

CONCEPT AND PROPERTIES OF LATTICE GAUGE THEORY*

Werner Kerler[†]
Stanford Linear Accelerator Center
Stanford University, Stanford, CA 94305

ABSTRACT

A finite lattice in 4 dimensions and correlation functions defined by integrals are used and the general concept is made precise. Former results on Schwinger-Dyson equations and Ward-Takahashi identities are extended and the much richer structure of quantities and relations, which arises necessarily on the lattice, is discussed. The mechanism of gauge fixing is analyzed and consequences for the advocated concept and for the axiomatic approach are pointed out. The implications of the generalized fermion degeneracy regularization for the position space propagator and in the relations for the various currents are shown. An explicit solution for open boundaries is presented and compared with that of the case of the otherwise used periodic conditions. The analogue of continuum methods for dynamical masses and particular decompositions of the fermion determinants are considered. The connection between degeneracy regularization and axial-vector anomaly and the situation for weak interactions are discussed. Further a number of important details is clarified.

Submitted to Physical Review D

* Work supported in part by the Deutsche Forschungsgemeinschaft and in part by the Department of Energy, contract DE-AC03-76SF00515.

† On sabbatical leave from Fachbereich Physik, Universität Marburg, D-3550 Marburg, Fed. Rep. of Germany.

I. INTRODUCTION AND SUMMARY

In view of the success of gauge theories it is a major task to develop a nonperturbative description of the underlying dynamics. This is crucial for color confinement as well as for dynamical symmetry breaking and mass generation. More generally, to find an ultimate unified theory in addition to group theory a better understanding of dynamical aspects could be essential. The introduction of a lattice¹ has turned out to be most promising for the nonperturbative analysis. With respect to the features related to confinement in QCD remarkable progress², though unfortunately without fermions, has been made. Lattice methods have also led to promising results³ for the dynamical breaking of chiral symmetry in QCD. So far, however, the extension to weak interactions, because of the need to handle the fermion spectrum degeneracy on the lattice⁴, has not been achieved. On the other hand, nonperturbative lattice methods are desirable in the electroweak case too, in particular because of the necessity to generate the masses dynamically which has become rather clear recently.^{5,6}

In this situation it appears important to develop lattice theory further and to make every effort to overcome the present difficulties. To do this, a clear concept and a formulation with a minimum of ingredients is essential. Within the latter respect the framework of a finite lattice with correlation functions defined by integrals, as used in the successful Monte Carlo calculations² in QCD, is most attractive. It has recently been shown to be advantageous for analytical calculations too by giving a nonperturbative derivation of the axial-vector anomaly⁷

and by investigating the lattice structure of Schwinger-Dyson equations and Ward-Takahashi identities.⁸ A consequent concept is now to consider the formulation seriously as a particular regularization of quantum field theory, with the goal of applying it to the gauge theory of all interactions. The next step then is to extend the previous results^{7,8} and to investigate a number of additional points which are important for the indicated concept. To do this is the aim of the present paper.

A major phenomenon to be encountered is the much richer structure of relations and quantities on the lattice which necessarily occurs (already for the simplest possible form of the lattice action). This suggests the possibility that by the advocated framework the structure of the ultimate theory is better resolved as it is by the present continuum theory. A further general feature is that, because quantization needs no gauge fixing, gauge invariant correlation functions become the natural objects and familiar quantities and relations with gauge fixing appear artificial. A problem to be considered within several respects is that related to the fermion degeneracy. Its direct nonperturbative connection⁷ to the axial-vector anomaly⁹ makes the view possible that one meets the questions related to the latter in a better resolved form.

In Sec. II the general concept is made precise and put in perspective to other formulations. It is pointed out that in the cases of QCD and QED the relation to continuum theory can already be considered as essentially established.

Apart from giving definitions (including a generalized degeneracy regularization) and some general properties, in Sec. III the transition to Minkowski space is discussed, which, being transparent in the present

notation, reveals a particular property of the degeneracy regularization.

In Sec. IV Schwinger-Dyson equations are considered which occur as an important tool also in later sections. The derivation of Wick's theorem and the specialization to a Wilson loop are indicated. The evaluation of the formerly introduced derivatives for the non-Abelian fields shows the rich structure of the equation of motion on the lattice.

The fermion propagator with the generalized degeneracy regularization is discussed in Sec. V. From its position space form, without referring to a spectrum degeneracy, a simple picture emerges, how the absence of the regularization leads to wrong results for fermion loops. In addition the properties of the symmetry group responsible for the degeneracy and its relation to the propagator are clarified.

In Sec. VI the case of open boundary conditions is investigated, firstly to study an alternative to the otherwise used periodic ones and secondly because the fermion spectrum then exhibits no degeneracy. The problem is explicitly solved and the propagator found to get a form which again leads to unacceptable results in perturbation theory. This provides an example that the spectrum alone does not guarantee the correct limit.

Exploiting an explicit relation between correlation functions with and without gauge fixing in Sec. VII first the mechanism of gauge fixing is analyzed. The discussion then leads to the more natural invariant correlation functions. Further, the general structure which emerges is shown to pose severe difficulties to the axiomatic approach. This reflects the deep difference between gauge theories and other quantum field theories.

Section VIII presents the derivation and discussion of various Ward-Takahashi identities and of the related currents. The role of gauge fixing is illustrated by an example. Properties of the occurring anomalous terms are pointed out. Again the richer structure of the lattice relations becomes apparent.

In Sec. IX first the lattice analogue of the usual continuum methods¹⁰ for dynamical mass generation is studied. It turns out that these methods then become rather unattractive. In addition one gets a drastic example of how familiar relations look on the lattice. Then decompositions of the fermion determinants according to their gauge field content are discussed and qualitative pictures for Wilson loop and fermion propagator given.

Finally in Sec. X the situation for weak interactions in view of the connection between fermion-degeneracy regularization and axial-vector anomaly is discussed.

II. BASIS OF THE FORMULATION

The lattice is considered as a particular regularization of quantum field theory which makes the definition of the latter independent of perturbation theory and which preserves gauge invariance. Euclidean space is used in the same sense as in continuum theory where it is introduced to define Minkowski space quantities properly.¹¹ On a finite lattice in four dimensions the representation of correlation functions then is well defined and simple. Thus one has a framework with a minimum of mathematical ingredients. On the other hand, it is the most immediate possi-

bility with respect to physical amplitudes in space-time.

It is to be realized that one can work in this framework without any further mathematical refinements. In fact, the Monte Carlo calculations² are numerical examples for this. In principle one can thus get any quantity and approach the continuum limit as close as one wishes (in practice, if an adequate computer exists). Also analytical calculations can be done to evaluate quantities or to decide general questions.^{7,8} Thus one can postpone the question of a mathematically more elegant way of handling the limit until more experience with the formulation in connection with physical facts shows what is really appropriate.

It is advocated here that to make contact to continuum quantum field theory one has only to consider perturbation theory, because the comparison with experiment so far rests on the perturbative form. This clearly reduces the mathematical requirements considerably. In addition, it appears advisable in view of the difficulties faced by the axiomatic approach which will be pointed out in Sec. VII.

For compact QED Sharatchandra¹² has shown that with Wilson's formulation¹ and degeneracy regularization⁴ one gets renormalizability and the correct continuum limit for perturbation theory. Actually, as has been pointed out recently,⁸ gauge fixing must be done in a more sophisticated way to obtain this result. Since the continuum perturbation expansion is the same for compact and noncompact QED, the usual continuum theory could be each of them. Accepting quantization on the basis of the gauge group measure as is done here, QED is compact. The Sharatchandra analysis is expected to go through in the non-Abelian case as well. Basic divergent diagrams for QED and QCD have been investigated for Wilson's lattice^{1,4}

by Karsten and Smit^{13,14} and found to have the correct limit. Thus, putting things together one sees that, though some details should be worked out further, in the cases of QED and QCD the connection of interest can be considered as essentially established.

A word on the relation to other approaches is in order here. Using the transfer matrix¹⁵ one arrives at a Hamiltonian and a space of states. This involves the continuum limit in one direction and necessarily gauge fixing. Thus considerable mathematical assumptions enter and things become more complicated. The formal functional integrals of continuum theory can be considered as properly defined by the integrals of the present framework in the indicated limit of the formulation. This is to be contrasted to defining the limit of such integrals individually either via a measure¹⁶ or a space of path¹⁷ which, apart from needing again assumptions, is anyway only possible in simple cases. The common feature of the alternatives is that already a more refined topology of the limit (manifest in the spaces of states or path or in the measure) is chosen.

III. DEFINITIONS AND GENERAL PROPERTIES

A finite lattice of $\mathcal{N} = 16 N_1 N_2 N_3 N_4$ sites in 4-dimensional Euclidean space is used. Periodicity for $n_\lambda \rightarrow n_\lambda + 2N_\lambda$ in the numbering of all variables is imposed as boundary condition. (An alternative condition is investigated and then discarded in Sec. VI.)

The action, with Grassmann variables describing fermions, is

$$S = v \sum_{n',n} \bar{\psi}_{n'} (\not{D} - X + M)_{n',n} \psi_n$$

$$+ v \sum_{n,\sigma,\lambda} \text{Tr}(1 - U_{\sigma n}^\dagger U_{\lambda,n+\sigma}^\dagger U_{\sigma,n+\lambda} U_{\lambda n}) / (g a_\sigma a_\lambda)^2 \quad (3.1)$$

n stands for all components n_μ , and $n + \lambda$ then means $n_\mu + \delta_{\mu\lambda}$. a_λ are the lattice spacings and $v = a_1 a_2 a_3 a_4$. One further has $\not{D} = \sum_\lambda \gamma_\lambda D_\lambda$ with

$$D_{\lambda n',n} = (U_{\lambda n'}^\dagger \delta_{n'+\lambda,n} - U_{\lambda n} \delta_{n',n+\lambda}) / (2a_\lambda) \quad (3.2)$$

and $M_{n',n} = \delta_{n',n}$. X in (3.1) is the fermion-degeneracy regularization which is here defined by

$$X = \eta \sum_\lambda W_\lambda \quad , \quad \eta = z \frac{1 + \gamma_5}{2} + z^* \frac{1 - \gamma_5}{2} \quad (3.3)$$

where z is a complex constant subject to $|z| > 0$ and

$$W_{\lambda n',n} = (U_{\lambda n'}^\dagger \delta_{n'+\lambda,n} + U_{\lambda n} \delta_{n',n+\lambda} - 2\delta_{n',n}) / (2a_\lambda) \quad (3.4)$$

For $z = 1$ (3.3) gives Wilson's original prescription⁴ and for $z = -i$ the alternative one of Osterwalder and Seiler.¹⁸

The gauge fields in (3.1) - (3.4) are given by $U_{\lambda n} = \exp\{iB_{\lambda n}\}$ with $B_{\lambda n} = \sum_\ell T^\ell B_{\lambda n}^\ell$ and the normalization $\text{Tr}(T^\ell T^j) = \frac{1}{2} \delta_{\ell j}$. In the Abelian case the operation Tr in (3.1) is to be replaced by a factor $\frac{1}{2}$ to conform with usual conventions. g is the coupling constant. To arrive at the field quantities of continuum theory the $B_{\lambda n}$ are to be expressed by $B_{\lambda n} = g a_\lambda A_{\lambda n}$.

A general correlation function is defined by

$$\langle P \rangle = \int e^{-S_P} / \int e^{-S} , \quad (3.5)$$

where \int means $\int_U \int_\psi$, with \int standing for the Grassmann-variable integrations $\prod_{n,\beta} \int d\psi_{n\beta} d\bar{\psi}_{n\beta}$ and \int_U similarly for the invariant integrations over the gauge group. The integrations in (3.5) include all variables on the lattice. In the Abelian case \int_U becomes simply $\prod_{n,\lambda} \int_{-\pi}^{\pi} dB_{\lambda n} / 2\pi$ and the dependence of P on U is reflected by the periodicity $P(B) = P(B + 2\pi)$. For P , specifying a particular function, not only a product of variables but also more general expressions are admitted. The technical device of introducing source terms, familiar in the formal applications of path integrals, is avoided here.

The fermion integrations in (3.5) have the important property

$$\int_\psi \vec{\partial}_{n\beta}^\psi Q = 0 , \quad \int_\psi Q \overleftarrow{\partial}_{n\beta}^\psi = 0 , \quad (3.6)$$

where $\vec{\partial}_{n\beta}^\psi$ denotes the left derivative and $\overleftarrow{\partial}_{n\beta}^\psi$ the right derivative with respect to the Grassmann variable $\psi_{n\beta}$. Analogous relations hold for $\bar{\psi}_{n\beta}$. To get similar equations for the gauge fields one has to introduce the derivatives⁸

$$\begin{aligned} \vec{\partial}_{\ell\lambda n}^U Q &= \lim_{\epsilon \rightarrow 0} \left(Q(\dots, \exp\{iT^\ell \epsilon\} U_{\lambda n}, \dots) - Q(\dots, U_{\lambda n}, \dots) \right) / \epsilon , \\ \overleftarrow{\partial}_{\ell\lambda n}^U Q &= \lim_{\epsilon \rightarrow 0} \left(Q(\dots, U_{\lambda n} \exp\{iT^\ell \epsilon\}, \dots) - Q(\dots, U_{\lambda n}, \dots) \right) / \epsilon , \end{aligned} \quad (3.7)$$

for which the property

$$\int_U \frac{\delta^+ U}{\delta \lambda_{\alpha n}} Q = 0 \quad , \quad \int_U Q \frac{\delta^+ U}{\delta \lambda_{\alpha n}} = 0 \quad (3.8)$$

follows. For the derivatives (3.7) the rule for products and chain rules for the cases of interest can be established such that one has a convenient tool for the non-Abelian fields. In the Abelian case both derivatives simplify to $\partial/\partial B_{\lambda n}$. From (3.6) and (3.8) Schwinger-Dyson equations are obtained in Sec. IV. Similar derivatives as (3.8) with respect to transformations give Ward-Takahashi identities in Sec. VIII.

The action (3.1) and the integration measure in (3.5) are invariant with respect to the gauge transformation

$$\psi'_n = V_n \psi_n \quad , \quad \bar{\psi}'_n = \bar{\psi}_n V_n^\dagger \quad , \quad U'_{\lambda n} = V_n + \lambda U_{\lambda n} V_n^\dagger \quad (3.9)$$

where

$$V_n = \exp \left\{ i \sum_{\ell} T^{\ell} \alpha_n^{\ell} \right\} \quad . \quad (3.10)$$

The behavior of P in (3.5) under (3.9) will be discussed in Sec. VII.

The classical continuum limit, i.e., that of S separately, is the correct one for (3.1) because then W_λ and thus X vanishes. In the quantum case, i.e., for correlation functions, X defined by (3.3) guarantees the correct limit as will be seen in Sects. V and VIII.

Because of the distinction between the a_λ made here, which goes easily through everywhere, the transition to Minkowski space is transparent at any stage. Thus to check the reality of S/v there one replaces

a_4 by ia_0 . Further A_{4n} is replaced by $-iA_{0n}$, such that (according to $B_{4n} = ga_4 A_{4n}$) B_{4n} becomes B_{0n} . Complex conjugation of the Grassmann variables is defined here as a one-to-one mapping of the Grassmann algebra (with generators $\psi_{n\beta}, \bar{\psi}_{n\beta}$) onto itself which satisfies $(\xi^*)^* = \xi$, $(\xi_1 \xi_2)^* = \xi_2^* \xi_1^*$ and $(\alpha \xi)^* = \alpha^* \xi^*$, where ξ is a Grassmann element and α a complex number (an involution¹⁹ would require much more). Now, with $R_0 = -\sum_{n',n} \bar{\psi}_{n',n} W_{0n',n} \psi_n$ being the W_0 contribution to S/v , it follows from (3.1) that $(S/v - R_0)^* = S/v - R_0$, however, that

$$R_0^* = -R_0 \quad . \quad (3.11)$$

Thus a peculiar property of the degeneracy regularization is found which will be met in a different form in Sec. VIII again.

In the quantum case one has more carefully to replace a_4 by $ia_0 + \epsilon$. This amounts to working in Minkowski space with an $i\epsilon$ -prescription which could be largely done instead of using Euclidean space. It leads, as it should, to Feynman propagators which, for example, can be immediately checked converting the equations in Sec. V.

For position space variables $x_\lambda = a_\lambda n_\lambda$, the replacement $a_4 \rightarrow ia_0$ means just $x_4 \rightarrow ix_0$. Momentum space variables are based on the r -representation of the occurring matrices which is obtained from the n -representation by the transformation

$$\mathcal{N}^{-1/2} \exp\left\{-\pi i \sum_\lambda r_\lambda n_\lambda / N_\lambda\right\} \quad . \quad (3.12)$$

Then one has $k_\lambda = \pi r_\lambda / (a_\lambda N_\lambda)$ and for $a_4 \rightarrow ia_0$, $r_4 \rightarrow -r_0$ one gets

$k_4 \rightarrow ik_0$. Thus the transition is completely consistent (and hyperbolic functions⁴ do not occur in the discussion of propagator poles).

IV. SCHWINGER-DYSON EQUATIONS

From the general relation (3.6) with $Q = -e^{-S}P$ one obtains

$$\int_{\psi} e^{-S} \left((\vec{\partial}_{n\beta}^{\psi} S)P - \vec{\partial}_{n\beta}^{\psi} P \right) = 0, \quad \int_{\psi} e^{-S} \left(P(S \overleftarrow{\partial}_{n\beta}^{\psi}) - P \overleftarrow{\partial}_{n\beta}^{\psi} \right) = 0. \quad (4.1)$$

Analogous relations hold for $\vec{\partial}_{n\beta}^{\bar{\psi}}$ and $\overleftarrow{\partial}_{n\beta}^{\bar{\psi}}$. With (4.1) one has by (3.5) the Schwinger-Dyson equation $\langle (\vec{\partial}_{n\beta}^{\psi} S)P - \vec{\partial}_{n\beta}^{\psi} P \rangle = 0$ and the respective ones for the other fermion derivatives. Because of the bilinearity of S in the fermion variables the second equation in (4.1) actually contains nothing new. This is seen by noting that $\vec{\partial}_{n\beta}^{\psi} S = -S \overleftarrow{\partial}_{n\beta}^{\psi}$ according to (3.1) and that from P only odd Grassmann elements contribute to the integrals. Due to the bilinearity it is also easy to read off from (3.1) that $\vec{\partial}_{n\beta}^{\bar{\psi}} S$ and $S \overleftarrow{\partial}_{n\beta}^{\bar{\psi}}$ is just what occurs in the lattice Dirac equations.

For the gauge field derivatives one gets from (3.8), putting $Q = -e^{-S}P$,

$$\int_U e^{-S} \left((\vec{\partial}_{\ell\sigma n}^U S)P - \vec{\partial}_{\ell\sigma n}^U P \right) = 0, \quad \int_U e^{-S} \left(P(S \overleftarrow{\partial}_{\ell\sigma n}^U) - P \overleftarrow{\partial}_{\ell\sigma n}^U \right) = 0, \quad (4.2)$$

which by (3.5) gives again Schwinger-Dyson equations. In (4.2) the two equations have different content. The evaluation of the derivatives of S gives

$$\frac{1}{v} \frac{\partial U}{\partial \ell_{\sigma n}} S = \sum_{\lambda} \left(\mathcal{F}_{\sigma\lambda, n}^{[4]\ell} - \mathcal{F}_{\sigma\lambda, n-\lambda}^{[3]\ell} \right) / (g a_{\sigma} a_{\lambda})^2 - \vec{J}_{\sigma n}^{\ell} / a_{\sigma} + \overset{\leftarrow}{J}_{\sigma n}^{\ell} / a_{\sigma} , \quad (4.3a)$$

$$\frac{1}{v} S \frac{\partial U}{\partial \ell_{\sigma n}} = \sum_{\lambda} \left(\mathcal{F}_{\sigma\lambda, n}^{[1]\ell} - \mathcal{F}_{\sigma\lambda, n-\lambda}^{[2]\ell} \right) / (g a_{\sigma} a_{\lambda})^2 - \overset{\leftarrow}{J}_{\sigma n}^{\ell} / a_{\sigma} + \vec{J}_{\sigma n}^{\ell} / a_{\sigma} , \quad (4.3b)$$

with the currents

$$\begin{aligned} \vec{J}_{\sigma n}^{\ell} &= \frac{i}{2} \left(\bar{\psi}_n \gamma_{\sigma} U_{\sigma n}^{\dagger} T^{\ell} \psi_{n+\sigma} + \bar{\psi}_{n+\sigma} \gamma_{\sigma} T^{\ell} U_{\sigma n} \psi_n \right) , \\ \overset{\leftarrow}{J}_{\sigma n}^{\ell} &= \frac{i}{2} \left(\bar{\psi}_n \gamma_{\sigma} T^{\ell} U_{\sigma n}^{\dagger} \psi_{n+\sigma} + \bar{\psi}_{n+\sigma} \gamma_{\sigma} U_{\sigma n} T^{\ell} \psi_n \right) , \end{aligned} \quad (4.4)$$

the quantities originating from X

$$\begin{aligned} \vec{J}_{\sigma n}^{\ell} &= \frac{i}{2} \left(\bar{\psi}_n \eta U_{\sigma n}^{\dagger} T^{\ell} \psi_{n+\sigma} - \bar{\psi}_{n+\sigma} \eta T^{\ell} U_{\sigma n} \psi_n \right) , \\ \overset{\leftarrow}{J}_{\sigma n}^{\ell} &= \frac{i}{2} \left(\bar{\psi}_n \eta T^{\ell} U_{\sigma n}^{\dagger} \psi_{n+\sigma} - \bar{\psi}_{n+\sigma} \eta U_{\sigma n} T^{\ell} \psi_n \right) , \end{aligned} \quad (4.5)$$

and the field-strength type components

$$\mathcal{F}_{\sigma\lambda, n}^{[\alpha]\ell} = 2 \text{Tr} \left(T^{\ell} \mathcal{F}_{\sigma\lambda, n}^{[\alpha]} \right) . \quad (4.6)$$

The matrices $\mathcal{F}_{\sigma\lambda, n}^{[\alpha]}$ in (4.6) are given by

$$\mathcal{F}_{\sigma\lambda, n}^{[\alpha]} = \left(\omega_{p(\alpha)}^{\dagger} - \omega_{p(\alpha)} \right) / (2i) \quad (4.7)$$

where $\omega_n = U_{\sigma n}^{\dagger} U_{\lambda, n+\sigma}^{\dagger} U_{\sigma, n+\lambda} U_{\lambda n}$ is the product starting with $U_{\lambda n}$ at point $p(1) = n$, and the other $\omega_{p(\alpha)}$ its cyclic permutations starting from $p(2) = n + \lambda$, $p(3) = n + \lambda + \sigma$, $p(4) = n + \sigma$, i.e., from the other

corners around the plaquette. They have the antisymmetry property

$$\mathcal{F}_{\sigma\lambda,n}^{[\alpha']} = - \mathcal{F}_{\lambda\sigma,n}^{[\alpha]} \quad (4.8)$$

valid for $\alpha' = \alpha = 1$, for $\alpha' = \alpha = 3$, and for $\alpha' = 2, \alpha = 4$. The products of gauge field factors around plaquettes in the $\mathcal{F}_{\sigma'\lambda',n'}^{[\alpha]}$, occurring in Eqs. (4.3a) and (4.3b) have the link from n to $n + \sigma$ in common and start from $n + \sigma$ and n , respectively, as is illustrated in Fig. 1. The occurring quantities (4.4) and (4.5) are also associated to this link.

In the continuum limit of the individual quantities, i.e., in the classical one, one has

$$\begin{aligned} \mathcal{F}_{\sigma\lambda,n}^{[\alpha]} / (g a_\sigma a_\lambda) &\rightarrow F_{\sigma\lambda}(x) \text{ for all } \alpha, \\ \vec{J}_{\sigma n}^l, \quad \overleftarrow{J}_{\sigma n}^l &\rightarrow J_\sigma^l(x), \\ \vec{J}_{\sigma n}^l, \quad \overleftarrow{J}_{\sigma n}^l &\rightarrow 0. \end{aligned} \quad (4.9)$$

In the quantum case, i.e., for correlation functions, it is to be stressed that all these quantities depend on the variables of the integrals, and the limit is the one discussed in Sec. II.

As an application of (4.1) the derivation of Wick's theorem is briefly indicated. Inserting $P = - \psi_{n,\beta} P_\psi$ and evaluating $\vec{\partial}_{n\beta}^\psi S$ the first equation in (4.1) becomes

$$\begin{aligned}
 & \sum_{n'', \beta''} \int_{\psi} e^{-S} \psi_{n', \beta'} \bar{\psi}_{n'' \beta''} P_{\psi} (\not{D} - X + M)_{n'' \beta'' n \beta} \\
 &= \frac{1}{v} \delta_{n' n} \delta_{\beta' \beta} \int_{\psi} e^{-S} P_{\psi} - \frac{1}{v} \int_{\psi} e^{-S} \psi_{n', \beta'} \overrightarrow{\delta}_{n \beta}^{\psi} P_{\psi} . \quad (4.10)
 \end{aligned}$$

It is to be noted that from P_{ψ} only even Grassmann elements contribute to the integrals. With $G = (\not{D} - X + M)^{-1}$ one obtains from (4.10)

$$\begin{aligned}
 \int_{\psi} e^{-S} \psi_{n', \beta'} \bar{\psi}_{n \beta} P_{\psi} &= \frac{1}{v} G_{n', \beta' n \beta} \int_{\psi} e^{-S} P_{\psi} \\
 &- \frac{1}{v} \sum_{n'', \beta''} \int_{\psi} e^{-S} \psi_{n', \beta'} \overrightarrow{\delta}_{n'' \beta''}^{\psi} P_{\psi} G_{n'' \beta'' n \beta} . \quad (4.11)
 \end{aligned}$$

Now, inserting $P_{\psi} = 1$ one gets

$$\int_{\psi} e^{-S} \psi_{n', \beta'} \bar{\psi}_{n \beta} = \frac{1}{v} G_{n', \beta' n \beta} \int_{\psi} e^{-S} . \quad (4.12)$$

Next, putting $P_{\psi} = \psi_{n''', \beta'''} \bar{\psi}_{n'' \beta''}$ in (4.11) and using the result (4.12) one obtains

$$\begin{aligned}
 & \int_{\psi} e^{-S} \psi_{n''', \beta'''} \bar{\psi}_{n'' \beta''} \psi_{n', \beta'} \bar{\psi}_{n \beta} \\
 &= \frac{1}{v} \left(G_{n', \beta' n \beta} G_{n''', \beta''' n'' \beta''} - G_{n', \beta' n'' \beta''} G_{n''', \beta''' n \beta} \right) \int_{\psi} e^{-S} . \quad (4.13)
 \end{aligned}$$

Obviously one can proceed in this way up to P_{ψ} with the maximal number of pairs $\psi \bar{\psi}$ on the lattice (for other types of P_{ψ} the integral vanishes identically). This gives the Wick expansion for fermions. It can be used

with respect to S_0 in $e^{-S} = e^{-S_0 - S_1} = e^{-S_0} \sum_{\nu=0}^{\infty} \frac{(-S_1)^\nu}{\nu!}$ to set up perturbation theory.

As an example for the application of (4.2) it is specialized to a Wilson loop. According to Fig. 1a for the first equation one has to choose $P = 2\text{Tr}(T^\ell L_{n+\sigma})$, where $L_{n+\sigma}$ is the product of gauge field factors along a closed loop starting and ending at point $n+\sigma$. Then summing over all ℓ one gets

$$\sum_{\ell} \mathcal{F}_{\sigma\lambda, n}^{[4]\ell} P(\ell) = 2\text{Tr} \left(\mathcal{F}_{\sigma\lambda, n}^{[4]} L_{n+\sigma} \right) \quad (4.14)$$

and similar expressions for the other terms. These forms can be interpreted as deformations of the loop as is illustrated in Fig. 2 for a particular case. It is thus seen that the type of equations which has been recently of interest with respect to strings²⁰ in the present formulation can be obtained in a general and easy way.

V. FERMION DEGENERACY AND PROPAGATOR

From (4.12) without gauge field one obtains for the free-fermion two-point function

$$\mathcal{P}_{n'\beta' n\beta}(z) = \frac{1}{v} G_{n'\beta' n\beta} \text{ with } U_{\lambda n} = 1 \quad (5.1)$$

By using the transformation (3.12) this can be written

$$\mathcal{P}_{n'\beta' n\beta}(z) = (\mathcal{N}_v)^{-1} \sum_r \exp \left\{ \pi i \sum_{\lambda} r_{\lambda} (n'_{\lambda} - n_{\lambda}) / N_{\lambda} \right\} \mathcal{K}_r(1, 0, z) \quad (5.2)$$

where for later convenience the abbreviation

$$\mathcal{K}_r(h, \alpha, z) = - \frac{i \sum_{\mu} h_{\mu} \gamma_{\mu} s_{\mu} + \eta^{\dagger} (w - \alpha) - m}{\sum_{\lambda} s_{\lambda}^2 + |z|^2 (w - \alpha)^2 + m^2 - (z + z^*)^m (w - \alpha)} \quad (5.3)$$

with $s_{\lambda} = \sin(\pi r_{\lambda} / N_{\lambda}) / a_{\lambda}$, $w = \sum_{\lambda} (\cos(\pi r_{\lambda} / N_{\lambda}) - 1) / a_{\lambda}$ and sign factors $h_{\lambda} = \pm 1$ is introduced.

To study the denominator in (5.2) it is advantageous to express it in terms of $u_{\lambda} = 2 \sin(\pi r_{\lambda} / (2N_{\lambda})) / a_{\lambda}$ which in contrast to s_{λ} is uniquely related to $k_{\lambda} = \pi r_{\lambda} / (a_{\lambda} N_{\lambda})$ within the full r -interval. This gives

$$\begin{aligned} \sum_{\lambda} s_{\lambda}^2 + |z|^2 w^2 + m^2 - (z + z^*)^m w &= \sum_{\lambda} u_{\lambda}^2 + m^2 \\ + (|z|^2 - 1) \sum_{\lambda} \frac{a_{\lambda}^2}{4} u_{\lambda}^4 + |z|^2 \sum_{\lambda \neq \sigma} \frac{a_{\lambda} a_{\sigma}}{4} u_{\lambda}^2 u_{\sigma}^2 + (z + z^*)^m \sum_{\lambda} \frac{a_{\lambda}}{2} u_{\lambda}^2. \end{aligned} \quad (5.4)$$

From (5.4) it is seen that for $|z| = 1$ the u_{λ}^4 term drops out. Such a choice of z thus generalizes the one advocated by Wilson⁴ to suppress the additional poles already at the finite stage. For the limit only $|z| > 0$ is needed as will be seen for the propagator in the following and for the anomaly term in Sec. VIII.

So far the summation over r_{λ} in (5.2) is from $-N_{\lambda} + 1 + c_{\lambda}$ to $N_{\lambda} + c_{\lambda}$, where c_{λ} is some integer. By an appropriate choice of c_{λ} and a shift of the summation indices in one half of the intervals by N_{λ} , \sum_r can be replaced by \sum_r' for which the summations are restricted to $-N_{\lambda}/2 < r_{\lambda} \leq N_{\lambda}/2$. In this way instead of (5.2) one obtains

$$\begin{aligned}
 \mathcal{P}_{n', \beta', n\beta}(z) &= (\mathcal{N}_v)^{-1} \sum_r' \exp\left\{ \pi i \sum_\lambda r_\lambda \Delta_\lambda / N_\lambda \right\} \left[\mathcal{K}_r(1, 0, z) \right. \\
 &+ \sum_\lambda (-1)^{\Delta_\lambda} \mathcal{K}_r(h(\lambda), m_\lambda, z) + \sum_{\sigma > \lambda} (-1)^{\Delta_\lambda + \Delta_\sigma} \mathcal{K}_r(h(\lambda, \sigma), m_\lambda + m_\sigma, z) \\
 &+ \sum_{\rho > \sigma > \lambda} (-1)^{\Delta_\lambda + \Delta_\sigma + \Delta_\rho} \mathcal{K}_r(h(\lambda, \sigma, \rho), m_\lambda + m_\sigma + m_\rho, z) \\
 &\left. + (-1)^{\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4} \mathcal{K}_r(-1, m_1 + m_2 + m_3 + m_4, z) \right]
 \end{aligned} \tag{5.5}$$

where $\Delta_\lambda = n'_\lambda - n_\lambda$ and $m_\lambda = 2 \cos(\pi r_\lambda / N_\lambda) / a_\lambda$. The sign factors h in (5.5) are defined by the requirement that $h_\mu = -1$ in (5.3) if one of the arguments of h equals μ and that $h_\mu = 1$ otherwise. The crucial point now is that for \sum_r' , in which k_λ and s_λ are uniquely related, the continuum limit can safely be performed. Then, as long as $|z| > 0$, all terms of (5.5) except the first one vanish because the m_λ become infinite. One thus remains with the limit of

$$- (\mathcal{N}_v)^{-1} \sum_r' \exp\left\{ \pi i \sum_\lambda r_\lambda \Delta_\lambda / N_\lambda \right\} \frac{i \sum_\mu \gamma_\mu s_\mu - m}{\sum_\lambda s_\lambda^2 + m^2} \tag{5.6}$$

which gives the correct result.

The additional poles occurring in the fermion problem have been interpreted as further particles.^{4, 14} From (5.5) it is seen that the sign factors h are an obstacle to this. For each term one could get rid of them by switching to another set of γ -matrices. One has, however, to decide for one set to describe the system as a whole properly.

It is interesting to look at what happens for $z = 0$. In that case one can derive from (5.5) that

$$\mathcal{P}_{n', \beta', n\beta}^{(0)} = -(\mathcal{N}_V)^{-1} \sum_{\mathbf{r}} \exp\left\{ \pi i \sum_{\lambda} r_{\lambda} \Delta_{\lambda} / N_{\lambda} \right\} \frac{i \sum_{\mu} f_{\mu}(\Delta) \gamma_{\mu} s_{\mu}^{-f(\Delta)m}}{\sum_{\lambda} s_{\lambda}^2 + m^2} \quad (5.7)$$

where

$$f_{\mu}(\Delta) = \begin{cases} 16 & \text{if } \Delta_{\mu} \text{ odd and } \Delta_{\lambda} \text{ with } \lambda \neq \mu \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \quad (5.8)$$

$$f(\Delta) = \begin{cases} 16 & \text{if all } \Delta_{\lambda} \text{ even,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus averaging n' in (5.7) over the 16 corners of an elementary cube one gets the same result as for (5.6). This holds for the limit too if (5.7) is averaged with a test function or occurs in a diagram combined with boson lines. It is, however, seen that wrong results arise in the limit as soon as functions (5.7) multiply, as they do in the case of fermion loops in perturbation theory. Then due to the factors (5.8) an additional (wrong) factor 16 arises for each multiplication. This is a simple example that one can get the spectrum and further features correctly though the problem is actually not solved.

The fermion degeneracy is related to the invariance of S under the replacement $\psi_n \rightarrow \gamma_5 \gamma_{\lambda} (-1)^{n_{\lambda}} \psi_n$ as was observed by Chodos and Healy²¹ considering 2 + 1 dimensions and which was recently also discussed for 4 dimensions.¹⁴ To find out more precisely here how this symmetry works, first four mappings T_{λ} of the Grassmann algebra onto itself are defined,

the action of which on the generators is given by

$$\psi_n^r = i\gamma_\lambda \gamma_5 (-1)^{n_\lambda} \psi_n, \quad \bar{\psi}_n^r = \bar{\psi}_n i\gamma_\lambda \gamma_5 (-1)^{n_\lambda}. \quad (5.9)$$

It is to be noted that on the r -representation [which is obtained by (3.12)] the T_λ then act as

$$\tilde{\psi}_r^r = i\gamma_\lambda \gamma_5 \tilde{\psi}_{r \pm N_\lambda}, \quad \tilde{\bar{\psi}}_r^r = \tilde{\bar{\psi}}_{r \pm N_\lambda} i\gamma_\lambda \gamma_5. \quad (5.10)$$

It is seen that S given by (3.1) with $z = 0$ as well as the integrations defining the correlation functions in (3.5) are invariant under (5.9) (the presence of the gauge field does not matter within this respect). Using (5.9) it follows that the mappings have the general property

$$T_\lambda T_\mu + T_\mu T_\lambda = 2\delta_{\lambda\mu}. \quad (5.11)$$

Thus the group generated by the T_λ is a well known one with 32 elements (16 basis elements of a Clifford algebra equipped with plus and minus signs).

Next the implications of this transformation group for the free fermion propagator with $z = 0$ are considered. Because the action and the integrations are invariant, a transformation of the variables by (5.9) in $\int_\psi e^{-S} \psi_{n'\beta'} \bar{\psi}_{n\beta} / \int_\psi e^{-S}$ shows that $\mathcal{S}_{n'\beta'n\beta}(0)$ becomes

$$(-1)^{\Delta_\lambda} \sum_{\alpha', \alpha} (\gamma_\lambda \gamma_5)_{\beta'\alpha'} \mathcal{S}_{n'\alpha'n\alpha}(0) (\gamma_5 \gamma_\lambda)_{\alpha\beta}. \quad (5.12)$$

On the other hand, the value of $\mathcal{S}_{n'\beta'n\beta}(0)$ is, of course, not changed

by using different integration variables, thus it must equal (5.12). This means that $\mathcal{P}_{n'\beta'n\beta}(0)$ is invariant under the transformation given by (5.12). The detailed mechanism is that the 16 terms in (5.5) with $z = 0$ transform among themselves as can be checked by considering the effect of (5.12) on each of them. Further one can create all of them by repeatedly applying (5.12) to (5.6). It is to be noted that T_λ as well as $-T_\lambda$ leads to (5.12) which explains why (5.12) generates only 16 transformations.

VI. FERMION PROBLEM AND BOUNDARIES

The periodic boundary conditions introduced in Sec. III (and exclusively used except in the present section) are particularly convenient because of the explicit solutions available in the free case. Thus (5.2) has been easily obtained from (5.1) by diagonalizing $D_{\lambda n'n}$ using the transformation (3.12). Considered for one direction λ , the crucial point is that with periodic conditions the matrix

$$D_{\nu'\nu} = \left(\delta_{\nu'+1,\nu} - \delta_{\nu',\nu+1} \right) / (2a) \quad (6.1)$$

has eigenvectors

$$f_{\alpha\nu} = \frac{1}{\sqrt{L}} e^{2\pi i \alpha \nu / L} \quad (6.2)$$

which correspond to eigenvalues

$$\lambda_\alpha / (2a) = 2i \sin(2\pi\alpha/L) / (2a) \quad (6.3)$$

Due to the identity $\sum_{\nu=1}^L e^{2\pi i(\alpha' - \alpha)\nu/L} = N\delta_{\alpha',\alpha}$ with the normalization introduced in (6.2) the relation

$$\sum_{\nu=1}^L f_{\alpha',\nu}^* f_{\alpha\nu} = \delta_{\alpha',\alpha} \quad (6.4)$$

is satisfied.

An immediate alternative possibility is the use of open boundary conditions. In the following it will be shown that in this case the fermion propagator can be explicitly calculated too. The clue to this is the exploitation of particular properties of the determinants which occur in the calculation of the eigenvalues of the matrix

$$M_{\nu',\nu}^L = \delta_{\nu'+1,\nu} - \delta_{\nu',\nu+1} \quad \text{with } \nu',\nu = 1,\dots,L. \quad (6.5)$$

One has to note that more explicitly this matrix for open boundaries reads

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & -1 & 0 \end{pmatrix} \quad \text{in contrast to} \quad \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & -1 \\ -1 & 0 & 1 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & -1 & 0 \end{pmatrix}$$

for periodic ones. Thus defining

$$d_L(\lambda) = \det(M^L - \lambda I) \quad (6.6)$$

for the open case one can derive the recursion relation

$$d_L(\lambda) = -\lambda d_{L-1}(\lambda) + d_{L-2}(\lambda) \quad . \quad (6.7)$$

From (6.7) the $d_L(\lambda)$ for all L can be readily calculated. Further, (6.7) can be used to show that the representation²²

$$d_L(\lambda) = (-1)^L \prod_{\alpha=1}^L \left(\lambda + 2i \cos \left(\frac{\pi\alpha}{L+1} \right) \right) \quad (6.8)$$

holds. This can be checked by using as an intermediate step the representation

$$d_L(\lambda) = \frac{(-\lambda + \sqrt{\lambda^2 + 4})^{L+1} - (-\lambda - \sqrt{\lambda^2 + 4})^{L+1}}{2^{L+1} \sqrt{\lambda^2 + 4}}, \quad (6.9)$$

which will be useful later too.

From (6.8) one reads off that the eigenvalues of \mathcal{M}^L are

$$\lambda_{\alpha} = -2i \cos \left(\frac{\pi\alpha}{L+1} \right). \quad (6.10)$$

Instead of $\alpha = 1, \dots, L$ one may use

$$\hat{\alpha} = \alpha - \frac{L+1}{2}, \quad \hat{\alpha} = -\frac{L-1}{2}, \quad -\frac{L-1}{2} + 1, \dots, \frac{L-1}{2}, \quad (6.11)$$

by which (6.10) becomes

$$\lambda_{\hat{\alpha}} = 2i \sin \left(\frac{\pi\hat{\alpha}}{L+1} \right). \quad (6.12)$$

Now, comparing (6.12) with (6.3) one observes that the factor 2 in the argument of the sine, responsible for the fermion degeneracy, has disap-

peared. The experience with (5.7), however, teaches that before drawing conclusions one should at least calculate the propagator. In order to be able to do this one has to find the eigenvectors of $\mathcal{M}_{\nu',\nu}^L$.

To get the eigenvectors $\phi_{\alpha\nu}$ in

$$\sum_{\nu'} \mathcal{M}_{\nu\nu'}^L \phi_{\alpha\nu'} = \lambda_{\alpha} \phi_{\alpha\nu} \tag{6.13}$$

one first observes that for fixed α the L equations (6.13) can be rearranged as

$$\begin{aligned} \phi_{\alpha 2} &= \lambda_{\alpha} \phi_{\alpha 1} \\ \phi_{\alpha 3} &= \lambda_{\alpha} \phi_{\alpha 2} + \phi_{\alpha 1} \\ \phi_{\alpha 4} &= \lambda_{\alpha} \phi_{\alpha 3} + \phi_{\alpha 2} \\ &\dots\dots\dots \\ \phi_{\alpha L} &= \lambda_{\alpha} \phi_{\alpha, L-1} + \phi_{\alpha, L-2} \\ 0 &= \lambda_{\alpha} \phi_{\alpha L} + \phi_{\alpha, L-1} \end{aligned} \tag{6.14}$$

The r.h.s. of (6.14) is then expressed by λ_{α} and $\phi_{\alpha 1}$ alone using the equations (6.14) to replace the $\phi_{\alpha\nu}$ with $\nu > 1$ there. Comparing the occurring polynomials in λ_{α} with those arising from the calculation of the $d_L(\lambda)$ by (6.7) it turns out that one gets

$$\phi_{\alpha\nu} = (-1)^{\nu-1} d_{\nu-1}(\lambda_{\alpha}) \phi_{\alpha 1} \tag{6.15}$$

To determine the $d_{\nu-1}(\lambda_{\alpha})$ one inserts (6.10) into (6.9) which gives

$$d_{\nu-1}(\lambda_\alpha) = i^{\nu-1} \sin\left(\frac{\nu\pi\alpha}{L+1}\right) / \sin\left(\frac{\pi\alpha}{L+1}\right). \quad (6.16)$$

Then with $\phi_{\alpha 1} = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi\alpha}{L+1}\right)$ one has

$$\phi_{\alpha\nu} = \sqrt{\frac{2}{N+1}} (-i)^{\nu-1} \sin\left(\frac{\nu\pi\alpha}{L+1}\right), \quad (6.17)$$

the normalization of which is such that

$$\sum_{\nu=1}^L \phi_{\alpha'\nu}^* \phi_{\alpha\nu} = \delta_{\alpha'\alpha}. \quad (6.18)$$

By (6.11) for (6.17) the form

$$\phi_{\alpha\nu}^{\hat{\alpha}} = \sqrt{\frac{2}{L+1}} \left(i \frac{1+(-1)^\nu}{2} \sin\left(\frac{\nu\pi\hat{\alpha}}{L+1}\right) + \frac{1-(-1)^\nu}{2} \cos\left(\frac{\nu\pi\hat{\alpha}}{L+1}\right) \right) \quad (6.19)$$

is obtained.

Now using (6.19) for all components of $D_{\lambda n' n}$, with $\hat{\alpha}$ corresponding to \hat{r}_λ , the propagator can be calculated. Up to terms with a factor $\exp\left\{\pi i \sum_\lambda \hat{r}_\lambda (n'_\lambda + n_\lambda) / (2N_\lambda + 1)\right\}$ which, having alternating signs from site to site, do not contribute to the limit, one obtains

$$-(\hat{\mathcal{N}}_\nu)^{-1} \sum_{\hat{r}} \exp\left\{\pi i \sum_\lambda \hat{r}_\lambda \Delta_\lambda / (2N_\lambda + 1)\right\} \frac{i \sum_{\mathbf{r}} f_\mu(\Delta) \gamma_\mu \hat{s}_\mu - f(\Delta)m}{\sum_\lambda \hat{s}_\lambda^2 + m^2}, \quad (6.20)$$

where $\hat{s}_\lambda = \sin(\pi \hat{r}_\lambda / (2N_\lambda + 1)) / a_\lambda$ and $\hat{\mathcal{N}} = \prod_\lambda (2N_\lambda + 1)$. Thus it turns out that one gets essentially the same unacceptable result as in (5.7).

There the doubling disappeared by the replacement of $\sum_{\mathbf{r}}$ by $\sum_{\mathbf{r}}'$.

Here it is not there because one has \hat{s}_λ instead of s_λ . If f_μ and f are suitably averaged one gets again correct results, however, a wrong limit if factors (6.20) multiply. From a more general point of view the present case is an example where the matrix corresponding to the derivative of the continuum has been chosen to get a nondegenerate spectrum, which is, however, not sufficient as the explicit calculation here shows.

VII. CORRELATION FUNCTIONS AND GAUGE INVARIANCE

To make contact to the functions and relations of continuum theory gauge fixing is to be studied. The usual procedure²³ can be performed on the lattice in a general and well-defined way. Denoting the group integrations over the transformations (3.10) by \int_V , for a given gauge fixing function \mathcal{L} the invariant function ϕ is given by

$$\phi(U) \int_V \mathcal{L}(U') = 1 \quad . \quad (7.1)$$

Using (7.1) it follows that

$$\int_U e^{-S} = \int_U e^{-S_\phi} \int_V \mathcal{L}(U') = \int_V \int_U e^{-S_\phi} \mathcal{L}(U) \quad . \quad (7.2)$$

The definition of correlation functions with gauge fixing is then

$$\langle P \rangle_{\mathcal{L}} = \int e^{-S_\phi} \mathcal{L} P / \int e^{-S_\phi} \mathcal{L} \quad . \quad (7.3)$$

To investigate the nature of the latter, in the numerator as well as in the denominator of (7.3) steps as in (7.2) can be done in the opposite direction, which gives

$$\langle P \rangle_f = \int_V \int e^{-S_\phi} f P / \int_V \int e^{-S_\phi} f = \int e^{-S_\phi} \int_V f(U') P(U', \psi', \bar{\psi}') / \int e^{-S} . \quad (7.4)$$

By inserting ϕ from (7.1) into (7.4) one thus obtains

$$\langle P \rangle_f = \langle \int_V f(U') P(U', \psi', \bar{\psi}') / \int_V f(U') \rangle \quad (7.5)$$

as the general relation between correlation functions with and without gauge fixing.

From (7.5) it is seen that gauge fixing amounts to the use of an effective

$$P_{\text{eff}} = \int_V f(U') P(U', \psi', \bar{\psi}') / \int_V f(U') , \quad (7.6)$$

which is obtained from a given P by averaging it with the gauge fixing function. Clearly P_{eff} is invariant. If P is invariant one has $P_{\text{eff}} = P$ and for the exact result gauge fixing actually does not matter. In numerical calculations it has been observed² that with gauge fixing the convergence becomes slower. The case of invariant P has been envisaged by Creutz²⁴ when discussing gauge fixing on a lattice. If P is not invariant, as for example for the usual fermion two-point function, it becomes obvious from (7.6) that the role of an appropriate f is to provide factors such that invariant contributions arise which do not vanish under \int_V . At the same time f must also contain an invariant term to guarantee a nonvanishing denominator in (7.6). Thus the choice of the gauge fixing function on the lattice needs some care. An example of an appropriate f , corresponding to $\exp \left\{ -\frac{1}{2\alpha} \int d^4x \sum_\mu (\partial_\mu A_\mu)^2 \right\}$ in Abelian

continuum theory, has recently been given.⁸

Using (7.3) it follows from (4.1) that with gauge fixing the Schwinger-Dyson equations with fermion derivatives,

$$\langle (\vec{\partial}_{n\beta}^{\psi} S) P - \vec{\partial}_{n\beta}^{\psi} P \rangle_{\not{f}} = 0 \quad , \quad \langle P (S \overleftarrow{\partial}_{n\beta}^{\psi}) - P \overleftarrow{\partial}_{n\beta}^{\psi} \rangle_{\not{f}} = 0 \quad , \quad (7.7)$$

look essentially as before, while replacing P by $\phi \not{P}$ in (4.2) one obtains

$$\langle \left(\vec{\partial}_{\ell\sigma n}^U (S - \ln(\phi \not{f})) \right) P - \vec{\partial}_{\ell\sigma n}^U P \rangle_{\not{f}} = 0 \quad , \quad \langle P \left((S - \ln(\phi \not{f})) \overleftarrow{\partial}_{\ell\sigma n}^U \right) - P \overleftarrow{\partial}_{\ell\sigma n}^U \rangle_{\not{f}} \quad (7.8)$$

for the ones with gauge field derivatives.

It is to be noted that gauge fixing has already in continuum theory some unpleasant features. A well-known one of these is the Gribov ambiguity.²⁵ Another one, which in the nonperturbative case becomes important, is that related to nonlinear transformations (or to ordering in operator language) which has been pointed out some time ago²⁶ and for which now examples have been calculated.²⁷ The present analysis on the lattice shows that by (7.6) gauge fixing constructs gauge invariant P_{eff} from noninvariant P in a complicated and physically unmotivated way. For example, for the fermion propagator in general an average of products of gauge-field factors (along paths between the two fermion points) with gauge-field dependent weights is formed. Thus it appears much more reasonable to start with gauge invariant P from the very beginning.

The only gauge invariant function which has so far been extensively studied in literature is the Wilson loop in the absence of fermions. With respect to dynamical mass generation the propagators of the full theory are of interest (including the case of infinite mass this could

describe confinement too). For the gauge field one can simply consider the correlation between two plaquettes (minimal Wilson loops) which in the continuum limit means to deal with field strength rather than with vector potentials. For the fermions, however, one has necessarily to include gauge field factors in P too. The most natural choice is a product $\mathcal{P}_{nn'}^\omega$ of them along a path ω from n' to n such that $\bar{\psi}_n \mathcal{P}_{nn'}^\omega \psi_{n'}$ is invariant. The generalization of $P_0 = \psi_{n'\beta'} \bar{\psi}_{n\beta}$ of the free case is then

$$P = - \bar{\psi}_{n\beta} \mathcal{P}_{n[\beta]n'[\beta']}^\omega \psi_{n'\beta'} \quad (7.9)$$

where $[\beta]$, $[\beta']$ mean internal symmetry indices only. There is in general, of course, a large number of possible paths from n' to n . To restrict it an immediate requirement is $\mathcal{P}_{nn}^\omega = 1$. Then it seems reasonable to exclude paths going through the lattice boundaries. Further one can admit only those of minimal length, which on the lattice, however, leaves still many degenerate possibilities. Thus there are some questions which deserve study in connection with the dynamics.

The structure of quantized gauge theory which emerges here leads to severe difficulties for the concept of axiomatic field theory to consider correlation functions (or Green's functions) as distributions. To point out how this comes about, first the case is considered where P depends on p variables $\phi_{\ell n}$. There, using test functions f on \mathbb{R}^{4p} , one can define $\langle P \rangle$ as distribution considering it as the linear continuous functional the action of which on the f is specified by

$$\int_{n^1, \dots, n^p} \langle P(\phi_{n^1 \ell^1}, \dots, \phi_{n^p \ell^p}) \rangle f(x^{(n^1)}, \dots, x^{(n^p)}) \quad (7.10)$$

For the continuum limit the fact can be exploited that if (7.10) converges for all test functions, the limit itself defines a distribution. Thus the connection can be established using only the weak topology of distributions. In appropriate examples it can be explicitly checked that (7.10) gives the usual expressions. However, all this does not work for gauge theory because of two reasons. Firstly, for correlation functions as the fermion propagator or the Wilson loop the number of gauge-field factors becomes infinite. This would mean $p \rightarrow \infty$ in (7.10) in which case the distribution formalism is no longer applicable. Secondly, because gauge invariance requires these factors to be along a path or a loop, the n -summation in (7.10) is no longer possible.

VIII. WARD-TAKAHASHI IDENTITIES

To get Ward-Takahashi identities the integration variables in $\int e^{-S} \mathcal{I} \mathcal{P}$ are transformed. An invariant factor I is included for later convenience. The transformations to be considered have in common that the integration measure is invariant with respect to them. From the transformed expressions the identities of interest follow by applying appropriate derivatives.

In the case of the transformation V defined by (3.10) the derivative $\frac{\delta^{\dagger V}}{\delta \lambda_n}$ is used which is related to V_n in the same way as $\frac{\delta^{\dagger U}}{\delta \lambda_n}$ in (3.7) is to U_{λ_n} . If only the fermion variables are transformed, i.e., for $\psi'_n = V_n \psi_n$, $\bar{\psi}'_n = \bar{\psi}_n V_n^{\dagger}$ one obtains

$$\int_{\psi'} e^{-S} I \left(P(S \frac{\delta^{\dagger V}}{\delta \lambda_n}) - P \frac{\delta^{\dagger V}}{\delta \lambda_n} \right) = 0 \quad . \quad (8.1)$$

By evaluation of the derivatives, using the chain rules for Grassmann variables, (8.1) expressed in $\psi_n, \bar{\psi}_n$ becomes

$$\int_{\psi} e^{-S} \mathbb{I} \left[P \left(\sum_{\lambda} (\overset{\leftarrow}{J}_{\lambda n}^{\ell} - \vec{J}_{\lambda, n-\lambda}^{\ell}) / a_{\lambda} - \sum_{\lambda} (\overset{\leftarrow}{j}_{\lambda n}^{\ell} - \vec{j}_{\lambda, n-\lambda}^{\ell}) / a_{\lambda} \right) - \frac{i}{v} \sum_{\beta} \left((\overset{\rightarrow}{\delta}_{n\beta}^{\psi} P) (T^{\ell} \psi_n)_{\beta} + (\bar{\psi}_n T^{\ell})_{\beta} \overset{\rightarrow}{\delta}_{n\beta}^{\bar{\psi}} P \right) \right] = 0 . \quad (8.2)$$

By (3.5) this gives the Ward-Takahashi identity with the quantities (4.4) and (4.5) which occur in the equations of motion too. Alternatively one can transform only the gauge field variables, i.e., consider

$U'_{\lambda n} = V_{n+\lambda} U_{\lambda n} V_n^{\dagger}$, which leads to

$$\int_{U'} e^{-S} \mathbb{I} \left(P(S \overset{\leftarrow}{\delta}_{\ell n}^{\leftarrow V}) - P \overset{\leftarrow}{\delta}_{\ell n}^{\leftarrow V} \right) = 0 . \quad (8.3)$$

By using the relation $Q \overset{\leftarrow}{\delta}_{\ell n}^{\leftarrow V} = \sum_{\lambda} \left(Q \overset{\leftarrow}{\delta}_{\ell \lambda n}^{\leftarrow U} - \overset{\rightarrow}{\delta}_{\ell \lambda, n-\lambda}^{\leftarrow U} Q \right)$, and after the evaluation of the derivatives, (8.3) gets the form

$$\int_U e^{-S} \mathbb{I} \left[P \left(- \sum_{\lambda} (\overset{\leftarrow}{J}_{\lambda n}^{\ell} - \vec{J}_{\lambda, n-\lambda}^{\ell}) / a_{\lambda} + \sum_{\lambda} (\overset{\leftarrow}{j}_{\lambda n}^{\ell} - \vec{j}_{\lambda, n-\lambda}^{\ell}) / a_{\lambda} \right) - \frac{1}{v} \sum_{\lambda} \left(P \overset{\leftarrow}{\delta}_{\ell \lambda n}^{\leftarrow U} - \overset{\rightarrow}{\delta}_{\ell \lambda, n-\lambda}^{\leftarrow U} P \right) \right] = 0 . \quad (8.4)$$

If the full transformation (3.9) is used, instead of (8.2) and of (8.4) one obtains

$$\int e^{-S} I \left[\frac{1}{v} \sum_{\lambda} (P \overset{\leftarrow}{\delta}_{\ell\lambda n}^U - \overset{\rightarrow}{\delta}_{\ell\lambda, n-\lambda}^U P) \right. \\ \left. + \frac{i}{v} \sum_{\beta} \left((\overset{\rightarrow}{\delta}_{n\beta}^{\psi} P) (T^{\ell} \psi_n)_{\beta} + (\bar{\psi}_n T^{\ell})_{\beta} \overset{\rightarrow}{\delta}_{n\beta}^{\bar{\psi}} P \right) \right] = 0 \quad , \quad (8.5)$$

which gives the Ward-Takahashi identity involving no currents.

For (8.2) the relation to the corresponding continuum equations is obvious. It gives the proper definition of the formal path integral relations there in the sense indicated in Sec. II. Putting $I = \phi$ according to (7.3) gauge fixing can be readily introduced. Further one may combine $\vec{J}_{\lambda n}^{\ell}$ with $\vec{J}_{\lambda n}^{\ell}$ and $\overset{\leftarrow}{J}_{\lambda n}^{\ell}$ with $\overset{\leftarrow}{J}_{\lambda n}^{\ell}$ to come closer to the usual appearance. The analogue of (8.4) is not used in continuum theory. To obtain correspondence of (8.5) to continuum relations one has to introduce gauge fixing, i.e., to put $I = \phi$, to replace P by \not{P} , and to apply (7.3).

It is instructive to consider (8.5) in the simplest case of interest with and without gauge fixing. For $I = \phi$ and $P = \not{\psi}_n, \beta, \bar{\psi}_n, \beta''$ one has

$$\int e^{-S} \phi \not{P} \left[\frac{1}{v} \sum_{\lambda} \left((\not{\ell n}) \overset{\leftarrow}{\delta}_{\ell\lambda n}^U - \overset{\rightarrow}{\delta}_{\ell\lambda, n-\lambda}^U (\not{\ell n}) \right) \psi_n, \beta, \bar{\psi}_n, \beta'' \right. \\ \left. + \frac{i}{v} \left(- \delta_{n n'} (T^{\ell} \psi_n)_{\beta} \bar{\psi}_n, \beta'' + \delta_{n n''} \psi_n, \beta (\bar{\psi}_n, T^{\ell})_{\beta''} \right) \right] = 0 \quad . \quad (8.6)$$

By (7.3) this gives the identity which corresponds to the continuum one involving the divergence of the fermion-untruncated vertex function and fermion propagators multiplied by δ -functions. The gauge field dependence of the vertex function in (8.6) is seen to originate from \not{P} . On the other hand, for $I = 1$ and P given by (7.9) one gets

$$\int e^{-S} \left[\frac{1}{v} \sum_{\lambda} \left(\mathcal{P}_{n''[\beta'']n'[\beta']}^{\omega} \overset{\leftarrow}{\partial}_{\ell\lambda n}^U - \overset{\leftarrow}{\partial}_{\ell\lambda, n-\lambda}^U \mathcal{P}_{n''[\beta'']n'[\beta']}^{\omega} \right) \psi_{n'\beta'} \bar{\psi}_{n''\beta''} \right. \\ \left. + \frac{1}{v} \mathcal{P}_{n''[\beta'']n'[\beta']}^{\omega} \left(-\delta_{nn'} (T^{\ell} \psi_{n'})_{\beta'} \bar{\psi}_{n''\beta''} + \delta_{nn''} \psi_{n'\beta'} (\bar{\psi}_{n''\beta''})_{\beta''} \right) \right] = 0. \quad (8.7)$$

Since the derivatives in (8.7) can be immediately evaluated, in this case the identity becomes trivial.

The transformation $\psi'_n = \exp\{i\alpha_n\} \psi_n$, $\bar{\psi}'_n = \bar{\psi}_n \exp\{-i\alpha_n\}$, using the derivative $\partial/\partial\alpha_n$, leads to the identity

$$\int_{\psi} e^{-S} \mathbb{I} \left[\mathbb{P} \left(\sum_{\lambda} (J_{\lambda n} - J_{\lambda, n-\lambda})/a_{\lambda} - \sum_{\lambda} (j_{\lambda n} - j_{\lambda, n-\lambda})/a_{\lambda} \right) \right. \\ \left. - \frac{1}{v} \sum_{\beta} \left((\overset{\leftarrow}{\partial}_{n\beta}^{\psi P}) \psi_{n\beta} + \bar{\psi}_{n\beta} \overset{\leftarrow}{\partial}_{n\beta}^{\bar{\psi} P} \right) \right] = 0 \quad (8.8)$$

for the singlet current

$$J_{\sigma n} = \frac{1}{2} \left(\bar{\psi}_n \gamma_{\sigma} U_{\sigma n}^{\dagger} \psi_{n+\sigma} + \bar{\psi}_{n+\sigma} \gamma_{\sigma} U_{\sigma n} \psi_n \right) \quad (8.9)$$

The quantity

$$j_{\sigma n} = \frac{1}{2} \left(\bar{\psi}_n \eta U_{\sigma n}^{\dagger} \psi_{n+\sigma} - \bar{\psi}_{n+\sigma} \eta U_{\sigma n} \psi_n \right) \quad (8.10)$$

originates from X. The relation to the continuum equations is for (8.8) in the same sense as for (8.2) obvious. Similarly as in (4.9) one has the classical limits $J_{\sigma n} \rightarrow J_{\sigma}(x)$ and $j_{\sigma n} \rightarrow 0$.

In the case of the transformation $\psi'_n = V_n^5 \psi_n$, $\bar{\psi}'_n = \bar{\psi}_n V_n^5$ with $V_n^5 = \exp\{i\gamma_5 \sum_{\ell} T^{\ell} \alpha_n^{\ell}\}$ the derivative $\overset{\leftarrow}{\partial}_{\ell n}^{V5}$, differing from $\overset{\leftarrow}{\partial}_{\ell n}^V$ by having

$\exp\{i\gamma_5 T^\ell \varepsilon\}$ instead of $\exp\{iT^\ell \varepsilon\}$ in its definition, is used to get the V^5 -analogue of (8.1). Now, however, the contribution resulting from X can no longer be decomposed in a divergence-like manner. Its terms are therefore expressed by (4.11) and by the analogous equation with $\frac{\vec{\partial}\bar{\psi}}{\partial n^\lambda}$. Thus one arrives at the identity

$$\int_{\psi} e^{-S} \mathbb{I} \left[P \left(\sum_{\lambda} (\overset{\leftarrow}{J}_{\lambda n}^{5\ell} - \vec{J}_{\lambda, n-\lambda}^{5\ell}) / a_{\lambda} - 2im\bar{\psi}_n \gamma_5 T^\ell \psi_n - \frac{i}{v} \text{tr} \left(\gamma_5 T^\ell (GX + XG)_{nn} \right) \right) \right. \\ \left. - \frac{i}{v} \sum_{n', \beta} \left((\overset{\rightarrow}{\partial}_{n', \beta} \psi, P) \left((1+GX)_{n', n} \gamma_5 T^\ell \psi_n \right)_{\beta} - \left(\bar{\psi}_n \gamma_5 T^\ell (1+XG)_{nn'} \right)_{\beta} \frac{\vec{\partial}\bar{\psi}}{\partial n', \beta} P \right) \right] = 0 \quad (8.11)$$

with the axial currents

$$\vec{J}_{\lambda n}^{5\ell} = \frac{i}{2} (\bar{\psi}_n \gamma_{\lambda} \gamma_5 U_{\lambda n}^{\dagger} T^\ell \psi_{n+\lambda} + \bar{\psi}_{n+\lambda} \gamma_{\lambda} \gamma_5 T^\ell U_{\lambda n} \psi_n) \quad , \\ \overset{\leftarrow}{J}_{\lambda n}^{5\ell} = \frac{i}{2} (\bar{\psi}_n \gamma_{\lambda} \gamma_5 T^\ell U_{\lambda n}^{\dagger} \psi_{n+\lambda} + \bar{\psi}_{n+\lambda} \gamma_{\lambda} \gamma_5 U_{\lambda n} T^\ell \psi_n) \quad , \quad (8.12)$$

and where tr refers to γ -matrices in addition to the internal symmetry ones.

By the transformation $\psi'_n = \exp\{i\gamma_5 \alpha_n\} \psi_n$, $\bar{\psi}'_n = \bar{\psi}_n \exp\{i\gamma_5 \alpha_n\}$, using $\partial/\partial \alpha_n$, one gets in a completely analogous way the identity

$$\int_{\psi} e^{-S} \mathbb{I} \left[P \left(\sum_{\lambda} (J_{\lambda n}^5 - J_{\lambda, n-\lambda}^5) / a_{\lambda} - 2im\bar{\psi}_n \gamma_5 \psi_n - \frac{i}{v} \text{tr} \left(\gamma_5 (GX + XG)_{nn} \right) \right) \right. \\ \left. - \frac{i}{v} \sum_{n', \beta} \left((\overset{\rightarrow}{\partial}_{n', \beta} \psi, P) \left((1+GX)_{n', n} \gamma_5 \psi_n \right)_{\beta} - \left(\bar{\psi}_n \gamma_5 (1+XG)_{nn'} \right)_{\beta} \frac{\vec{\partial}\bar{\psi}}{\partial n', \beta} P \right) \right] = 0 \quad (8.13)$$

for the singlet axial current

$$J_{\lambda n}^5 = \frac{i}{2}(\bar{\psi}_n \gamma_\lambda \gamma_5 U_{\lambda n}^\dagger \psi_{n+\lambda} + \bar{\psi}_{n+\lambda} \gamma_\lambda \gamma_5 U_{\lambda n} \psi_n) \quad . \quad (8.14)$$

For (8.11) and (8.13) the relation to the corresponding continuum equations is again clear provided that one has in the limit

$$\frac{1}{v} \text{tr} \left(\gamma_5 T^\ell (GX + XG)_{nn} \right) \rightarrow \frac{g^2}{16\pi^2} \text{Tr} \left(T^\ell \sum_{\mu\nu\lambda\rho} \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x) \right) \quad , \quad (8.15)$$

$$\frac{1}{v} \text{tr} \left(\gamma_5 (GX + XG)_{nn} \right) \rightarrow \frac{g^2}{16\pi^2} \text{Tr} \left(\sum_{\mu\nu\lambda\rho} \epsilon_{\mu\nu\lambda\rho} F_{\mu\nu}(x) F_{\lambda\rho}(x) \right) \quad , \quad (8.16)$$

$$\frac{1}{v} (1 + GX)_{n,n} \quad , \quad \frac{1}{v} (1 + XG)_{n,n} \rightarrow \delta^4(x' - x) \quad , \quad (8.17)$$

i.e., one has to get the Adler-Bell-Jackiw term⁹ in (8.16), the result of Bardeen²⁸ in (8.15), and the same limit as for $\frac{1}{v} \delta_{n,n}$ alone in (8.17). For the degeneracy regularizations of Wilson⁴ and of Osterwalder and Seiler¹⁸ it has been recently shown⁷ that (8.16) and (8.17) indeed hold. The presented demonstration⁷ generalizes to the case of the regularization (3.3) with arbitrary $z \neq 0$. In addition it can be modified to give (8.15) too.

The limits of the identities for correlation functions given by (8.11) and (8.13) with (3.5) are again to be considered in the sense of Sec. II, and in particular provide the proper definition of the corresponding formal path integral relations of continuum theory. Thus, for example, (8.13) is the adequate description of what has been considered by Fujikawa.²⁹ With respect to (8.15) and (8.16) it is to be realized

that the limit is only the naive one for classical gauge fields. $F_{\mu\nu}(x)$ and $F_{\lambda\rho}(x)$ there arise as the limit of averages of four quantities⁷ which are related to the four plaquettes having the point x in common. In the full quantum case these quantities depend, of course, on the gauge-field variables. With respect to the currents it is to be noted that classically one has again $\vec{J}_{\lambda n}^{5\ell}, \overleftarrow{J}_{\lambda n}^{5\ell} \rightarrow J_{\lambda}^{5\ell}(x)$ and $J_{\lambda n}^5 \rightarrow J_{\lambda}^5(x)$.

The currents considered here are seen to be related to the links of the lattice just as the gauge fields are. Thus their nonlocality does not exceed the minimally necessary one. On the other hand, it is remarkable that they automatically get forms which are reminiscent of the pointsplitting ones introduced³⁰ in continuum theory to overcome difficulties. The regularization quantities $\vec{J}_{\lambda n}^{\ell}, \overleftarrow{J}_{\lambda n}^{\ell}, j_{\lambda n}$ may be combined with the respective currents. It is, however, to be noted that they inherit the property (3.11) of the degeneracy regularization. Thus, while one has $J_{kn}^* = J_{kn}$ for $k = 1, 2, 3$ and $J_{4n}^* = -J_{4n}$ for the currents, which with $J_{4n} = iJ_{0n}$ gives $J_{0n}^* = J_{0n}$, for the regularization quantities one gets $j_{\lambda n}^* = j_{\lambda n}$ for $\lambda = 1, \dots, 4$ and thus with $j_{4n} = ij_{0n}$ the result $j_{0n}^* = -j_{0n}$.

IX. METHODS FOR DYNAMICAL MASSES

The usual method¹⁰ in continuum theory to investigate dynamical mass generation is based on Schwinger-Dyson equations and in addition uses Ward-Takahashi identities. The solution of the integral equations presents, however, still considerable problems.³¹ Therefore, having now the corresponding equations on the lattice it appears worthwhile to study the respective possibilities there, since everything then is well

defined.

To get the analogues of the usual equations first the $\vec{\bar{\psi}}_{n,\beta}$ version of (7.7) with $P = \bar{\psi}_{n\beta}$ is considered which is

$$\langle (\vec{\bar{\psi}}_{n,\beta}, S) \bar{\psi}_{n\beta} \rangle_f = \delta_{n',n} \delta_{\beta',\beta} \quad . \quad (9.1)$$

Introducing $\mathcal{G}_{n',\beta',n\beta} = \langle \psi_{n',\beta'} \bar{\psi}_{n\beta} \rangle_f$, (9.1) can be written $\mathcal{P}^{-1} \mathcal{G} + \mathcal{V} = 1$, or after multiplication with \mathcal{G}^{-1} ,

$$\mathcal{G}^{-1} = \mathcal{P}^{-1} + \mathcal{V} \mathcal{G}^{-1} \quad , \quad (9.2)$$

where \mathcal{P} is the free propagator given by (5.1) and

$$\begin{aligned} \mathcal{V}_{n',\beta',n\beta} = & \nu \sum_{\lambda} \left(\left\langle \left((U_{\lambda n'}^{\dagger} - 1) (\gamma_{\lambda} - \eta) \psi_{n'} + \lambda \right)_{\beta'} \bar{\psi}_{n\beta} \right\rangle_f \right. \\ & \left. - \left\langle \left((U_{\lambda, n'} - \lambda - 1) (\gamma_{\lambda} + \eta) \psi_{n'} - \lambda \right)_{\beta'} \bar{\psi}_{n\beta} \right\rangle_f \right) / (2a_{\lambda}) \quad . \end{aligned} \quad (9.3)$$

The basic elements of (9.3) are three-point functions of the form

$$\langle Y_{\lambda n}^{\ell} \psi_{n',\beta'} \bar{\psi}_{n''\beta''} \rangle_f \quad \text{with } Y_{\lambda n}^{\ell} = 2\text{Tr}(T^{\ell}(U_{\lambda n} - 1)/a_{\lambda}) \quad \text{and similarly for } U_{\lambda n}^{\dagger}.$$

To relate these to vertex functions one has to put

$$\langle Y_{\lambda n}^{\ell} \psi_{n',\beta'} \bar{\psi}_{n''\beta''} \rangle_f = \sum \mathcal{D}_{\lambda n \bar{\lambda} n}^{\ell \bar{\ell}} \mathcal{G}_{n',\beta',\bar{n}',\bar{\beta}'} \Gamma_{\lambda n n',\bar{\beta}',\bar{n}''\bar{\beta}''}^{\bar{\ell}} \mathcal{G}_{\bar{n}''\bar{\beta}'' n''\beta''} \quad (9.4)$$

where one may choose $\mathcal{D}_{\lambda n \bar{\lambda} n}^{\ell \bar{\ell}} = \langle \mathcal{A}_{\lambda n}^{\ell} \mathcal{A}_{\bar{\lambda} n}^{\bar{\ell}} \rangle_f$ with

$$\mathcal{A}_{\lambda n}^{\ell} = 2\text{Tr}(T^{\ell}(U_{\lambda n} - U_{\lambda n}^{\dagger})) / (2iga_{\lambda}) \quad \text{to be close to continuum theory.}$$

The usual approximation corresponds then to setting

$$\Gamma_{\lambda n n',\beta',n''\beta''}^{\ell} \approx \text{const.} (T^{\ell} \gamma_{\lambda})_{\beta',\beta''} \delta_{nn'} \delta_{nn''} \quad . \quad (9.5)$$

If (9.4) with (9.5) is inserted, (9.2) becomes the analogue of the equation on which the usual method^{10,31} is based.

It is obvious that gauge fixing is essential in the indicated approach. Therefore, from the lattice point of view as discussed in Sec. VII it is not natural. In addition the ansatz (9.4) with the approximation (9.5) is deeply rooted in perturbation theory and can hardly be justified at the nonperturbative level. The Schwinger-Dyson equation associated with $\mathcal{D}_{\lambda n \lambda \bar{n}}^{\ell \bar{\ell}}$, namely (7.8) with $P = \mathcal{A}_{\lambda' n'}^{\ell'}$, due to the compactness of the gauge field only in the limit gets a workable form. Thus on the lattice the method becomes rather unattractive within several respects.

Following the usual line of argument¹⁰ further, to conclude via the Ward-Takahashi identity from the fermion propagators with mass about a vertex pole, (8.6) is to be considered. Then the obtained knowledge is to be used for the vertex in the gauge-field Schwinger-Dyson equation (7.8) with $P = \mathcal{A}_{\lambda' n'}^{\ell'}$, i.e., in the three-point function

$$\left\langle \frac{v}{a_\lambda} \left(-\tilde{J}_{\lambda n}^\ell + \tilde{J}_{\lambda n}^{\bar{\ell}} \right) \mathcal{A}_{\lambda' n'}^{\ell'} \right\rangle, \quad (9.6)$$

to come towards a vector boson mass. However, the gauge field dependences of the vertex functions contained in (8.6) and (9.6) are of a rather different nature, and in (9.3) one has even a third version of this phenomenon. The relation between these functions is thus scarcely useful for practical purposes. From a more general point of view one sees that familiar continuum quantities in a nonperturbative formulation can have different analogues in different situations.

A central question for concrete calculations which remains is

how to deal with fermions. Actually, due to the bilinearity of S in the Grassmann variables, $\int_{\psi} e^{-S_f}$, with S_f denoting the fermion part of S , is nothing else than $\det(vG^{-1})$, $\int_{\psi} e^{-S_f} \psi_{n'\beta'} \bar{\psi}_{n\beta}$ is its minor without row n' , β' and column n, β , and the corresponding integrals with more pairs $\psi \bar{\psi}$ similarly are minors with more rows and columns deleted. In contrast to their conceptual simplicity, however, the actual solution of determinants is a major problem, in particular if the result is needed for the subsequent gauge field integration. A selection of the occurring contributions according to their gauge field content appears therefore advantageous. To get it one has to note that one can write

$$S_f = \sum_{n, \beta', \beta} \left(\sum_{\lambda} (\bar{u}_{\lambda n \beta' \beta} - u_{\lambda n \beta' \beta}) + u_{n \beta' \beta} \right) \quad (9.7)$$

where

$$\begin{aligned} u_{\lambda n \beta' \beta} &= v \bar{\psi}_{n+\lambda, \beta'} \left((\gamma_{\lambda} + \eta) U_{\lambda n} \right)_{\beta' \beta} \psi_{n\beta} / (2a_{\lambda}) \quad , \\ \bar{u}_{\lambda n \beta' \beta} &= v \bar{\psi}_{n\beta'} \left((\gamma_{\lambda} - \eta) U_{\lambda n}^{\dagger} \right)_{\beta' \beta} \psi_{n+\lambda, \beta'} / (2a_{\lambda}) \quad , \\ u_{n \beta' \beta} &= v \bar{\psi}_{n\beta'} \left(m + \eta \sum_{\lambda} \frac{1}{a_{\lambda}} \right)_{\beta' \beta} \psi_{n\beta} \quad . \end{aligned} \quad (9.8)$$

This allows the representation

$$e^{-S_f} = \prod_{n, \beta', \beta} \left(\prod_{\lambda} (1 - \bar{u}_{\lambda n \beta' \beta}) (1 + u_{\lambda n \beta' \beta}) \right) (1 - u_{n \beta' \beta}) \quad . \quad (9.9)$$

From (9.9) only the products contribute to $\int_{\psi} e^{-S_f}$ which contain all components $\psi, \bar{\psi}$ (and only once). Thus visualizing $u_{\lambda n \beta' \beta}$, $\bar{u}_{\lambda n \beta' \beta}$ and

$\psi_{n\beta}, \beta$ as depicted in Fig.3 one must have $4N$ ingoing and $4N$ outgoing lines at each point, where N denotes the number of internal symmetry components. Similarly to $\int_{\psi} e^{-S_f} \psi_{n,\beta}, \bar{\psi}_{n\beta}$ only those products from (9.9) with all components except $\psi_{n,\beta}$, and $\bar{\psi}_{n\beta}$ give a contribution.

For simplicity the discussion of the emerging picture is now restricted to $N = 1$. It is seen that the contributing paths formed by the elements of Figs. 3a,b are just the ones allowed for factors $U_{\lambda n}, U_{\lambda n}^{\dagger}$ by gauge invariance. Actually, these paths are even there without the gauge field and have then equal weight. In the presence of the gauge field they are weighted according to their dependence on the latter and to the coupling strength. In Fig. 4 it is illustrated that loops to compensate a Wilson loop and to invalidate the area law argument¹ are readily available.

An example of a contribution to the fermion propagator is shown in Fig. 5. The path from the minor completes ω given by (7.9) to a loop. Thus for stronger coupling minor paths close to ω are preferred. It is to be remembered that the freedom in the choice of $\mathcal{P}_{nn}^{\omega}$, is what remains here from the gauge dependence of the propagator in continuum theory. With respect to a dynamically generated mass the implications of this choice are of prime interest. It is felt that the indicated picture can provide a guide for the development of quantitative methods.

X. SITUATION FOR WEAK INTERACTIONS

The fact that the fermion degeneracy regularization breaks chiral symmetry has so far been an obstacle to extending the lattice formula-

tion to weak interactions and thus to the electroweak theory. This is unsatisfactory from the practical as well as from the conceptual point of view. Thus it appears necessary to look more carefully where the problem is actually located.

The central point to be realized is the connection between the degeneracy regularization and the axial-vector anomaly,⁹ which has been shown in perturbation theory at the one-loop level by Karsten and Smit^{13,14} and which has been established independently of perturbation theory in a general way by the present author.⁷ In Sec. VIII it is manifest by the limits (8.16) and (8.17) in Eq. (8.13). Because one must have the anomaly term, one is thus forced to break chiral invariance. There is no contradiction to the SLAC approach³², which starts in a chiral invariant way, since for the crucial test objects, fermion loop and axial-vector current, the limit there cannot be performed without further specification.³³ In this context a degeneracy regularization may be viewed as a parametrized limit prescription (technically somewhat similar, for example, to the $i\epsilon$ -prescription for Green's functions).

Next it has to be noted that also in continuum theory the notion of chiral symmetry is to be qualified. For example, the chiral U(1) symmetry in QCD with massless quarks is one without a conserved gauge-invariant current. Thus this symmetry is spoiled if one insists on gauge invariance as one has to do according to the discussion in Sec. VII. With respect to perturbation theory it is to be remembered that to avoid anomalies means to build the theory such that the anomalous contributions cancel³⁴, and, of course, not that the individual terms are not there. Therefore, it is not surprising that the lattice formulation, being non-

perturbative, shows these features in a more detailed form. This form has here the virtue that the effects of the anomaly are explicitly prescribed by the degeneracy regularization.

Then it is to be observed that what forbids putting the degeneracy regularization into the electroweak action is nothing new but just what does not allow mass terms there. Following the conventional way this would lead to inserting the regularization via a Higgs coupling too. To circumvent the nonrenormalizability of this coupling one could use a subsequent limit of parameters^{13,35} which in the present notation amounts to letting a_λ in X go to zero later. The unpleasant features of this way are, apart from the somewhat artificial limit, the same as for the masses. There are the reasons^{5,6} making elementary Higgs fields unwelcome which need not be repeated here. Further, to avoid trivially vanishing expectation values of the Higgs fields, gauge fixing is needed, which is not natural on the lattice. Thus, summing up, just as for the masses,^{6,36} looking for a dynamical mechanism appears more reasonable for the degeneracy regularization too, though this is certainly still more difficult.

ACKNOWLEDGEMENTS

I wish to thank S. Drell and the SLAC Theoretical Physics Group for their kind hospitality. I am grateful for many stimulating discussions with colleagues here. This work was supported in part by the Deutsche Forschungsgemeinschaft and in part by the Department of Energy, contract DE-AC03-76SF00515.

REFERENCES

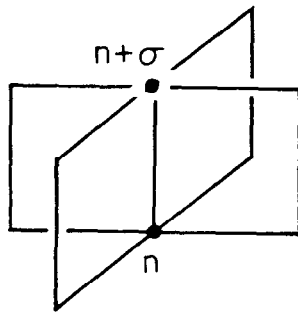
1. K. G. Wilson, Phys. Rev. D10, 2445 (1974).
2. M. Creutz, Phys. Rev. Lett. 45, 313 (1980); Phys. Rev. D21, 2308 (1980); and references cited therein.
3. B. Svetitsky, S. D. Drell, H. R. Quinn, and M. Weinstein, Phys. Rev. D22, 490 (1980); 22, 1190 (1980); J. Smit, Nucl. Phys. B175, 307 (1980); J. Greensite and J. Primack, Nucl. Phys. B180, 170 (1981); J.-M. Blairon, R. Brout, F. Englert and J. Greensite, University of Brussels preprint (September 1980).
4. K. G. Wilson, in Proceedings of the International School, Erice, 1975, ed. by A. Zichichi (Plenum, New York, 1977), Part A, p. 69.
5. S. Weinberg, Phys. Rev. D19, 1277 (1979).
6. L. Susskind, Phys. Rev. D20, 2619 (1979).
7. W. Kerler, Phys. Rev. D (to be published), and SLAC-PUB-2634.
8. W. Kerler, Phys. Lett. B (to be published), and SLAC-PUB-2646.
9. S. L. Adler, Phys. Rev. 177, 2426 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
10. R. Jackiw and K. Johnson, Phys. Rev. D8, 2386 (1973); J. M. Cornwall and R. E. Norton, Phys. Rev. D8, 3338 (1973); F. Englert, J.-M. Frère, P. Nicoletopoulos, Phys. Lett. 52B, 433 (1974); 59B, 346 (1975).
11. K. Symanzik, J. Math. Phys. 7, 510 (1966); lectures, Varenna 1968, ed. by R. Jost (Academic Press, New York, 1969), p. 152; K. Osterwalder and R. Schrader, Commun. Math. Phys. 31, 83 (1973); 42, 281 (1975).

12. H. S. Sharatchandra, Phys. Rev. D18, 2042 (1978).
13. L. H. Karsten, Ph.D. Thesis, Amsterdam (1979).
14. L. H. Karsten and J. Smit, Stanford University preprint, ITP-677 (1980).
15. T. D. Schultz, D. C. Mattis and E. H. Lieb, Rev. Mod. Phys. 36, 856 (1964).
16. N. Wiener, J. Math. and Phys. 58, 131 (1923); V. P. Maslov, A. M. Chebotarev, Sov. Math. Dokl. 17, 975 (1976); Theor. Math. Phys. 28, 793 (1976).
17. C. de Witt-Morette, Commun. Math. Phys. 28, 47 (1972); 37, 63 (1974); S. Albeverio, R. Høegh-Krohn, Lecture Notes in Mathematics 523 (Springer, Berlin, Heidelberg, New York, 1976).
18. K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110, 440 (1978).
19. F. A. Berezin, The Method of Second Quantization (Academic Press, New York, 1966).
20. D. Foerster, Phys. Lett. 87B, 87 (1979); T. Eguchi, Phys. Lett. 87B, 91 (1979); D. Weingarten, Phys. Lett. 87B, 97 (1979); 90B, 280 (1980).
21. A. Chodos and J. B. Healy, Nucl. Phys. B127, 426 (1977).
22. T. Muir, The Theory of Determinants (Dover Publ., New York, 1960).
23. L. D. Faddeev and V. N. Popov, Phys. Lett. 25B, 29 (1967).
24. M. Creutz, Phys. Rev. D15, 1128 (1977).
25. V. N. Gribov, Nucl. Phys. B139, 1 (1978).
26. W. Kerler, Phys. Lett. 76B, 423 (1978).

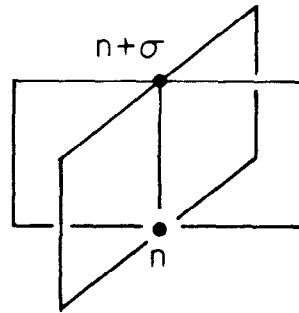
27. T. N. Todoron, Phys. Rev. D21, 2348 (1980); N. H. Christ and T. D. Lee, Phys. Rev. D22, 939 (1980).
28. W. A. Bardeen, Phys. Rev. 184, 1848 (1969).
29. K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D21, 2848 (1980).
30. J. Schwinger, Phys. Rev. 82, 664 (1951); K. Johnson, Nucl. Phys. 25, 431 (1961).
31. H. Pagels, Phys. Rev. D21, 2336 (1980).
32. S. D. Drell, M. Weinstein and S. Yankielowicz, Phys. Rev. D14, 1627 (1976).
33. J. M. Rabin, SLAC-PUB-2676, February 1981.
34. C. Bouchiat, J. Iliopoulos, and Ph. Meyer, Phys. Lett. 38B, 519 (1972); D. J. Gross and R. Jackiw, Phys. Rev. D6, 477 (1972).
35. J. Smit, private communication.
36. For a recent review of such developments see: P. Sikivie, CERN preprint TH-2951 (October 1980).

FIGURE CAPTIONS

- Fig. 1. Gauge field factors around plaquettes illustrated in 3 dimensions for (a) Eq. (4.3a), and (b) Eq. (4.3b).
- Fig. 2. Deformations in the $\sigma\lambda$ -plane of a particular Wilson loop (situated in the $\sigma\lambda$ -plane and going through the link from n to $n+\sigma$) within the Schwinger-Dyson equation (4.2) due to the terms from (4.3a) of (a) $\mathcal{F}_{\sigma\lambda}^{[4]}$, (b) $\mathcal{F}_{\sigma\lambda, n-\lambda}^{[3]}$, and (c) $\vec{J}_{\sigma n}$.
- Fig. 3. Graphical representation of the quantities (a) $u_{\lambda n\beta'\beta}$, (b) $\bar{u}_{\lambda n\beta'\beta}$, and (c) $u_{n\beta'\beta}$ in Eq. (9.8).
- Fig. 4. Example of a loop from the fermion determinant which can compensate a Wilson loop.
- Fig. 5. Example of a path from the fermion minor which completes the prescribed path ω (in dashed line) of $\mathcal{P}_{nn}^{\omega}$, of the fermion propagator to a loop (the enclosed area is shaded).



3-81 (a)



(b) 4069A1

Fig. 1

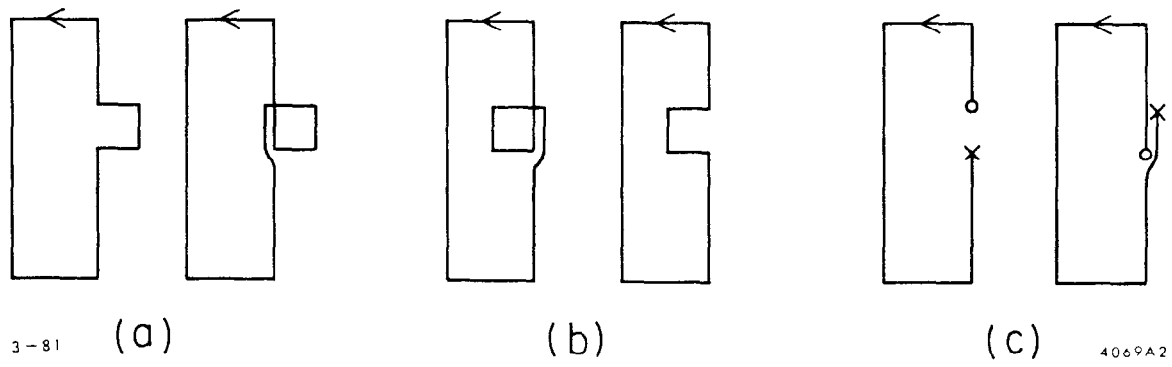


Fig. 2

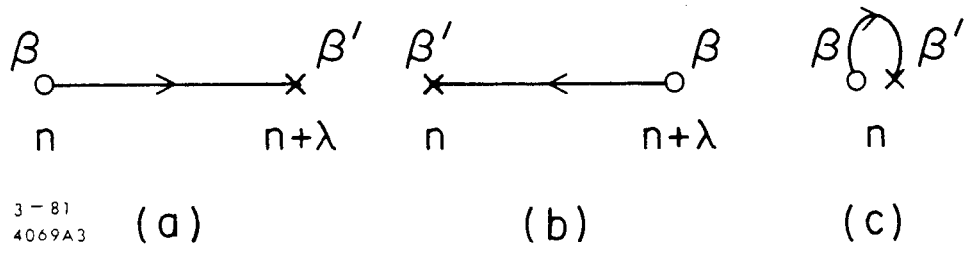
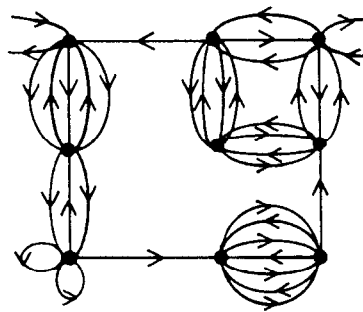


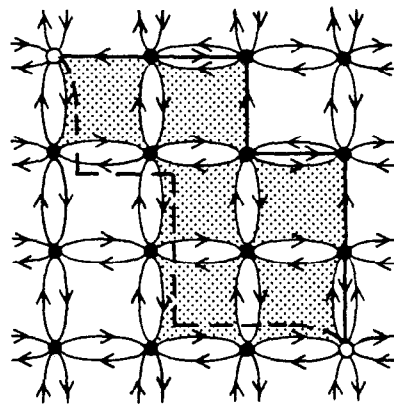
Fig. 3



3 - 81

4069A4

Fig. 4



3 - 81

4069A5

Fig. 5