Coulomb phase factor with a p-dependent phase arbitrariness, while attempts to define lepton number operators $n_i^{out}(p)$ should encounter no such difficulties.

B. Soft Photon Emission

As has long been understood,[2,4] collision processes in QED involving leptons are typically accompanied by the emission and absorption of infinite numbers of soft photons. We now consider a model in which the same phenomenon arises from a classical current. Our discussion is partly based on a discussion appearing in Ref. 14.

We consider Maxwell's equations $\partial_\mu F^{\mu\nu} = J^\nu$ and $\varepsilon_{\kappa\lambda\mu\nu}\, \partial^\lambda F^{\mu\nu} = 0$ with $F^{\mu\nu}$ quantized and J^ν an external c-number field satisfying $\partial_\mu J^\mu = 0$. A solution is easily found: Let ^ denote the Fourier transform and define

$$\hat{\Delta}^{\mu\nu\alpha}_{ret}(k) = i\, \frac{g^{\mu\alpha}k^\nu - g^{\nu\alpha}k^\mu}{k^2 - i\varepsilon k^o} \quad , \tag{9}$$

where ε is "infinitesimally small." Then, if we let $F^{\mu\nu}_{in}$ denote the free electromagnetic field in the Fock representation,

$$F^{\mu\nu} \equiv F^{\mu\nu}_{in} + \Delta^{\mu\nu\alpha}_{ret} * J_\alpha \tag{10}$$

solves Maxwell's equations. (Other solutions may be obtained by choosing non-Fock representations for $F^{\mu\nu}_{in}$. We do not consider these, but, with slight modification, the following discussion applies to the choice of any reasonable generalized coherent state representation.)

The subscript "in" used above is not misleading. Because of the $i\varepsilon$ prescription in Eq. (9), $F^{\mu\nu}$ converges in an appropriate sense to $F^{\mu\nu}_{in}$ as $t \to -\infty$. Similarly, we have

$$F^{\mu\nu} = F^{\mu\nu}_{out} + \Delta^{\mu\nu\alpha}_{adv} * J_\alpha \quad , \tag{11}$$

where $\Delta^{\mu\nu\alpha}_{adv}$ is defined like $\Delta^{\mu\nu\alpha}_{ret}$ but with the opposite $i\varepsilon$ prescription. Subtracting Eq. (10) from Eq. (11) yields $F^{\mu\nu}_{out} - F^{\mu\nu}_{in} = (\Delta^{\mu\nu\alpha}_{ret} - \Delta^{\mu\nu\alpha}_{adv}) * J_\alpha$ which in turn implies

$$a^{out}_\lambda(k) - a^{in}_\lambda(k) = f_\lambda(k) \quad , \tag{12}$$

where $f_\lambda(k) = -i\varepsilon^*_\lambda(k) \cdot \hat{J}(k)$. The $\varepsilon_\lambda(k)$ are the usual free field polarization vectors.

By assumption, $a_\lambda^{in}(k)$ acts in the Fock representation, and Eq. (12) solves the problem of constructing $a_\lambda^{out}(k)$. It remains for us to develop scattering theory. The usual procedure is to note that Eq. (12) has the solution $a_\lambda^{out} = S^{-1} a_\lambda^{in} S$, where

$$S \equiv e^{a^{out\dagger}(f) - a^{out}(f^*)} = e^{-\frac{1}{2}\sum_\lambda \int \overline{dk}\,|f_\lambda(k)|^2} e^{a^{out\dagger}(f)} e^{-a^{out}(f^*)} \tag{13.a}$$

$[a^{out\dagger}(f) \equiv \sum_\lambda \int \overline{dk}\; f_\lambda(k)\; a_\lambda^{out}(k)^\dagger$, etc.]. It follows that

$$S_{\alpha\beta} = \langle \alpha \text{ out}|S|\beta \text{ out}\rangle \quad , \tag{13.b}$$

where $S_{\alpha\beta} \equiv \langle \alpha \text{ out}|\beta \text{ in}\rangle$ and α and β label points in the phase spaces of the out- and in-representations.

However, Eq. (13) makes sense only if $\sum_\lambda \int \overline{dk}|f_\lambda(k)|^2 < \infty$. This is not the case if J^ν represents a nontrivial collision process, in which case $f(k)$ goes like $1/|\vec{k}|$ near $\vec{k} = 0.$[14] If in fact $\sum_\lambda \int \overline{dk}|f_\lambda(k)|^2 = \infty$, which we henceforward assume, we have several alternative procedures:

(a) Use Eq. (13) but first introduce an infrared cutoff of some sort. The S-matrix elements will then be cutoff dependent, but scattering probabilities inclusive over soft photon emission will be finite as the cutoff is released. This is the procedure presently used in almost all practical QED calculations. It is unjustified, but it works.

(b) Note that the failure of Eq. (13) arises from the fact that $F^{\mu\nu}_{out}$ acts in a non-Fock representation.[7] In fact, since $a_\lambda^{out}(k)|0 \text{ in}\rangle = f_\lambda(k)|0 \text{ in}\rangle$, $F^{\mu\nu}_{out}$ acts in a generalized coherent state representation which would be unitarily equivalent to the Fock representation if $\sum_\lambda \int \overline{dk}|f_\lambda(k)|^2 < \infty$. Thus, the α appearing in $\langle \alpha \text{ out}|\beta \text{ in}\rangle$ should label points in a non-Fock phase space. $\langle \alpha \text{ out}|\beta \text{ in}\rangle$ will then be well defined, and scattering probabilities inclusive over soft photon emission can in principle be computed by means of a non-Fock phase space integration. This must yield the correct answer. However, I don't know how to practically parametrize points in a non-Fock phase space.

(c) Use the Araki-Haag approach as developed in Section II. Recall that photon number operators were defined to be $a_\lambda^{out}(k)^\dagger a_\lambda^{out}(k)$ for $k \neq 0$. Consequently, using Eq. (12),

$$n_\lambda^{out}(k) = (a_\lambda^{in}(k)^\dagger + f_\lambda(k)^*)(a_\lambda^{in}(k) + f_\lambda(k)) \quad . \qquad (14)$$

One can then apply Eq. (4). As an example, let $\rangle$ be a coherent state $|g\ in\rangle$, which is defined by $a_\lambda^{in}(k)|g\ in\rangle = g_\lambda(k)|g\ in\rangle$, where $\sum_\lambda \int \overline{dk}|g_\lambda(k)|^2 < \infty$.

Using Eq. (14), we see immediately that

$$\langle n_{\lambda_1}^{out}(k_1) \ldots n_{\lambda_m}^{out}(k_m)\rangle' = \prod_{i=1}^m |f_{\lambda_i}(k_i) + g_{\lambda_i}(k_1)|^2 \quad .$$

(The prime is crucial!) Inserting this result into Eq. (4) yields

$$P_m(k_1,\lambda_1; \ldots ; k_m,\lambda_m) = e^{-\sum_\lambda \int_{hard} \overline{dk}|f_\lambda(k) + g_\lambda(k)|^2} \prod_{i=1}^m |f_{\lambda_i}(k_i) + g_{\lambda_i}(k_i)|^2 \qquad (15)$$

Equation (15) is the correct result. It is obtained simply, without infrared cutoffs or non-Fock soft-photon phase space summations, using the scattering theory developed in Section II.

C. Infraparticles

A single particle state is defined to be an eigenstate of the mass operator. However, loosely speaking, electrons are always accompanied by a cloud of soft photons, and this prevents the occurrence of mass eigenstates at the electron "mass." Consequently, electrons are not particles and are more properly known as infraparticles. This is troublesome for the LSZ approach but does not, apparently, present difficulties for the Araki-Haag approach. To see this, we consider a simple model in which "electrons" are always accompanied by a soft photon cloud.

We first construct the Hilbert space containing the soft photon clouds. For simplicity, we take the "photon" field

$$A_{ph}(x) = \int \overline{dk} \left[a(k)\, e^{ik\cdot x} + a(k)^{\dagger} e^{-ik\cdot x}\right]$$

to be a free massless real scalar field. Let $f(k) = |\vec{k}|^{-1}\, e^{-\vec{k}^2}$. Given an integer n, we define a generalized coherent state $|n\rangle$ by $a(k)|n\rangle = nf(k)|n\rangle$, and we let $\mathscr{H}_{ph,n}$ denote the Hilbert space generated by applying creation operators to $|n\rangle$. On each $\mathscr{H}_{ph,n}$ we can also define as usual the energy-momentum tensor $T^{\mu\nu}_{ph}$ and the momentum P^{μ}_{ph}. On $\mathscr{H}_{ph} \equiv \bigoplus_{n=-\infty}^{\infty} \mathscr{H}_{ph,n}$ we define the unitary operator U by $U^{-1}a(k)U = a(k) + f(k)$ and $U|n\rangle = |n+1\rangle$. The crucial consequence of $\int \overline{dk}\,|f(k)|^2 = \infty$ which we shall need below is that, for $x \neq 0$,

$$\underset{\lambda\to\infty}{\text{w-lim}}\; U(\lambda x) = 0 \quad , \tag{16}$$

where $U(x) \equiv e^{-ix\cdot P}\, U\, e^{ix\cdot P}$. Equation (16) holds between states from a dense domain which we presume to be large enough to justify the applications below. We do not here present the proof of Eq. (16), but merely note that it hinges on the result

$$\lim_{\lambda\to\infty} e^{-2\int \overline{dk}\,|f(k)|^2 \sin^2\left(\frac{\lambda x\cdot k}{2}\right)} = 0 \quad .$$

This result is easily verified. Furthermore, one can check that the result fails if f(k) is replaced with the less singular and square integrable (with respect to $\overline{dk}$) function $|\vec{k}|^{-1+\varepsilon}\, e^{-\vec{k}^2}$, $\varepsilon > 0$.

We next consider the "leptons". For simplicity we consider a free massive complex scalar field

$$\psi_{el}(x) = \int \overline{dp} \left[b_1(p)\ e^{ip\cdot x} + b_2(p)^\dagger\ e^{-ip\cdot x} \right]$$

defined on the Hilbert space $\mathscr{H}_{el}$. $\mathscr{H}_{el,n}$ denotes the charge n sector of $\mathscr{H}_{el}$. The energy-momentum tensor $T^{\mu\nu}_{el}$, the momentum P^{μ}_{el}, and the electron current J^{μ}_{el} are defined as usual.

On $\mathscr{H}_{el} \otimes \mathscr{H}_{ph}$ we define

$$\begin{aligned} \psi(x) &= \psi_{el}(x) \otimes U(x) \\ A &= 1 \otimes A_{ph} \\ P^{\mu} &= P^{\mu}_{el} \otimes 1 + 1 \otimes P^{\mu}_{ph} \\ T^{\mu\nu} &= T^{\mu\nu}_{el} \otimes 1 + 1 \otimes T^{\mu\nu}_{ph} \\ J^{\nu} &= J^{\nu}_{el} \otimes 1. \end{aligned}$$

Note that these operators all map $\mathscr{H} \equiv \bigoplus_{n=-\infty}^{\infty} \mathscr{H}_{el,n} \otimes \mathscr{H}_{ph,n}$ into itself. (ψ maps $\mathscr{H}_{el,n} \otimes \mathscr{H}_{ph,n}$ into $\mathscr{H}_{el,n+1} \otimes \mathscr{H}_{ph,n+1}$ while the rest map $\mathscr{H}_{el,n} \otimes \mathscr{H}_{ph,n}$ into itself.) These operators acting on $\mathscr{H}$ constitute our model. Modulo technical difficulties, our model satisfies all the Wightman axioms except Lorentz invariance and locality. (In particular, the vacuum is cyclic; that is, the domain D consisting of all linear combinations of arbitrary products of ψ and A fields applied to the vacuum is dense in $\mathscr{H}$.)

The point of all this is that when ψ acts on the vacuum, ψ_{el} creates a lepton while U creates an accompanying cloud of soft photons. Thus, our leptons are infraparticles. We now consider the LSZ and Araki-Haag approaches applied to this situation. [Warning: in what follows, the electron vacuum, the photon vacuum, and their tensor product are all denoted by $|0\rangle$.]

We first consider LSZ scattering theory. From the LSZ approach, we would expect for example to have

$$\text{w-}\lim_{x^o\to\infty}\int d\vec{x}\; e^{ip\cdot x}\psi(x) = \frac{1}{2p^o}\, b_{out}(p)^\dagger \quad . \tag{17}$$

[Eq. (17) is somewhat unconventional, but entirely appropriate, and its use simplifies (but is not crucial for) our discussion.] In our model, the left-hand side of Eq. (17) vanishes. To show this, we first consider the special case of the left-hand side of Eq. (17) sandwiched between $\langle 0|\psi^*(y)$ and $|0\rangle$.

$$\lim_{x^o\to\infty}\langle 0|\psi^*(y)\int d\vec{x}\; e^{ip\cdot x}\;\psi(x)|0\rangle$$

$$= \lim_{x^o\to\infty}\langle 0|\psi^*_{el}(y)\int d\vec{x}\; e^{ip\cdot x}\;\psi_{el}(x)|0\rangle\;\langle 0|U(y)^{-1}U(x)|0\rangle$$

$$= \langle 0|\psi^*_{el}(y)\,\frac{1}{2p^o}\, b_2(p)^\dagger|0\rangle \lim_{x^o\to\infty}\langle 0|U(y)^{-1}U(x)|0\rangle$$

$$= 0 \quad ,$$

where we have used both the analog of Eq. (17) for the field ψ_{el} acting on $\mathscr{H}_{el}$ and Eq. (16). This argument may be easily generalized to show that the left-hand side of Eq. (17) vanishes between any pair of states in D. Thus, the LSZ approach cannot find the electron inside its soft photon cloud.

We now consider Araki-Haag scattering theory. We expect that n_i^{out} will turn out to be $b_i^\dagger b_i \otimes 1$. The analogs of Eq. (1.a) and (1.b) are therefore

$$\text{w-}\lim_{t\to\infty}\int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right)\; T^{\mu o}(t,\vec{x}) = \int \overline{dp}\; h\left(\frac{\vec{p}}{p^o}\right)\; p^\mu[b_1^\dagger b_1 + b_2^\dagger b_2]\otimes 1 \tag{18.a}$$

$$\text{w-}\lim_{t\to\infty}\int d\vec{x}\; h\left(\frac{\vec{x}}{t}\right)\; J^o(t,\vec{x}) = \int \overline{dp}\; h\left(\frac{\vec{p}}{p^o}\right)[b_1^\dagger b_1 - b_2^\dagger b_2]\otimes 1 \tag{18.b}$$

We first verify Eq. (18.b) sandwiched between the states $\langle 0|\psi^*(y)$ and $\psi(z)|0\rangle$. The left-hand side then equals

$$\lim_{t\to\infty} \langle 0|\psi^*_{el}(y) \int d\vec{x}\ h(\tfrac{\vec{x}}{t})\ J^o_{el}(t,\vec{x})\ \psi_{el}(z)|0\rangle\ \langle 0|U(y)^{-1}U(z)|0\rangle$$

$$= \langle 0|\psi_{el}(y) \int \overline{dp}\ h(\tfrac{\vec{p}}{p^o})[b_1^\dagger b_1 - b_2^\dagger b_2]\ \psi_{el}(z)|0\rangle\ \langle 0|U(y)^{-1}U(z)|0\rangle \quad ,$$

where we have used the analog of Eq. (18.b) for the field ψ_{el} acting on $\mathscr{H}_{el}$. And the result clearly equals the right-hand side of Eq. (18.b) sandwiched between $\langle 0|\psi^*(y)$ and $\psi(z)|0\rangle$. This completes the verification. The arguments used may be easily generalized to verify Eq. (18.b) between any pair of states in D.

The verification of Eq. (18.a) between $\langle 0|\psi^*(y)$ and $\psi(z)|0\rangle$ proceeds like the above except for the appearance of the unwanted term

$$\langle 0|\psi^*_{el}(y)\psi_{el}(z)|0\rangle\ \lim_{t\to\infty} \langle 0|U(y)^{-1} \int d\vec{x}\ h(\tfrac{\vec{x}}{t})\ T^{\mu o}_{ph}(t,\vec{x})\ U(z)|0\rangle \quad .$$

Fortunately, this term vanishes due to the support properties of h, as can be shown using a stationary phase argument. In fact, we expect that $\text{w-}\lim_{t\to\infty} \int d\vec{x}\ h(\frac{\vec{x}}{t})\ T^{\mu o}_{ph}(t,\vec{x}) = 0$ holds in general for h supported inside the unit ball, but we do not prove it. If so, the above argument may be generalized to verify Eq. (18.a) between any pair of states in D.

So, Eq. (18) is essentially verified, and we see that Araki-Haag scattering theory is perfectly capable of finding infraparticles.

4. Disadvantages

As noted, the disadvantages of Araki-Haag scattering theory applied to QED are that our conjecture remains unproven and that we have no practical computation scheme. We now briefly discuss these difficulties, considering the latter first.

The difficulties involved in practically applying our scattering theory are threefold.

A. Spin

For obvious reasons, the direct application of Eq. (2) would be very impractical. However, most scattering probabilities computed include a sum over final spins. In that case, only the combinations $n_1^{out} + n_2^{out}$ and $n_3^{out} + n_4^{out}$ are of interest, and these may be obtained directly from Eqs. (1.a) and (1.b). Thus, the observation of spins in the final state represents a considerable complication, and we shall henceforward consider only spin summed scattering probabilities.

B. Representations

Let ϕ denote a generic local observable. If the expectation values $\langle \phi_1 \cdots \phi_n \rangle$ are at our disposal, then spin summed scattering probabilities may be computed using only Eqs. (1) and (4). The problem arises, however, as to how to represent the state $\rangle$.

We have taken our theory of the universe to be the Wightman field theory associated with the local gauge-invariant fields of QED. Every state in our theory may be approximated arbitrarily well by linear combinations of products of observable fields applied to the vacuum. We shall call this the Wightman representation. (Note that our theory includes only states of total charge zero. This is no handicap, although "behind the moon" arguments will be necessary to describe many scattering processes of interest.[9])

The problem with the Wightman representation is that it is not known in practice how to use it. Given an experimentally well defined state (e.g. colliding beams in an accelerator), it is not known how to represent that state in either the Wightman representation or, for that matter, in any other. The most elementary notion of quantum mechanics -- i.e., the representation of physical states as vectors in a Hilbert space -- has yet to be practically implemented in QED, except in special cases. This is the deep problem alluded to in the introduction, and it merits attention. For now, though, we focus our attention on those situations for which we do know how to represent the state.

For field theories with mass gap and a complete set of scattering states there is in fact no problem: scattering theory itself then provides both the in- and the out-representations, and the in-representation may be used to specify the state. In the case of QED we are not so fortunate. We of course have the operators n_i^{in} (defined like the n_i^{out} except with $t \to -\infty$.) But, just as $\vec{P}_{out}(0)$ did not comprise an irreducible set of operators for nonrelativistic Coulomb scattering, neither will the n_i^{in} be irreducible for QED. Consequently, we have no in-representation.[16]

All is not lost, however. In addition to the n_i^{in}, we have the photon creation operators $a_\lambda^{in}(k)^\dagger$ at our disposal. These may be applied to the vacuum to construct states consisting entirely of a finite number of incoming photons. For such states, and for those states whose Wightman representations are somehow known, our scattering theory may be practically applied.

C. Perturbation theory

In those situations to which our scattering theory is practically applicable, it is necessary to manipulate expressions of the form $\langle 0|\phi_1 \ldots \phi_n|0\rangle$, where some of the ϕ's are used to construct the initial state and the rest are used to construct the n_i^{out}. These vacuum expectation values may be obtained from perturbation theory by analytically continuing the Euclidean Green's functions in coordinate space.[13] It would be very convenient to have a set of "Feynman rules" for the result of this analytic continuation, and, indeed, I know of one such prescription.[10] However, when applied to QED, this prescription generates spurious infrared divergences which then cancel when the graphs are summed. Thus, I know of no graphical expansion of (nontime-ordered) vacuum expectation values which is infrared finite graph by graph. That is why this paper contains no worked out examples of Araki-Haag scattering theory applied to QED.

Regarding the second disadvantage -- that our conjecture remains unproven -- we have little to say. A rigorous proof is at present impossible since QED is not yet known to exist. Axiomatic considerations, which would attempt to establish similar results in more general field theories, might or might not be interesting. And attempts to prove the conjecture at the level of perturbation theory are at present hampered by the lack of a reasonable graphical expansion for the nontime-ordered vacuum expectation values.

Nevertheless, the conjecture is quite likely to be true.

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While a student at the 1977 Ettore Majorana International School of Subnuclear Physics, I asked Sidney Coleman for a gauge invariant definition of the S-matrix. He replied that I should tag along after the experimenters for a day or two taking notes of everything they did. The result would be my definition.

At the time I thought his reply rather arch, but it expressed an essential truth: Scattering experiments are performed by making a finite number of observations in a finite volume during a finite time. Clearly, then, scattering theory should in principle be derivable from a theory of local observables...

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16. The preceding remarks are unduly pessimistic. A derivation similar to that leading to Eq. (4) may be used to construct initial state phase space projection operators. In a subsequent paper (with any luck), these operators will be used to construct states with which cross sections may be defined, and the results should be equivalent to the usual Feynman rules. This equivalence, once demonstrated, will to some extent justify both the conjecture of this paper and the usual Feynman rules.