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EVOLUTION AND COVARIANCE IN CONSTRAINT DYNAMICS*

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ABSTRACT

Two problems of first class relativistic constraint dynamics of direct interaction are considered. The first is the relation of the gauge motion to the physical motion. In a manifestly covariant formulation they are shown to be exactly equivalent, the latter duplicating the former. The physical motion arises as a consequence of the explicit dependence on an (invariant) time parameter introduced by the gauge fixations (specification of "equal time" surfaces). No simple relation exists in general between the evolution generator and the translation generators. The second problem deals with the covariance of the physical world lines. It is satisfied trivially for Lorentz covariance (since the formalism is covariant) and nontrivially for translations in the equal time surface orthogonal to the total momentum. Necessary conditions on the fixations are given to ensure this.

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I. BACKGROUND

The attempts to formulate relativistic particle dynamics with interactions that are not necessarily field mediated go back half a century. But the desirability of using more than the minimum number of variables together with suitable constraints is of much more recent vintage. Constraint formulations in this context were suggested independently and to various degrees of generality and explicitness by at least four different authors.[1-4] The subsequent development resulted in a large number of papers which have recently focussed on the problem of evolution of the dynamical system and on the covariance of its world lines.[5-9]

Of special interest is the constraint dynamics with first class constraints because these permit a quantization of this classical system.[10] The elimination of first class constraints requires "fixations" which are time dependent constraints. Such constraints have recently been studied in their own right.[11]

The problem of evolution deals with two questions. One is the fixation dependence of the dynamics and the other is the relation of the gauge dynamics (generated by the first class constraints within an equivalence class of points in phase space) and the physical dynamics. The two problems are related.

The problem of covariance is raised by the requirement of relativity which demands that different inertial observers see the same world lines. This condition is not trivial to satisfy and the so-called "world-line condition" has played an important role ever since the no-go theorem of

1965.[12]

In the present paper we shall study the problems of evolution and covariance from a slightly different point of view than has been done recently.[5,6,8,9] While our approach is in many respects equivalent to what was done by these authors and our results overlap, the following differences should be noted.

The present work deals with N-particle systems ($N \geq 2$) and is throughout manifestly Lorentz covariant. In view of the results by J. N. Goldberg[7] this covariance ensures Lorentz covariance in both the phase space and configuration space (Minkowski space). As we shall see, even this formulation leads to nontrivial problems of the type mentioned above. Furthermore, use will be made of the Bergmann-Komar reduced variables (star variables).[13] These are a much more powerful tool than Dirac brackets only. And finally we permit very general fixations which we cast into standard forms that are found to be convenient.

After a brief review of those aspects of first class constraint dynamics that are of interest here (Sec. II) fixations are introduced and standardized (Sec. III). Evolution and covariance are then discussed in Secs. IV and V, respectively. The last section summarizes our conclusions.

II. THE QUOTIENT SPACE

Classical relativistic dynamics in Hamiltonian form is based on an 8N dimensional symplectic space Γ for N spinless particles of masses $m_a > 0$ ($a = 1,\dots,N$). It is spanned by the canonical variables q_a, p_a

which are fourvectors in 3 + 1 dimensional Minkowski space with metric η (trace + 2). They satisfy the covariant Poisson bracket algebra

$$\left\{q_a^{\mu}\ ,\ p_{b\nu}\right\} = \delta_{ab}\delta^{\mu}_{\nu} \quad . \tag{1}$$

The Poincaré generators are determined by

$$P_{\mu} = \sum_a p_{a\mu}, \qquad M^{\mu}{}_{\nu} = \sum_a \left(q_a \wedge p_a\right)^{\mu}{}_{\nu} \tag{2}$$

so that Eq. (1) is a realization of the Poincaré algebra via (2). The N constraints

$$K_a \approx 0 \qquad (a = 1,\ldots,N) \tag{3}$$

(the symbol $\approx$ indicating weak equality, valid in a subspace of Γ only) are assumed to be first class,

$$\left\{K_a, K_b\right\} \approx 0 \qquad \forall\ a,b \tag{4}$$

since this is the interesting case for the purpose of quantization. The variables K_a are the interacting mass shells

$$K_a = p_a^2 + m_a^2 + \phi_a(\gamma) \tag{5}$$

with $\gamma \equiv (q_1,\ldots q_N, p_1,\ldots p_N) \in \Gamma$ and ϕ_a a reasonably well behaved interaction function consistent with Eq. (4).

The constraints (3) restrict $\Gamma(8N)$ to a constraint hypersurface of 7N dimensions, $\mathcal{M}(7N)$, which is the system mass shell hypersurface.

Within $\mathcal{M}$ each K_a generates a trajectory for every point $\gamma \in \mathcal{M}$, parametrized by λ_a, such that

$$\frac{d\gamma}{d\lambda_a} = \left\{\gamma, K_a\right\} \tag{6}$$

and these trajectories remain in $\mathcal{M}$ for all λ_a because the trajectory generators are first class, (4). The N trajectories from any point $\gamma \in \mathcal{M}$ span an N dimensional surface $\Sigma(N)$. $\mathcal{M}(7N)$ is thus foliated by these $\Sigma(N)$ and one can define a quotient space $\Phi(6N)$ by

$$\Phi(6N) = \mathcal{M}(7N)/\Sigma(N) \quad . \tag{7}$$

This quotient space is the physical phase space of 6N dimensions. The above construction ensures that the reduced 6N dimensional phase space $\Phi(6N)$ carries a nondegenerate symplectic structure, i.e., there exist 6N canonical variables on it by Darboux's theorem.

The physical motion takes place entirely in Φ and obeys the constraints (3) which satisfy (4) and which now hold strongly on Φ. All points on a given Σ then are physically equivalent and correspond to a single point on Φ. The surfaces Σ are thus equivalence classes. The "motions" (6) in Σ are gauge motions of no physical significance.

How, then, does one determine the physical motion of the system when all the physical information about the interaction between the particles is buried in the constraint functions K_a, (5)?

III. THE FIXATIONS

The theory developed so far is still lacking an essential ingredient. It is necessary to specify the observer, i.e., a parametrization by which one can describe the evolution of the system. In Minkowski space Hamiltonian dynamics requires the specification of three-dimensional (usually spacelike) hypersurfaces parametrized by some parameter τ, say, so that the infinitesimal evolution takes the system from the surface τ to the surface $\tau + \delta\tau$. A corresponding specification of hypersurfaces in phase space has not yet been made. In Komar's terminology[14] the syntactic aspect of the theory has been stated but the semantic one is still missing.

If the intended meaning of the q_a is a position fourvector for particle _a_, then q_a^o is its coordinate time. The hypersurfaces to be specified are therefore expected on physical grounds to relate the q_a^o to τ.

In a non-covariant way the simplest such specification would thus be

$$\chi_a \equiv q_a^o - t \approx 0 \quad . \tag{8}$$

In this case $\tau = t$ is not a Lorentz invariant parameter. Since the physical (intended) meaning of the generators P_μ, (2), of space-time translations is that of total momentum and energy, a Lorentz covariant statement instead of (8) would be

$$\chi_a \equiv - q_a \cdot \hat{P} - \tau \approx 0 \tag{9}$$

where the unit vector $\hat{P}$ is defined as $P/\sqrt{-P^2}$. A still more sophisticated choice would be the one advocated in Ref. 15 which has the form (at least for N = 2),

$$\chi_a \equiv - q_a \cdot \hat{P} + p_a \cdot \hat{P}\tau \approx 0 \quad . \tag{10}$$

All these are examples of "fixations", constraints that specify the generalized "equal time surface". In general, they are of the form

$$\chi_a(\gamma , \tau) \approx 0 \qquad (a = 1,\ldots,N) \tag{11}$$

but since they are to be monotonic functions of τ (on physical grounds) it is reasonable to assume that (11) can be solved with respect to τ and we shall restrict ourselves to that case. The fixations can then be brought to the "first standard form",

$$\chi_a \equiv \sigma_a(\gamma) - \tau \approx 0 \qquad (a = 1,\ldots,N) \tag{12}$$

and the examples (8) and (9) are of this form. Equation (10) can obviously also be put into the form (12). A "second standard form" is obtained by retaining χ_N but replacing the other χ_a by $\chi_a - \chi_N$,

$$\begin{aligned} \bar{\chi}_a &\equiv \chi_a - \chi_N = \sigma_a(\gamma) - \sigma_N(\gamma) \approx 0 \qquad (a = 1,\ldots,N-1) \\ \bar{\chi}_N &= \chi_N = \sigma_N(\gamma) - \tau \approx 0 \quad . \end{aligned} \tag{13}$$

This second standard form has the advantage that τ occurs in only one of the N fixations.

Geometrically, the N fixations specify an N dimensional surface.

When used simultaneously with the constraints (3) they form a set of 2N second class constraints. We require that the matrix

$$\Delta = \left(\left\{\chi_\alpha,\ K_b\right\}\right) \quad (\alpha,\ b = 1,\ldots,N) \tag{14}$$

is non-singular,

$$\det \Delta \neq 0 \quad . \tag{15}$$

The phase space $\Gamma(8N)$ is thus reduced by the complete set of 2N second class constraints to a nondegenerate phase space $\Phi^*(6N)$ of 6N dimensions. This space is diffeomorphic to the quotient space Φ for every fixed τ, but gives in general a different diffeomorphism for each value of τ.

In terms of the geometrical picture of a foliated space $\mathscr{M}(7N)$ which we had before the introduction of fixations, the N-dimensional fixation surface (for a fixed τ) can be considered as restricting each surface $\Sigma(N)$ to a single point, which can be characterized by $q^*_a(\tau)$ and $p^*_a(\tau)$, $\gamma^*(\tau) \equiv (q^*_1,\ldots q^*_N,\ p^*_1,\ldots p^*_N)$. The τ is necessary because different τ give different fixation surfaces.

The point $\gamma^* \epsilon\ \Phi^*$ can be constructed by a method due to Bergmann and Komar[13] in which the τ dependence becomes explicit in γ^*. This construction is simply the reduction of the 8N component γ to that particular 6N component γ^* which is compatible with the 2N constraints. For fixed τ

$$\gamma^* = \gamma - \sum_{m=1}^{2N} \sum_{n=1}^{2N} \left\{\gamma, C_m\right\} (D^{-1})_{mn} C_n \tag{16}$$

where

$$C_m = \chi_m \quad (m = 1,\ldots,N)$$

$$C_m = K_m \quad (m = N+1,\ldots,2N) \tag{17}$$

and the 2N × 2N dimensional matrix D consists of four N × N submatrices

$$D = \begin{pmatrix} \theta & \Delta \\ -\tilde{\Delta} & 0 \end{pmatrix} = \left(\{C_m, C_n\}\right) \tag{18}$$

$$\theta = \left(\{\chi_\alpha, \chi_\beta\}\right) \quad (\alpha,\beta = 1,\ldots,N) \tag{19}$$

and Δ was defined earlier in (14). In (18) 0 indicates the N × N zero matrix and is there because of (4). The inverse matrix D^{-1} which occurs in (16) follows from (18) to be [it exists in view of (15)]

$$D^{-1} = \begin{pmatrix} 0 & -\tilde{\Delta}^{-1} \\ \Delta^{-1} & \Delta^{-1}\,\theta\,\tilde{\Delta}^{-1} \end{pmatrix} \tag{20}$$

so that (16) can be written more explicitly as

$$\begin{aligned} \gamma^* = \gamma &- \left\{\gamma, \chi_\alpha\right\} \left(-\tilde{\Delta}^{-1}\right)_{\alpha a} K_a \\ &- \left\{\gamma, K_a\right\} \left[\left(\Delta^{-1}\right)_{a\alpha}\chi_\alpha + \left(\Delta^{-1}\,\theta\,\tilde{\Delta}^{-1}\right)_{ab} K_b\right] \quad . \end{aligned} \tag{21}$$

Here and in the following we use the summation convention for simplicity, summing all repeated indices from 1 to N.

The reduced variables γ^* are now functions of τ via χ_α in the third term on the right. The τ dependence disappears in D if we assume one of the standard fixation forms (12) or (13). But for any choice of fixations the γ^* are thus uniquely determined functions of τ.

The reduction of γ to γ^* via (16) ensures that

$$\left\{\gamma^*, \chi_\alpha\right\} = 0, \quad \left\{\gamma^*, K_a\right\} = 0 \quad \forall\, \alpha, a \tag{22}$$

and thus identifies the γ^* as physical variables.

IV. THE EVOLUTION

Since the fixations are to be preserved throughout the motion, the evolution operator should not change them. Now in first class constraint theory the evolution operator is not known: any combination of the first class constraints

$$H = \omega_a K_a \tag{23}$$

is acceptable. The ω_a can be almost arbitrary functions of the q and p but will be assumed positive.[16] The requirement of conservation of fixations,

$$\frac{d\chi_\alpha}{d\tau} = \frac{\partial\chi_\alpha}{\partial\tau} + \left\{\chi_\alpha, H\right\} = 0$$

now fixes the ω_a. From (14) and

$$\left\{\chi_\alpha, K_a\right\} \omega_a \approx - \frac{\partial\chi_\alpha}{\partial\tau} \tag{24}$$

follows ω_a uniquely on Φ^*,

$$\omega_a = -\left(\Delta^{-1}\right)_{a\alpha} \frac{\partial \chi_\alpha}{\partial \tau} \quad . \tag{25}$$

If we adopt the first standard form, (12), this expression simplifies to

$$\omega_a = \sum_\alpha \left(\Delta^{-1}\right)_{a\alpha} \tag{26a}$$

and if we adopt the second standard form, (13), of the fixations

$$\omega_a = \left(\Delta^{-1}\right)_{aN} \quad . \tag{26b}$$

Thus, H is uniquely determined by the fixations,

$$H = K_a \left(\Delta^{-1}\right)_{aN} \quad . \tag{27}$$

But this H does not seem to provide for the evolution of the physical variables γ^*. These commute (weakly) with H in view of (22). One does observe though that the original variables γ now describe a unique trajectory on Σ:

$$\frac{d\gamma}{d\lambda} = \left\{\gamma \, , H\right\} \approx \left\{\gamma, \, K_a\right\} \left(\Delta^{-1}\right)_{aN} \quad . \tag{28}$$

The choice of fixations selects exactly one particular trajectory on the N-dimensional surface Σ. This is, however, a gauge dynamics since it relates equivalent points (points on Σ) to one another, all of which referring to the same point on the quotient space. This is not the evolution of the physical variables. Where then is the evolution of the physical variables?

The answer to this puzzle lies in the τ dependence of the γ^* introduced by the reduction process (21) through the fixations:

$$\frac{d\gamma^*}{d\tau} = \frac{\partial\gamma^*}{\partial\tau} + \left\{\gamma^*, H\right\} = \frac{\partial\gamma^*}{\partial\tau} \quad . \tag{29}$$

Let us for convenience again adopt the second standard form of the fixations so that

$$\frac{\partial\chi_\alpha}{\partial\tau} = -\delta_{\alpha N} \quad . \tag{30}$$

Then (21) inserted into (29) gives, since the only explicit τ dependence in (21) occurs in the χ_α that are not in P.B.'s,

$$\frac{d\gamma^*}{d\tau} = -\left\{\gamma, K_a\right\}\left(\Delta^{-1}\right)_{a\alpha}\left(-\delta_{\alpha N}\right) = \left\{\gamma, K_a\right\}\omega_a \quad . \tag{31}$$

Thus one obtains from (28) the fundamental result

$$\frac{d\gamma^*}{d\tau} = \left\{\gamma, K_a\right\}\omega_a \approx \left\{\gamma, H\right\} = \frac{d\gamma}{d\lambda} \quad . \tag{32}$$

One is led to the important conclusion that <u>the evolution of the physical variables is exactly realized by the gauge dynamics</u>. We can thus identify τ and λ, and we have the following picture.

The τ-dependent realizations Φ^* of the quotient space Φ associate with each point of the trajectory of γ on Σ (generated by H) a different set of physical variables γ^*, each set being labelled by a different value of τ. There is (at least locally) a one-to-one map of the trajectory $\gamma(\lambda)$ to the trajectory $\gamma^*(\tau)$, the latter being generated by the reduction process for 2N second class constraints. This bijection is the fundamental mechanism that permits one to treat the gauge motion as if it were the physical motion. It thus resolves the puzzle created by papers in the literature that treat the gauge motion as physical motion without clear justification.

V. COVARIANCE

The physical trajectory is a one-dimensional object in a $6N+1$ dimensional space, the direct product space of $\Phi^* = \Phi^*(\gamma^*)$ and τ. A projection into a $3N+1$ dimensional Minkowski space yields one space-time trajectory (world line) for the N particles. When M_{3N+1} is mapped into M_{3+1} N trajectories result and the conventional space-time description emerges: one obtains N world lines in M_{3+1}.

It is now necessary to show that the first class constraint dynamics just described is consistent with Poincaré transformations according to the principle of relativity.

The generator of infinitesimal Poincaré transformations (Λ, a) is

$$G = L + T$$

$$L = \frac{1}{2}\,\omega_{\mu\nu}M^{\mu\nu} \qquad T = a^{\mu}P_{\mu} \quad . \tag{33}$$

Now all our constraints are manifestly Lorentz invariant, if we exclude the type exemplified by (8) which has been discussed before,[8]

$$\{\chi_\alpha, L\} = 0 \qquad \{K_a, L\} = 0 \quad . \tag{34}$$

In particular, τ is a Lorentz invariant parameter. Thus, the γ^* are Lorentz covariant. But translation invariance is not so easily dealt with.

The mass shell constraints are chosen to be translation invariant,

$$\left\{K_a, T\right\} = 0 \tag{35}$$

because one is interested only in translation invariant interactions ϕ_a; they can depend only on the position <u>differences</u> $q_b - q_c$. But the fixations are by nature <u>not</u> translation invariant because at least the variables q_a^o are involved individually rather than as differences. Otherwise "equal τ surfaces" could not be specified. Another way of saying this is the following: for spatial distances the interparticle separations are sufficient and no distinguished point needs to be specified; but for the evolution of the system all points in the three-space must agree on the same time $\tau = 0$, say. The examples (8) - (10) show this lack of T-invariance explicitly,

$$\left\{\chi_\alpha, T\right\} \neq 0 \quad . \tag{36}$$

The second standard form (13) does, however, have the advantage that at least in some cases [example (9) but not (10)] N-1 of the N χ_α <u>are</u> T-invariant and only χ_N is not.

What effect does this have on γ^*? One finds easily from (21)

$$\left\{\gamma^*, P_\mu\right\} \approx \left\{\gamma, P_\mu\right\} - \left\{\gamma, K_a\right\}\left(\Delta^{-1}\right)_{a\alpha}\left\{\chi_\alpha, P_\mu\right\} \tag{37}$$

because all other terms have vanishing P.B. with P_μ.

Now from the general theory of reduced variables one knows that

$$\{A^*, B\} \approx \{A, B^*\} \approx \{A^*, B^*\} \approx \{A, B\}^*$$

where the last bracket is the Dirac bracket. Therefore

$$\begin{aligned} \left\{\gamma, P_\mu^*\right\} &\approx \left\{\gamma, P_\mu\right\}^* \\ &= \left\{\gamma, P_\mu\right\} - \left\{\gamma, K_a\right\} \left(\Delta^{-1}\right)_{a\alpha} \left\{\chi_\alpha, P_\mu\right\} \qquad (38) \\ &= \left\{\gamma, P_\mu - K_a \left(\Delta^{-1}\right)_{a\alpha} \left\{\chi_\alpha, P_\mu\right\}\right\} \quad . \end{aligned}$$

By the same reduction process as used in (16) and (21)

$$P_\mu^* = P_\mu - K_a \left(\Delta^{-1}\right)_{a\alpha} \left\{\chi_\alpha, P_\mu\right\} \qquad (39)$$

and we see that (38) is in fact a strong equality. Thus, the lack of T-invariance of the fixations is exactly taken into account when one makes translations with the reduced generators P^*.

The reduced translation generators P^* can be simply related to H <u>if</u> the χ_α satisfy the condition

$$\left\{\chi_\alpha, P_\mu\right\} = C_\mu \frac{\partial \chi_\alpha}{\partial \tau} \quad . \qquad (40)$$

In that case, using (25) in (39)

$$P_\mu^* = P_\mu - C_\mu K_a \left(\Delta^{-1}\right)_{a\alpha} \frac{\partial \chi_\alpha}{\partial \tau} = P_\mu + C_\mu H \quad . \qquad (41)$$

This is a generalization of a result first obtained by Bergmann and Komar.[8] Since the assumption (40) is indeed satisfied in some cases [e.g., by the popular fixations (9)] it is desirable to consider its consequences. One now finds a relationship between the translations (generated by the P_μ) and H. From the fundamental relations (32) and (41) follows on Φ^*

$$\frac{d\gamma^*}{d\tau} = \frac{d\gamma}{d\lambda} = \left\{\gamma , H\right\} = \frac{C^\mu}{C^2}\left\{\gamma, P^*_\mu - P_\mu\right\} \quad . \tag{42}$$

In conventional Hamiltonian dynamics one identifies the Hamiltonian with P^o, the total energy of the system. But in those cases P^o is a given function of the $\vec{q}$ and $\vec{p}$, i.e., $P^o = P^o(\vec{\gamma})$. This is not the case here where $P^o = \sum_a p^o_a$ and the p^o_a are independent of the other variables; here

$$\left\{A, P^o\right\} = \sum_a \frac{\partial A}{\partial q^o_a} \tag{43}$$

which by itself implies no dynamics. The relation (42) is therefore of interest. If one uses the fixations (9)

$$C^\mu = \hat{P}^\mu, \qquad P^*_\mu = P_\mu\left(1 + \frac{H}{\sqrt{-P^2}}\right) \tag{44}$$

and

$$\frac{d\gamma^*}{d\tau} = \hat{P}^\mu\left\{\gamma, P_\mu - P^*_\mu\right\} \quad . \tag{45}$$

In general. however, (40) is not satisfied as can be seen by the example (10). Then <u>there is no simple relation between the dynamics of the physical variables $d\overset{*}{\gamma}/d\tau$ and their space-time translations.</u>

Now it is clear from (39) that the reduced generators $\overset{*}{P}_\mu$ and $\overset{*}{M}_{\mu\nu} = M_{\mu\nu}$ satisfy the Poincaré algebra on $\overset{*}{\Phi}$. Thus, the physical variables must have the appropriate transformation properties with respect to them. For $\overset{*}{M}_{\mu\nu}$ this is trivial as indicated above. For $\overset{*}{P}_\mu$ we have from (37)

$$\left\{ \overset{*}{q_a^\lambda} , \overset{*}{P}_\mu \right\} \approx \left\{ q_a^\lambda , P_\mu \right\} - \left\{ q_a^\lambda , K_b \right\} \left(\Delta^{-1} \right)_{b\alpha} \left\{ \chi_\alpha , P_\mu \right\} \quad . \tag{46}$$

At this point one must recall that each $\overset{*}{q}_a$ and each $\overset{*}{p}_a$ although a fourvector, involves only three independent variables, the space they span involving only 6N dimensions. This results from the fact that the constraint equations $K_a \approx 0$ and $\chi_a \approx 0$ are strong equations in $\overset{*}{\Phi}$. One can thus use them to eliminate the $q_a^{\parallel} \equiv -q_a \cdot \hat{P}$ and the $p_a^{\parallel} \equiv -p_a \cdot \hat{P}$ in favor of the $q_a^{\perp}$ and the $p_a^{\perp}$. Thus the $\overset{*}{q}_a$ and $\overset{*}{p}_a$ can be expressed entirely in terms of the $q_a^{\perp}$ and the $p_a^{\perp}$ and $\hat{P}$.

One can therefore require correct transformation properties under translations only in the hyperplane orthogonal to P^μ. Specifically, the $\overset{*}{q}_a$ must transform <u>co</u>variantly, the $\overset{*}{p}_a$ <u>in</u>variantly under such translations,

$$\left\{ q_a^{*\lambda} , P_\mu \right\} P_\perp^{\mu\nu} = P_\perp^{\lambda\nu} \tag{47}$$

$$\left\{ p_a^{*\lambda} , P_\mu \right\} P_\perp^{\mu\nu} = 0 \tag{48}$$

where the projection orthogonal to P^{μ} is

$$P_{\perp}^{\mu\nu} = \eta^{\mu\nu} + \hat{P}^{\mu}\hat{P}^{\nu} \quad . \tag{49}$$

The relations (47) and (48) state that in the center of mass frame ($\vec{P} = 0$) the threevectors $\vec{q}_a^{\,*}$ and $\vec{p}_a^{\,*}$ transform under translations by $\vec{c}$, respectively, as

$$\vec{q}_a^{\,*} \to \vec{q}_a^{\,*} + \vec{c} \quad \text{and} \quad \vec{p}_a^{\,*} \to \vec{p}_a^{\,*} \quad . \tag{50}$$

No statement is made about the q_a^{o*} and p_a^{o*} since their transformation properties are fixed in terms of those of the $\vec{q}_a^{\,*}$ and $\vec{p}_a^{\,*}$ by the constraints.

In order to satisfy the requirements (47) and (48) it is both necessary and sufficient that the fixations are restricted by

$$\left\{ \chi_\alpha , P_\mu \right\} = \theta_\alpha \hat{P}_\mu \tag{51}$$

a condition which is indeed satisfied by (9) as well as (10). This requirement ensures that P_μ^* points into the same direction as P_μ, as can be seen from (39).

This completes the proof that the physical variables transform correctly under Poincaré transformations which exclude translations in the direction of P^{μ}.

The world line conditions specified in some papers are therefore identically satisfied and need not be imposed here as an additional restriction.

VI. CONCLUSIONS

The manifestly covariant formulation of relativistic constraint dynamics presented in this paper for first class constraints exhibits the following results.

The gauge dynamics within a leaf $\bar{\Sigma}(N)$ of the foliation of the mass shell constraint hypersurface $\mathcal{M}(7N)$ is shown to be weakly equal to the physical dynamics of the reduced variables as an explicit function of the invariant time parameter τ introduced by the fixations. This is expressed by Eq. (32). No translations are involved in establishing this relationship.

The fixations are not arbitrary. They must not only be Lorentz covariant but must also satisfy (52) the condition that the P^*_μ and P_μ are parallel. In addition, they must yield ω_a that are translation invariant and that are consistent with a cluster decomposition.[15] The fixations (10) satisfy both of these requirements at least for $N=2$. The fixations (9) satisfy <u>neither</u>. Indeed, in general they do not even give translation invariant ω_a (necessary for translation invariance of $d\gamma^*/d\tau$): $\{\chi_a, K_b\}$ involves $\{\chi_a, \phi_b\}$ which contains a term

$$\frac{\partial(q_a \cdot \hat{P})}{\partial p_c} \cdot \frac{\partial \phi_b}{\partial q_c} = \frac{q_a^\perp}{\sqrt{-P^2}} \cdot \frac{\partial \phi_b}{\partial q_c}$$

which is not translation invariant unless the ϕ_b depend on momenta only. Thus, Δ is not so invariant and consequently neither are the ω_a. The fixations (9) are good only for position independent interactions.

There is in general no simple relation between the evolution generator H and the translation generators P_μ. A relation like (41) is spurious because it is based on the assumption (40) that is not satisfied in general. (The commonly used fixations (9) which do satisfy it are not admissible except for position independent interactions.)

Finally, full Poincaré covariance for $L + T(\mathbb{R}^3)$ is demonstrated for the q_a^* and L-covariance and $T(\mathbb{R}^3)$ invariance for the p_a^*. Here $\mathbb{R}^3$ is the three-dimensional hypersurface in M_{3+1} orthogonal to P_μ. Covariance of the world lines is thus ensured.

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REFERENCES

1. R. Arens, Nuovo Cimento 21B, 395 (1974).
2. P. Droz-Vincent, Rep. Math. Phys. 8, 79 (1975).
3. I. T. Todorov, Comm. JINR E2-10125, Dubna (1976).
4. A. Komar, Phys. Rev. D18, 1881, 1887 (1978).
5. A. Kihlberg, R. Marnelius, and N. Mukunda, "Relativistic potential models as systems with constraints and their interpretation", University of Göteborg preprint, April 1980.
6. N. Mukunda and E. C. G. Sudarshan, "Forms of relativistic dynamics with world-lines", University of Texas (Austin) preprint, Aug. 1980.
7. J. N. Goldberg, "Relativistically interacting particles and world-lines", Syracuse University preprint, October 1980.
9. L. Lusanna, "Gauge-fixings, evolution generators and world-line conditions in relativistic classical mechanics with constraints", University of Geneva preprint, April 1981.
10. L. P. Horwitz and F. Rohrlich, "Constraint relativistic quantum dynamics", Syracuse University preprint, March 1981.
11. N. Mukunda, Phys. Scripta 21, 801 (1980).
12. D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. 35, 350 (1965); H. Leutwyler, Nuovo Cimento 37, 556 (1965).
13. P. G. Bergmann and A. Komar, Phys. Rev. Lett. 4, 432 (1960).
14. A. Komar, Phys. Rev. D18, 3617 (1978).
15. F. Rohrlich, Phys. Rev. D23, 1305 (1981).

16. For a given choice of the ω_a and a trajectory parameter λ one has

$$\frac{d\gamma}{d\lambda} \approx \omega_a \left\{ \gamma , K_a \right\}$$

so that comparison with (6) leads to the identification $\omega_a \approx d\lambda_a/d\lambda$. It is for this reason that one assumes $\omega_a > 0$.