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ASYMPTOTIC FREEDOM*

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Maurice Goldhaber first introduced me to the physics of the nucleus in 1946 at the University of Illinois, when the mesons of Yukawa were viewed as supplying the nuclear "glue" and the challenge to understand the saturation of nuclear forces was the analog of today's studies of "asymptotic freedom." Nuclear physics was then still in a relatively primitive state, but in his discussions and formal presentations, Goldhaber always traversed the path directly to the basic physical ideas. It is in that spirit that I dedicate to Maurice Goldhaber this discussion of asymptotic freedom in quantum chromodynamics (QCD) for his 70th birthday festschrift.

Asymptotic freedom means the weakening of the effective color charge in QCD at small distances. We are familiar with the opposite behavior in quantum electrodynamics (QED), in which the effective charge, or interaction strength, increases at short distances. This QED behavior is understood as a normal shielding phenomenon in a quantum description of the interaction of charged virtual particle–antiparticle pairs with the radiation field. These virtual pairs are polarized in the neighborhood of a charged source, thereby creating a shielding cloud. A charge viewed (i.e., probed) from a distance greater than the electron Compton wavelength of about 10^{-11} cm will be shielded by this polarized cloud of virtual electron–positron pairs; within this cloud, a probe will "see" the stronger bare charge e_0 (FIGURE 1):

$$e_0 \equiv \left| \int \rho_0(r) d^3r \right| > \left| \int [\rho_0(r) + \rho_p(r)] d^3r \right| \equiv |e.$$

Precise experimental confirmation of this shielding phenomenon has existed for 30 years since very early measurements of the Lamb shift in the hydrogen fine structure.[1]

The analogous shielding of the color charge occurs in QCD. In this case, we say that virtual quark pairs are polarized and shield the color charge sources; the more different flavors of quarks there are, the greater the shielding. However, a new physical phenomenon occurs in QCD and is the source of the "asymptotic freedom" behavior.[2] Expressed most simply in rough physical terms, QCD is a non-Abelian gauge theory, and by contrast to QED, its quantums (the gluons) themselves carry the color charges, thereby spreading out the color charge of the sources of matter. As is well known, the interaction between two distributed static charges is weaker than that between two point charges of the same strength when their charge clouds overlap one another. Herein lies the origin of asymptotic freedom in QCD.

In more precise terms, the interaction between two static point charges is exactly Coulomb in QED if only the radiation field is quantized, but the charged particles are treated classically, so that no shielding cloud of virtual pairs is created. However, in

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QCD we no longer can introduce the concept of static point charges. A point charge may be anchored in space as infinitely massive, but its different color components will fluctuate in the internal symmetry, or color, space as a result of the commutation relations of non-Abelian gauge theory. There is an analog of this behavior in nonrelativistic quantum mechanics: the individual spin vectors cannot both be fixed in a system of two interacting spins, namely, if

$$H = A\mathbf{J}_1 \cdot \mathbf{J}_2$$

$$\dot{\mathbf{J}}_1 \equiv i[\mathbf{J}_1, H] = A\mathbf{J}_1 \times \mathbf{J}_2 \neq 0 \tag{1}$$

$$\dot{\mathbf{J}}_2 \equiv i[\mathbf{J}_2, H] = -\dot{\mathbf{J}}_1 \neq 0.$$

Thus, in QCD with two fixed color charges, which we represent as vectors in the internal symmetry (color) space, we have

$$\rho(x) \equiv \rho_1(x) + \rho_2(x)$$

$$\dot{\rho}_1 = i[\rho_1, H_{\mathrm{QCD}}] \neq 0 \tag{2}$$

$$\dot{\rho}_2 = i[\rho_2, H_{\mathrm{QCD}}] \neq 0.$$

Equation 2 expresses the non-Abelian character of QCD. The fact that QCD is a gauge theory tells us that we have a differential law of current conservation relating

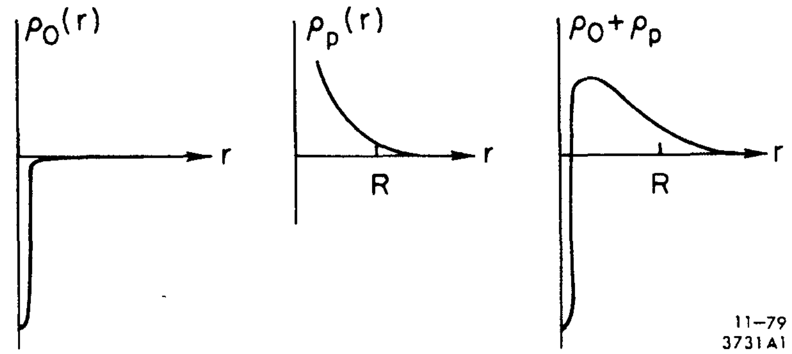

FIGURE 1. Polarization contribution to the charge density. ρ_o is the charge density of the bare charge, and ρ_p is the charge density of the induced polarization cloud of virtual pairs.

the time change of the (color) charge density with the divergence of a local current. Thus, for two (approximately) nonoverlapping charges, we can write in QCD

$$\dot{\rho}_1 + \mathrm{div}\,\mathbf{J}_1 = 0$$

$$\dot{\rho} + \mathrm{div}\,\mathbf{J}_2 = 0. \tag{3}$$

It is now evident from Equations 3 that, even though we may introduce two fixed point charges ρ_1 at x_1 and ρ_2 at $x_2 \neq x_1$, these point charges will necessarily be surrounded by current distributions. Furthermore, these current distributions cannot be localized since they are carried by the massless gluons of the color gauge field. Inevitably, the color sources $\rho(x)$ are spread out in space, and we are describing the interaction of two distributed charge distributions. This follows necessarily from the non-Abelian (Equations 2) and gauge nature (Equations 3) of QCD. The net effect is that in QCD, we are necessarily describing the interaction between two distributed charges, and, as is well known, when they overlap, the interaction is reduced in strength and, relative to the Coulomb force, grows weaker as their separation decreases.

This is the end of the physical story about why the running coupling constant in QCD may weaken at small distances between two fixed (but not static) point charges. This antishielding correction, along with a partially compensating but smaller shielding, appears because of fluctuations of the color gauge field. The normal shielding contributions due to the matter field, as discussed earlier, will also occur when the matter field is quantized. If there are too many degrees of freedom (flavors of quarks) in the matter field, the shielding will dominate, as is well known, and "asymptotic freedom" will be lost. However, the basic and dominant antishielding effect is due to spreading out of the sources of color charge due to quantum effects in the gauge field. They arise and can be studied, both formally and physically, without including the pair effects of the matter field, which can be described simply in terms of heavy sources anchored to fixed points in space.[3]

Our starting point is the Hamiltonian of the color gauge field (gluons) plus fixed sources of color charge:

$$\mathrm{H} = \int d^3x\, \mathcal{H}(x)$$

$$\mathcal{H}(x) = \frac{1}{2}\sum_{i=1}^{3} [\mathbf{E}_i^{\mathrm{T}} \times \mathbf{E}_i^{\mathrm{T}} + \mathbf{B}_i \times \mathbf{B}_i] - \frac{1}{2}\phi \cdot \nabla^2\phi. \tag{4}$$

This is identical in form to QED with static sources. There are, however, important differences, beyond the fact that the transverse electric field $\mathbf{E}_i^{\mathrm{T}}$, the magnetic field $\mathbf{B}_i$, and the scalar potential ϕ have different color components, here written in a vector notation (with $n^2 - 1$ components for an SU(n) of color; for notational simplicity, we assume that $n = 2$ and use a three-component vector notation in color space). The magnetic field, which in QED is the curl of the vector potential, is here given by a nonlinear relation

$$\mathbf{B}_i = \partial_j \mathbf{A}_k - \partial_k \mathbf{A}_j + g\mathbf{A}_j \times \mathbf{A}_k \qquad (i, j, k \text{ cyclic}) \tag{5}$$

and Gauss's law for the static potential ϕ, defined by $\mathbf{E}_i = E_i^{\mathrm{T}} - \partial_i\phi$, becomes

$$\begin{aligned} \nabla^2\phi &= -g\left[\rho - \sum_{i=1}^{3} \mathbf{A}_i \times \mathbf{E}_i\right] \\ &= -g\left[\rho - \sum_{i=1}^{3} (\mathbf{A}_i \times \mathbf{E}_i^{\mathrm{T}} - \mathbf{A}_i\partial_i \times \phi)\right]. \end{aligned} \tag{6}$$

We rewrite Equation 6 as

$$\nabla^2\phi = -g(1 + g\mathbf{A}_i\partial_i\Delta^{-1}\times)^{-1}(\rho - \mathbf{A}_i \times \mathbf{E}_i^{\mathrm{T}}), \tag{7}$$

where repeated indexes are summed and $\Delta \equiv \nabla^2$.

We work in radiation gauge: $\partial_i\mathbf{A}_i = 0$ and A_i, $\mathbf{E}_i^{\mathrm{T}}$ satisfy the familiar equal-time transverse commutation relations; that is,

$$[A_i^a(x, t), E_j^{\mathrm{T}b}(y, t)] = -i\delta_{ab}\delta_{ij}^{\mathrm{T}}(x - y). \tag{8}$$

In terms of the normal-mode expansion,

$$\mathrm{A}_i^a(x, t) = \sum_{\substack{l\\ \lambda=1,2}} \frac{\epsilon_i(l, \lambda)}{\sqrt{2lV}}\{C^a(l, \lambda)e^{i(l_j x_j - lt)} + \text{h.c.}\} \tag{9}$$

$$\sum_{i=1}^{3} l_i\epsilon_i(l, \lambda) = 0$$

$$\sum_{\lambda=1,2} \epsilon_i(l, \lambda)\epsilon_j(l, \lambda) = \delta_{ij} - \frac{l_i l_j}{l^2}$$

$$E_i^{\mathrm{T}a}(x, t) = -\dot{A}_i^a(x, t)$$

and

$$[C^{\dagger a}(l', \lambda'), C^b(l, \lambda)] = \delta_{ab}\delta_{\lambda\lambda'}\delta_{l,l'}.$$

The additional nonlinear terms in Equations 5 and 6, or 7, are characteristic of a Yang–Mills, or non-Abelian, gauge theory in which the directions in color space rotate under a local gauge transformation. The gauge-invariant Lagrangian in the presence of fixed color sources $\rho(x, t)$ is expressed as

$$\mathcal{L} = -\frac{1}{4}\sum_{\mu,\nu=0}^{3}\mathbf{f}_{\mu\nu}\cdot\mathbf{f}^{\mu\nu} - g\rho\cdot\mathbf{A}_o$$

with

$$\mathbf{f}_{\mu\nu} \equiv \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu.$$

The field equations are the Euler–Lagrange equations that are derived by the principle of least action, and the generalized Gauss's law is a constraint equation for the component with no-time development; that is, Equation 6 follows from

$$\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\frac{\partial\mathcal{L}}{\partial\dfrac{\partial\mathbf{A}_o}{\partial x_i}} - \frac{\partial\mathcal{L}}{\partial\mathbf{A}_o} = 0.$$

On the basis of Equations 4, 5, and 7, we can now simply and directly calculate the interactional energy between two fixed charges by perturbation theory in the color charge g.

To illustrate the antishielding, we need calculate only to order g^4, that is, the g^2 corrections to the Coulomb interactions due to the radiation-field fluctuations. This requires calculating the contributions to order g^4 to the interactional energy between two fixed charges as contained in Equation 4. First-order perturbation theory gives the contribution

$$\Delta E_1 = \langle 0| -\tfrac{1}{2} \int d^3x\, \phi \cdot \nabla^2\phi |0\rangle, \tag{10}$$

and second-order theory gives

$$\Delta E_2 = \sum_{n\neq 0} \frac{\langle 0| -\tfrac{1}{2}\int d^3x\, \phi \cdot \nabla^2\phi |n\rangle\langle n| -\tfrac{1}{2}\int d^3x\, \phi \cdot \nabla^2\phi |0\rangle}{E_o - E_n}, \tag{11}$$

where $0\rangle$ denotes the radiation-field vacuum and only the interactional energy between two fixed charges, defined by

$$\rho(x, t) = \rho_1(x, t) + \rho_2(x, t) \tag{12}$$

and forming a color singlet (the quark–antiquark potential) is retained in calculating Equations 10 and 11.

Introducing Equation 7 in Equation 10 and integrating by parts, we find for the interactional energy

$$\begin{aligned} \Delta E_1 &= -g^2 \int d^3x\, \Delta^{-1}\rho_1 \cdot \langle 0|(1 + gA_i\partial_i\Delta^{-1}\times)^{-2}|0\rangle\rho_2 \\ &\simeq -g^2\int d^3x[\rho_1 \cdot \Delta^{-1}\rho_2 + 3g^2\langle 0|\rho_1 \cdot \Delta^{-1}A_i\partial_i\Delta^{-1} \times A_j\partial_j\Delta^{-1} \times \rho_2|0\rangle]. \end{aligned} \tag{13}$$

The first term is recognized as the Coulomb interaction. The vacuum expectation value of the quadratic correction is readily evaluated in momentum space by use of Equation 9 and the Fourier expansion of the charge density

$$\rho_j(x) = \frac{1}{V}\sum_k \rho_j(k)\, e^{ik_i(x-X_j)_i}. \tag{14}$$

We find

$$\begin{aligned} \Delta E_1 &= g^2\frac{1}{V}\sum_k \frac{e^{ik_i(X_1-X_2)_i}}{k^2}\rho_1(k) \cdot \rho_2(k)\left\{1 + \frac{3}{2}g^2C_2\frac{1}{V}\sum_l \frac{[1 - (l_jk_j)^2/l^2k^2]}{l|l_i - k_i|^2}\right\} \\ &= g^2\frac{1}{V}\sum_k \frac{e^{ik_i(X_1-X_2)_i}}{k^2}\rho_1(k) \cdot \rho_2(k)\left\{1 + \frac{12g^2}{48\pi^2}C_2 \ln\frac{\Lambda^2}{k^2}\right\}, \end{aligned} \tag{15}$$

where Λ is the ultraviolet cutoff, and C_2 is the quadratic Casimir, which for SU(2) is equal to 2. Since the divergence is only logarithmic, there is no need for special care in treating it; it may be absorbed into the coupling-constant renormalization by standard means. Equation 15 shows antishielding since the effective potential strength is greater for large separations $R \equiv |X_1 - X_2|$ than for small ones; that is, the logarithmic enhancement in Equation 15 increases as the effective k value decreases or, equivalently, as $R \sim 1/k_{\text{eff}}$ increases. The diagrammatic representation of this

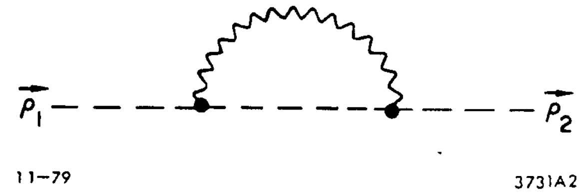

FIGURE 2. Diagram that corresponds to antishielding contribution to Equation 15. Dashed and wavy lines are, respectively, instantaneous Coulomb and transverse gluon propagators.

contribution can be inferred from Equation 13, recognizing that Δ^{-1} is the instantaneous Coulomb propagator, and is illustrated in FIGURE 2.

To the same g^4 order, we must also evaluate the second-order contribution of Equation 11. In this case, the static potential appears four times. As we see from Equation 7, two of the ϕs in Equation 11 must create and destroy dynamical gluon pairs via the field term $g\mathbf{A}_i \times \mathbf{E}_i^T$; thus, the intermediate state $|n>$ contains such a pair of transverse gluons. By straightforward calculation with Equations 7 and 9, we get

$$E_2 = 2g^4 \sum_{n=2\text{ gluons}} \frac{\langle 0| - \int d^3x\, \rho_2 \cdot \Delta^{-1}\mathbf{A}_i \times \mathbf{E}_i^T |n\rangle\langle n| \int d^3x \rho_1 \cdot \Delta^{-1}\mathbf{A}_j \times \mathbf{E}_j^T |0\rangle}{E_0 - E_n}$$

$$= g^2 \frac{1}{V} \sum_k \frac{e^{ik_i(X_1 - X_2)_i}}{k^2} \rho_1(k) \cdot \rho_2(k) \left\{ \frac{-g^2}{48\pi^2} C_2 \ln \frac{\Lambda^2}{k^2} \right\}. \quad \textbf{(16)}$$

The sign of Equation 16 is consistent with normal shielding, as it must be, according to the general spectral theorem, since it records the contribution of physical quantums—the transverse gluons—in the intermediate state. The corresponding graph is FIGURE 3. The sum of Equations 15 and 16 agrees with the well-known result[2,3] for the running coupling constant. If we normalize the charge at mass μ^2, so that

$$g^2 \rightarrow g_r^2(\mu) \equiv g^2 \left[1 + \frac{11g^2}{48\pi^2} C_2 \ln \frac{\Lambda^2}{\mu^2} \right], \quad \textbf{(17)}$$

the interaction can be rewritten

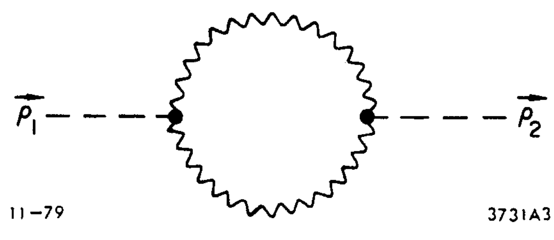

FIGURE 3. Diagram that corresponds to shielding contribution to Equation 16. Dashed and wavy lines are, respectively, instantaneous Coulomb and transverse gluon propagators.

$$\Delta E(R) = g_v^2(\mu)\frac{1}{V}\sum_k \frac{e^{ik_iR_i}}{k^2}\rho_1(k)\cdot\rho_2(k)\left[1+\frac{11g_v^2(\mu)}{48\pi^2}C_2\ln\frac{\mu^2}{k^2}\right], \tag{18}$$

indicating a logarithmic enhancement $\sim \ln \mu R$ relative to the Coulomb law with incrasing R. Equivalently, the running coupling constant

$$g_v^2(\mu)\left[1+\frac{11g_v^2(\mu)}{48\pi^2}C_2\ln\mu^2R^2\right] \tag{19}$$

decreases with decreasing separation. This is the crucial asymptotic freedom behavior of QCD.

There is another simple and intuitive way to illustrate the spreading of the color charge due to fluctuations of the radiation field. We interpret the right-hand side of Equation 7 as an effective charge density, $\rho_{eff}(r)$, and compute the mean square color radius for a fixed source. To order g^4, this computation, again, requires performing a second-order perturbation calculation of the ratio

$$\langle r^2\rangle = \frac{\langle 0|\int d^3xr^2\rho_{eff}|0\rangle + \sum_{n\neq 0} 2\frac{\langle 0|\int d^3xr^2\rho_{eff}|n\rangle\langle n|-\frac{1}{2}\int d^3x\phi\cdot\nabla^2\phi|0\rangle}{E_0-E_n}}{\langle 0|\int d^3x\rho_{eff}|0\rangle + \sum_{n\neq 0} 2\frac{\langle 0|\int d^3x\rho_{eff}|n\rangle\langle n|-\frac{1}{2}\int d^3x\phi\cdot\nabla^2\phi|0\rangle}{E_0-E_n}} \tag{20}$$

The two-gluon state $n>$ is created by the field source $g\mathbf{A}_i\times\mathbf{E}_i^T$ in Equation 7. Just as its contribution to the interactional energy corresponds to shielding and reduces the effect of charge spreading in Equation 16, here it also reduces the square radius. However, charge spreading due to the first term is the larger effect. We calculate as before, only here it is necessary to introduce an infrared cutoff to control the divergence from large-distance contributions that are weighted heavily by the r^2 factor in the numerator of Equation 20. We do this by the substitution

$$k^2 \rightarrow k^2 + b^2$$

in the effective charge density

$$\begin{aligned}\langle 0|-\nabla^2\phi|0\rangle &= g\,\langle 0|(1+g\mathbf{A}_i\partial_i\Delta^{-1}\times)^{-1}|0\rangle \\ &= g\frac{1}{V}\sum_k e^{ik_iX_i}\rho(k)\left[1+\frac{g^2}{12\pi^2}C_2\ln\frac{\Lambda^2}{k^2}\right].\end{aligned} \tag{21}$$

The final result to order g^2 is

$$\langle r^2\rangle = r_0^2 + \frac{g^2}{4\pi^2}C_2\frac{1}{b^2}, \tag{22}$$

where r_0^2 denotes the mean square radius of the bare source $\rho(x)$. The correction to the distribution of the color charge in Equation 22 is infrared divergent but positive, indicating a spreading of the source.

With the inclusion of matter pairs, there are additional shielding corrections according to the substitution[2]

$$11 \rightarrow 11 - 2/3\, n_f \qquad (23)$$

where n_f is the number of different flavors of quarks that contribute.[4] Finally, we remark that the energy-momentum tensor is an operator with zero anomalous dimensions, and this calculation can, in principle, be extended to higher orders in g^2 to exhibit the renormalization group series,[5] that is,

$$1 + x + x^2 + \cdots \rightarrow \frac{1}{1 - x}. \qquad (24)$$

Such extensions are clearly handled much more efficiently by covariant graph techniques, as invented by Feynman originally to escape from the cumbersome procedures of "old-fashioned perturbation theory." Here, as shown, the latter provide a direct and intuitive physical picture of the interactional energy and of "asymptotic freedom."

The breakdown of the perturbation expansion for large distances or stong coupling is evident in this calculation. Indeed, according to Equation 24, a singularity occurs eventually, a result not unrelated to the Gribov ambiguity[6] for strong fields. In this connection, it is interesting to note that to the lowest order in g^2, ln (Λ^2/k^2) corrections resulting from the non-linear term in the definition of the magnetic field of Equation 5 have no role. This is because the additional term lacks a derivative relative to the nonlinear terms in the generalized Gauss's law of Equation 7 that we have calculated. However, when one considers the large-distance behavior of QCD, it is important to include the effects of the low-frequency—even the zero-frequency—modes. It is, in fact, just the nonlinear terms in Equation 5, which introduce quartic amplitudes in the Hamiltonian, that presumably are central to the analysis of confinement.[7]

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Notes and References

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3. The following exercise of calculating the interactional energy between two fixed charges may be found explicitly and implicitly in the published and unpublished literature. In particular, see the beautiful lectures by V. N. Gribov at the 12th Winter School of the Leningrad Nuclear Physics Institute, 1977, which Dr. Nigel Parsons called to my attention after I discussed this approach at Oxford University in the spring of 1979 while I was on sabbatical leave from SLAC. (An English translation of these lectures was issued

as SLAC-Trans-176, further compounding my embarrassment.) More recently, a detailed analysis of the quark–antiquark static potential in QCD has been given by Daniel Stump (University of Indiana, in preparation). See also: KHRIPLOVICH, I. B. 1969. Yad. Fiz. **10:** 409 (1970. Sov. J. Nucl. Phys. **10:** 235); ALTUKHOV, A. M. & I. B. KHRIPLOVICH. 1970. Yad. Fiz. **11:** 902 (1970. Sov. J. Nucl. Phys. **11:** 504); DUNCAN, A. 1976. Phys. Rev. D **13:** 2866; APPELQUIST, T., M. DINE & I. J. MUZINICH. 1977. Phys. Lett. **69B:** 231; 1978. Phys. Rev. D **8:** 2074; FEINBERG, F. 1977. Phys. Rev. Lett. **39:** 316; 1978. Phys. Rev. D **17:** 2659; FRENKEL, J. & J. C. TAYLOR. 1978. Oxford University Preprint; 1979. Erratum; CAHILL, K. & D. STUMP. 1979. Indiana University Preprint (IUHET-37). I am also familiar with unpublished work by: BJORKEN, J. D. & by GILES, R. For a qualitative description see Weisskopf, V. 1974. International School of Subnuclear Physics. A. Zichichi, Ed. **12:** 307.

4. Equation 23 shows that when $n_f \geq 17$ flavors, the criterion for asymptotic freedom is violated.
5. I thank Richard Hughes of Oxford University for valuable discussions on the extension to higher orders and the possibility of deriving the renormalization group results.
6. See GRIBOV.[3]
7. I have enjoyed stimulating discussions on this point with T. D. Lee. See discussion in the 1979 SLAC Summer Institute Lectures by: BJORKEN, J. D.