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LATTICE PREDICTIONS FOR THE INTERQUARK POTENTIAL*

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Abstract: The measured values of Wilson loops in Monte Carlo simulations of SU(3) lattice gauge theories are used to predict the $q\bar{q}$ potential. The relationship between Λ_0 and the short distance scale of the $q\bar{q}$ potential is also calculated. The predictions are in agreement with theoretical expectations but indicate that fermions must be incorporated into the lattice calculations before any realistic results relevant to QCD can be derived.

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1. Introduction

The behaviour of Wilson loops on the lattice has been extensively studied in recent Monte Carlo simulations[1] of pure SU(3) lattice gauge theories.[2] In continuum QCD the potential between two static colour sources separated by a distance R is evaluated from the expectation values of Wilson loops, $\mathrm{Tr}\, P \exp\{ig \oint_C A^a_\mu T^a dx^\mu\}$, and is given by

$$V(R) = \lim_{T\to\infty} \left[-\frac{1}{T} \ln\left(\mathrm{Tr}\ \left\langle 0 \middle| P\left[\exp\left\{ ig\oint_C A^a_\mu T^a dx^\mu \right\}\right] \middle| 0 \right\rangle\right)\right] \quad . \tag{1}$$

Here P denotes path ordering and A^a_μ and T^a denote the colour gauge fields and group generators respectively, a being the colour index. Assuming that the weak coupling limit and the continuum limit of the lattice theory are identical, we can use (1) and the measured values of rectangular Wilson loops to obtain a lattice prediction for the $q\bar{q}$ potential in the absence of light quarks.[3]

We expect that at short distances perturbative calculations should be valid, so that (1) is dominated by single gluon exchange. Hence for SU(3) the $q\bar{q}$ potential is given by

$$V\left(\vec{q}^{\,2}\right) \sim -\frac{\frac{4}{3}\alpha_s\left(\vec{q}^{\,2}\right)}{\vec{q}^{\,2}} \quad , \qquad \alpha_s = \frac{g^2}{4\pi} \quad . \tag{2}$$

At long distances, potential models for heavy $q\bar{q}$ systems and string models imply a linearly confining potential $V(R) \sim KR$, where K is the string tension. The string model also predicts[4] that an additional contribution, proportional to 1/R, will be present as a consequence of string dynamics.

We note that lattice gauge theories are known to exhibit confinement naturally in the strong coupling limit. Furthermore, since the string tension is an undetermined parameter which must be chosen in order to set the QCD scale,[5] any calculation of the $q\bar{q}$ potential using lattice results is guaranteed to exhibit the desired behaviour at large R. The areas of interest lie in the short distance region where the lattice potential should exhibit Coulomb-like behaviour and in the comparison with phenomenologically determined potentials.

2. The Static $q\bar{q}$ potential at Short Distances

Incorporating the lowest order behaviour of $\alpha_s(\vec{q}^{\,2})$ into (2) and taking the Fourier transform we obtain[6]

$$V(R) = \frac{8\pi}{33R \ln(\Lambda R)} \quad , \qquad \Lambda R << 1 \quad . \tag{3}$$

Unfortunately, Λ is a scheme dependent quantity and is ambiguous in lowest order. In order to make a useful prediction for short distance behaviour of the lattice $q\bar{q}$ potential, we need to evaluate the perturbative expansion for the energy between two static colour sources to the one loop level, using as an expansion parameter $g_0(a)$, the coupling constant defined in the presence of the lattice cutoff. Since we know how to relate expansions in different renormalization schemes, the evaluation of this quantity in any scheme will suffice, provided all finite terms to a given order in g^2 are retained.

Using a minimal subtraction scheme (MS) in the Feynman gauge ($\alpha = 1$) we find the potential is given by[7)]

$$V_{MS}\left(\bar{q}^2\right) = -\frac{g_{MS}^2\left(\mu^2\right)C_2(R)}{\bar{q}^2}\left[1 + \frac{g_{MS}^2\left(\mu^2\right)C_2(G)}{16\pi^2}\left(\frac{11}{3}\,\ln\frac{\mu^2}{\bar{q}^2} - \frac{11}{3}\,\gamma + \frac{31}{9}\right)\right] \quad , \qquad (4)$$

where $C_2(R)$ and $C_2(G)$ are the usual Casimir factors for SU(N) and γ is Euler's constant.

The relationship between $g_0(a)$ and $g_{MOM}(\mu^2)$, the coupling constant defined in the momentum subtraction scheme, is given by[8)]

$$g_{MOM}^2\left(\mu^2\right) = g_0^2(a)\left[1 + g_0^2(a)\left(\frac{11C_2(G)}{48\pi^2}\,\ln\frac{\pi^2}{a^2\mu^2} + R(N)\right) + \ldots\right] \quad , \qquad (5)$$

where R(N) is a finite number depending on the gauge group SU(N).

Expressing $g_{MS}(\mu^2)$ in terms of $g_{MOM}(\mu^2)$ we obtain[9)]

$$g_{MS}^2\left(\mu^2\right) = g_{MOM}^2\left(\mu^2\right)\left[1 + A\left(\alpha,n_f\right)\frac{g_{MOM}^2\left(\mu^2\right)}{4\pi} + \ldots\right] \quad , \qquad (6)$$

where the $A(\alpha,n_f)$ are calculated[13)] for various values of α and n_f. Using (4), (5) and (6), we obtain

$$V_0\left(\bar{q}^2\right) = -\frac{C_2(R)g_0^2(a)}{\bar{q}^2}\left[1 + \frac{g_0^2(a)C_2(G)}{16\pi^2}\left(\frac{11}{3}\,\ln\frac{\pi^2}{a^2\bar{q}^2} - J\right)\right] \quad , \qquad (7)$$

where

$$J = \frac{11\gamma}{3} - \frac{31}{9} + \frac{4\pi A(1,0)}{C_2(G)} + \frac{16\pi^2 R(N)}{C_2(G)} \quad .$$

We can rewrite (7) by absorbing J into the logarithm and expressing $g_0^2(a)$ in terms of a and the QCD scale, Λ_0.[10)]

If we take the Fourier transform of the resulting expression in the short distance limit and as usual, absorb terms like $(\text{const}/\ln R)^2$ into the definition of Λ, we obtain a prediction for the short distance behaviour of the lattice potential given by

$$V_0(R) = \frac{4\pi C_2(R)}{11R \ln\left(\Lambda_P R\right)^2} , \tag{8}$$

where

$$\Lambda_P = 82.07\ \Lambda_0 \quad \text{for SU(3)} , \quad n_f = 0 ,$$

$$= 56.47\ \Lambda_0 \quad \text{for SU(2)} , \quad n_f = 0 .$$

The determination of Λ_P from the measured values of Wilson loops thus provides a consistency check on lattice calculations.

3. Calculation of the $q\bar{q}$ Potential from the Numerical Data

The Monte Carlo procedure measures the lattice average of a rectangular Wilson loop, $\langle W(I,J)\rangle$, as a function of the coupling constant, with I and J denoting the side lengths of the loop in fundamental lattice units. If we suppose for the moment that J corresponds to the time direction and convert (1) into lattice variables we obtain the potential

$$V(Ia) = \lim_{Ja\to\infty} \left\{\frac{1}{Ja}\ \left[-\ \ln\langle W(I,J)\rangle \right]\right\} . \tag{9}$$

For SU(3), the total lattice size is only 6^4 and the values of I and J are restricted to lie in the range, $1 \le I,J \le 3$. Finite size effects could thus be important and it may be necessary to include terms in (9) for finite J, that drop out in the limit $J\to\infty$. For example, correction terms involving powers of the cutoff and an important perimeter dependent contribution due to the self-energy of the static sources must be considered.

Two simple parametrizations of $\langle W(I,J)\rangle$ which include these additional contributions come to mind: (i) $\langle W(I,J)\rangle$ asymmetric in I,J and (ii) $\langle W(I,J)\rangle$ symmetric in I,J. Consider first the asymmetric case given by

$$-\ln\langle W(I,J)\rangle = V(Ia)\cdot Ja + P_I(a)\cdot Ia + P_J(a)\cdot J(a) + C(a) , \quad I \le J , \tag{10}$$

where $P_I(a)$, $P_J(a)$ and $C(a)$ are a dependent constants to be determined. The condition $I \le J$ is imposed with the idea that an $I\times J$ Wilson loop approximates (9) if J is chosen to be the time direction. Since there is nothing to distinguish the time direction, the Monte Carlo data for SU(3) is actually symmetric in I and J. In (10) we note that the self-energy contribution $P_J(a)$ can never be separated from V(Ia). This means that the potential is determined only up to an arbitrary a dependent constant.

For the values of I and J that are available, (10) generates an overdetermined system of equations. The extra equation is used to obtain an additional estimate of the value of one of the potentials which, in the absence of finite size effects and statistical fluctuations, would be equal to the other estimates.

The magnitude of finite size effects can also be estimated from (10) by observing that the perimeter dependent and constant terms can be eliminated to give

$$\left[V(Ia) + P_J(a)\right] = \frac{1}{a}\left\{-\ell n\langle W(I,J+1)\rangle + \ell n\langle W(I,J)\rangle\right\} \quad . \tag{11}$$

The larger value of J, the better we expect the estimate (11) of V(Ia) to be. For example, for $I,J \leq 3$, $[V(a) + P_J(a)]$ can be chosen to be

$$\frac{1}{a}\left\{-\ell n\langle W(1,3)\rangle + \ell n\langle W(1,2)\rangle\right\} \quad , \tag{12}$$

or

$$\frac{1}{a}\left\{-\ell n\langle W(1,2)\rangle + \ell n\langle W(1,1)\rangle\right\} \quad . \tag{13}$$

We expect (12) to give a better estimate of V(a) than (13). The spread in the values obtained using (12) and (13) also gives an indication of the errors involved in using small sized Wilson loops.

Next we consider the case of W(I,J) symmetric in I,J, which is given by

$$-\ell n\langle W(I,J)\rangle = \tfrac{1}{2}\left\{\left[V(Ia)+P(a)\right]Ja + \left[V(Ja)+P(a)\right]Ia + C(a)\right\} \quad . \tag{14}$$

Here we have adopted the idea that the lattice results average each of the possible interpretations of the time direction for an $I \times J$ loop. Again, for $I,J \leq 3$, the system is overdetermined. As before, we can estimate finite size effects by choosing some subset of the equations generated by (14) to determine our potentials. The two possibilities chosen for illustration are:

(a) $\{I,J\} = \{1,2\}$

(b) $\{I,J\} = \{1,3\}$.

Both sets of SU(3) data analysed here are measured with $g_0^2 < 1$. Hence the distance between the sources may be calculated using the known weak coupling behaviour.[11] It turns out that the distance scales differ by a factor of two, so that the unknown normalization due to the self-energy contribution can be removed by choosing $V(a_1) = V(2a_2)$, with $a_1 = 2a_2$.

4. Discussion

The results of the calculations described in Section 3 using parametrizations (10) and (14) are shown in Figures 1 and 2, respectively. Both figures

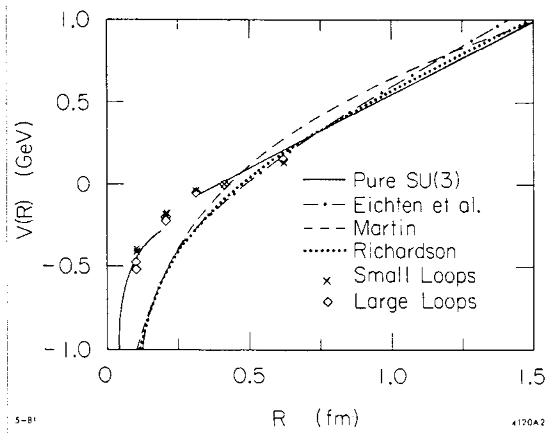

Fig. 1. The evaluation of the $q\bar{q}$ potential from lattice measurements of Wilson loops using parametrization (10). The calculated points are compared with various phenomenological potentials and with the prediction (15) for the $q\bar{q}$ potential in the absence of light quarks. The points labelled "large loops" and "small loops" correspond to choosing V(a) given by (12) and (13), respectively.

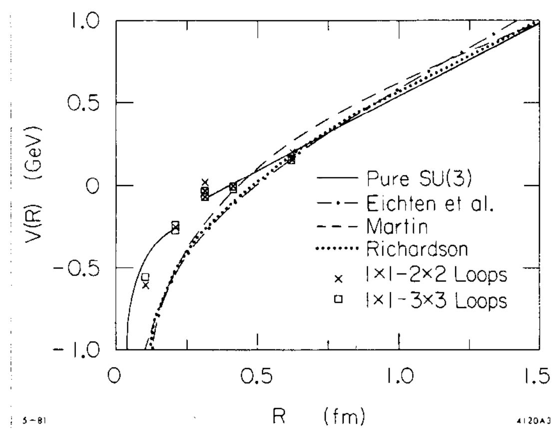

Fig. 2. As for Figure 1, but using parametrization (14) to evaluate the potential. The points labelled "1 × 1 - 2 × 2 loops" and "1 × 1 - 3 × 3 loops" correspond to alternatives (a) and (b) respectively, as discussed in Section 3.

include for comparison several phenomenological potentials determined from heavy $q\bar{q}$ bound state spectra.[6,12] The solid lines in each figure are plots of

$$V(R) = \frac{16\pi}{33R\,\ell n\left(\Lambda_P R\right)^2} \quad , \quad R < 0.2\ \mathrm{fm} \quad ,$$

and

$$V(R) = KR \quad , \quad R > 0.3\ \mathrm{fm} \quad . \tag{15}$$

These curves exhibit the expected long and short distance behaviour of the lattice $q\bar{q}$ potential. We see that in both cases the data points give a reasonable description of the short distance behaviour calculated in Section 2. It is also reassuring to note that the results are not very dependent on the detailed method of extraction of V(R) since there is reasonable agreement between the results for different methods of estimating the potential. In Figure 1, the values determined from (12) are in marginally better agreement with the prediction (15), but more data would be required to see a definite trend.

Presumably due to the absence of light fermions in the lattice analysis, the lattice potential is in poor agreement with the phenomenological potentials and consequently does not give a good fit to $c\bar{c}$ and $b\bar{b}$ spectra. These results thus indicate the need to include fermions in the lattice analysis before we can determine any realistic values for quantities relevant to QCD.

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References and Footnotes

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10. Recall that Λ_0 has been determined in Monte Carlo simulations of SU(2) and SU(3) lattice gauge theories. For SU(3) the value is given by $(5 \pm 1.5) \times 10^{-3}\sqrt{K}$.
11. These values are subject to a 30% uncertainty due to the uncertainty in Λ_0. There is also a systematic error in Λ_0 arising from using the value of K determined in the presence of light quarks instead of that for pure QCD.
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13. Note that in the MS scheme used in Ref. 7, $\ell n\, 4\pi$ terms are also subtracted. The values calculated for $\Lambda_{MOM}/\Lambda_{MS}$ in Ref. 9 must therefore be divided by $\sqrt{4\pi}$ to obtain the correct ratios for the MS scheme of Ref. 7. In particular for $\alpha = 1$ and $n_f = 0$, $\Lambda_{MOM}/\Lambda_{MS} = 2.170$.