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(A)

ESTIMATE OF COHERENT TUNE SHIFTS FOR PEP*

C. Y. Yao† and A. W. Chao
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

I. Introduction

Transverse and longitudinal instabilities for a bunched PEP beam with a Gaussian distribution are treated using the standard technique[1] in which instability problems are solved by looking for eigenvalues of the linearized Vlasov equation. The eigen solutions are conveniently expanded in terms of the Laquerre polynomials, and the eigenvalues are given by a symmetric matrix whose elements can be expressed in infinite series.

We will follow the well-known formalism[1] to obtain the matrix formula, and then apply it numerically to PEP ring to estimate the transverse coherent tune shifts. The impedance used is that estimated for the PEP RF cavities. The agreement with experimental data seems reasonable.

II. Analysis

(A) Transverse Case

We use the following coordinates to describe the motion of a particle: x and θ are the transverse and longitudinal positions; r_x, ϕ_x, r_s and ϕ_s are polar coordinates in transverse and longitudinal phase spaces; ω_β and ω_s are the unperturbed betatron and synchrotron angular frequencies.

The linearized Vlasov equation can be written in terms of the transverse impedance $Z_\perp(\omega)$ as[2]

$$-i\Delta\omega_m g_m(r_s) + \frac{cNe^2\omega_o}{4\pi E_o\nu_\beta} g_o(r_s)\int_0^\infty \sum_p Z_\perp\left(p\omega_o+\Omega_m\right) \times J_m\left[\left(p+\nu_m-\frac{\xi}{\alpha}\right)r_s\right] J_m\left[\left(p+\nu_m-\frac{\xi}{\alpha}\right)r_s'\right] \times g_m(r_s')\, r_s'\, dr_s' = 0 \quad . \qquad (1)$$

Here assumption has been made that the bunch distribution is:

$$\psi = f_o(r_x)g_o(r_s) + f_1(r_x)g_m(r_s)e^{i\phi_x+im\phi_s-i\Omega_m t} \qquad (2)$$

with the second term as a perturbation term consisting of a dipole oscillation for the transverse motion and a m-th mode oscillation for the longitudinal motion. We have also assumed that the tune shift $\Delta\omega_m$ is smaller than ω_s. In the absence of perturbation, the mode frequency is given by $\Omega_m = \omega_\beta + m\omega_s$. The other quantities in (1) are:

e = electron charge
E_o = particle energy
ω_o = revolution angular frequency
ν_β = betatron tune ω_β/ω_o
ν_s = synchrotron tune ω_s/ω_o
$\nu_m = \Omega_m/\omega_o \approx \nu_\beta + m\nu_s$ for transverse case or $m\nu_s$ for longitudinal case
c = speed of light
$\Delta\omega_m = \Omega_m - \omega_\beta - m\omega_s$ for transverse case or $\Omega_m - m\omega_s$ for longitudinal case
ξ = chromaticity $\Delta\nu_\beta/(\Delta p/p)$
α = momentum compaction factor
N = number of electrons in the bunch
$J_m(x)$ = Bessel function.

Equation (1) can be rewritten as

$$-i\Delta\omega_m \rho(r_s)\, g_m(r_s) + \int_0^\infty k(r_s,r_s')\, g_m(r_s')\, dr_s' = 0 \qquad (3)$$

where

$$\rho(r_s) = g_o^{-1}(r_s)\, r_s = \text{weight function}$$

$$k(r_s,r_s') = \frac{cNe^2\omega_o}{4\pi E_o\nu_\beta}\sum_p Z_\perp\left(p\omega_o+\Omega_m\right) J_m\left[\left(p+\nu_m-\frac{\xi}{\alpha}\right)r_s\right] \times J_m\left[\left(p+\nu_m-\frac{\xi}{\alpha}\right)r_s'\right] r_s r_s' \quad . \qquad (4)$$

Note that the weight function defined here is the reciprocal of Sacherer's.[1] This choice of weight function makes Eq. (3) self adjoint.[3] With this property the eigen solutions belonging to different eigenvalues of Eq. (3) are othorgonal to each other provided the scalar product of two arbitrary functions is defined as:

$$(f,g) = \int_0^\infty \rho(r_s)\, f(r_s)\, g(r_s)\, dr_s \quad . \qquad (5)$$

A standard way to solve Eq. (3) is to first expand the solution in terms of a set of base functions. For a Gaussian bunch with rms bunch length σR (R = average machine radius), the base functions are most conveniently chosen, according to the Storm-Liouville procedure,[3] to be

$$Y_n = \mathscr{L}_n\left(\frac{r_s^2}{2\sigma^2}\right)\exp\left(-\frac{r_s^2}{2\sigma^2}\right) \quad . \qquad (6)$$

Here $\mathscr{L}_n$ are the Laguerre polynomials.[3]

We expand the eigen solution of Eq. (3) as:

$$g_m(r_s) = \sum_n A_n^m\, Y_n(r_s) \quad . \qquad (7)$$

By making use of the orthogonal condition of Y_n's, we obtain from Eq. (3) a matrix equation for the coefficients A_n^m:

$$\sum_n H_{kn}^{(m)} A_n^m - i\Delta\omega_m A_k^m = 0 \qquad (8)$$

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† Visiting Scientist from the University of Science and Technology, Hefei, China.

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where

$$H_{kn}^{(m)} = \frac{ce^2 N\omega_o}{8\pi^2 E_o \nu_\beta} \sum_p Z_\perp\left(p\omega_o + \Omega_m\right) I_k^{(m)} I_n^{(m)} \quad , \tag{9}$$

and the bunch spectrum $I_n^{(m)}$ is defined as:

$$I_n^{(2\ell+1)} = \frac{\sqrt{\pi}}{n!} \sum_{k=0}^{\infty} \left(\frac{a}{2}\right)^{2k+2\ell+1}$$

$$\times \frac{(-1)^{k+n}(k+\ell+\frac{1}{2})!(k+\ell+\frac{1}{2}) \ldots (k+\ell+\frac{3}{2}-n)}{k!\ (k+2\ell+1)!}$$

$$(\text{For odd } m = 2\ell+1) \tag{10}$$

and

$$I_n^{(2\ell)} = \frac{1}{n!} \sum_{k=n-\ell}^{\infty} \left(\frac{a}{2}\right)^{2k+2\ell} \frac{(-1)^{k+n}[(k+\ell)!]^2}{k!\ (k+2\ell)!\ (k+\ell-n)!}$$

$$(\text{For even } m = 2\ell) \tag{11}$$

where $a = \sqrt{2}\,\sigma[p + \nu_m - (\xi/\alpha)]$ and $(n+\frac{1}{2})$ is understood to be $(n+\frac{1}{2})(n-\frac{1}{2}) \ldots \frac{1}{2}$.

Series' (10) and (11) converge rapidly for parameters that concern us. Particularly, for $m=0$ case (the 'rigid' mode), the summation can be worked out, we have:

$$H_{kn}^{(0)} = \frac{Ne^2 c\omega_o}{8\pi^2 E_o \nu_\beta} \sum_p Z_\perp\left(p\omega_o + \Omega_m\right) \frac{(a/2)^{2(k+n)}}{k!\ n!} e^{-a^2/2} \ . \tag{12}$$

Having these matrix elements calculated, the coherent tune shift and growth rate are given by the eigenvalues of the matrix $iH^{(m)}$. The real part gives coherent tune shift while the imaginary part gives growth rate (damping rate if negative).

The mode number m specifies "azimuthal" distribution of the perturbation, as indicated by Eq. (2). In our treatment we assume the mode frequency shifts are smaller than the synchrotron frequency, and therefore we can ignore the coupling among different azimuthal modes. This assumption restricts our calculation to be valid for relatively weak beam currents. [For strong beam currents, different azimuthal modes do couple, and one has to treat a "transverse turbulence" case[4].]

For a given azimuthal mode m, there are infinite number of "radial" modes, each corresponds to an eigen solution of Eq. (3). The Eq. (8) gives in principle the frequency shifts and growth rates of all the radial modes for a given azimuthal mode number m.

(B) Longitudinal Case

The above recipe can be repeated for longitudinal motion. The trick here is that for a Gaussian bunch the longitudinal equation has exactly the same weight function as Eq. (3). Therefore the whole treatment will be the same as the transverse case except that the impedance $Z_\perp$ is replaced by $Z_\parallel/p$.

We present here only the results for the matrix equation for the radial modes with azimuthal mode number m:

$$\sum_n G_{kn}^{(m)} A_n^m - i\Delta\omega_m A_k^m = 0 \tag{13}$$

where

$$G_{kn}^{(m)} = \frac{\alpha N e^2 \omega_o^2 m}{4\pi^2 \sigma^2 \nu_s E_o} \sum_p \frac{Z_\parallel\left(p + m\nu_s\right)}{p} I_k^{(m)} I_n^{(m)} \quad . \tag{14}$$

and $I_n^{(m)}$ is defined in Eqs. (10) and (11), but with $p + \nu_m - (\xi/\alpha)$ replaced by $p + m\nu_s$. Equation (13) is valid for the instability growth rates of the radial modes. To obtain the frequency shifts, one must include an additional term which has been ignored.[1] The additional term in frequency shift comes from the fact that the unperturbed Gaussian distribution produces a non-zero longitudinal wakefield. The same situation does not happen in the transverse case because there the unperturbed distribution does not produce any transverse wakefield.

III. Computation

(A) Impedance

We calculated transverse mode frequency shifts and growth rates for PEP. As we will see later, the coherent frequency shifts are comparable in the horizontal and vertical betatron motions for PEP. This leads us to the hypothesis that the PEP impedance mainly comes from the RF cavities (which have basically round cross sections) rather than from the vacuum chamber (which may have a flat elliptical cross section).[5] In our calculation, we have used an estimated transverse impedance consisting of an algebraic sum of resonator impedances, each of which represented by

$$Z_\perp(\omega) = R_s \frac{\omega_R/\omega}{1 - iQ\left[\dfrac{\omega}{\omega_R} - \dfrac{\omega_R}{\omega}\right]} \quad . \tag{15}$$

There are 120 cavities in total around the ring, lumped in three RF stations; each cavity contributes approximately 23 resonator impedance terms to the total impedance. The 23 values of R_s and ω_R for one of the PEP model cavity was measured before installation.[6] The frequencies ω_R ranged from about 0.6 GHz to about 2 GHz. Considering the diversity among these cavities, we have assumed a random distribution in ω_R with average variation of ±2.5%. For simplicity, the Q values are assumed the same for all the impedance terms of all cavities: $Q = 9000$.

(B) Results

The calculation involved the summation of infinite series'. To speed up the computation we used integral to replace the summation for smooth parts of impedance. For bunch spectrum $I_n^{(m)}$, we used an spline function to approximate the summation.[7]

In contrast with the usual head-tail calculations,[8] our results of instability growth rates do not depend sensitively on the chromaticity ξ. This is due to the fact that we have used an impedance that contains narrow-band peaks. In our calculation, we have set $\xi = 0$.

We have calculated the transverse coherent mode tune shift $\Delta\nu$ and the growth rate τ^{-1} as functions of the unperturbed tune ν_β. The results are shown in Figs. 1 and 2 for the lowest radial mode for the case of $m=0$ (rigid bunch mode). It is found that the coupling among radial modes does not significantly affect the results for the lowest radial mode. To obtain Figs. 1 and 2, it is only necessary to calculate $H_{oo}^{(0)}$ from Eq. (12) although a more elaborate calculation was done to justify this approximation. For higher order radial modes, however, coupling effects are important. The growth rate vanishes at integral and half-integral tunes, in consistence with Ref. 9.

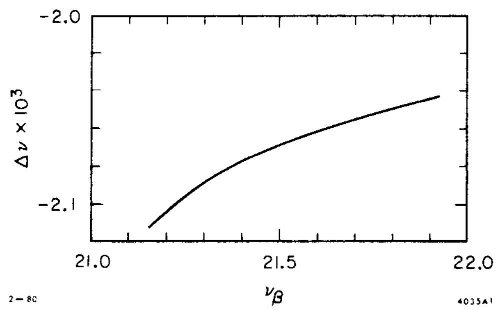

Fig. 1. Coherent betatron tune shift versus the unperturbed betatron tune for the rigid bunch mode m = 0 with a 14.5 GeV, 1 ma single-bunch beam.

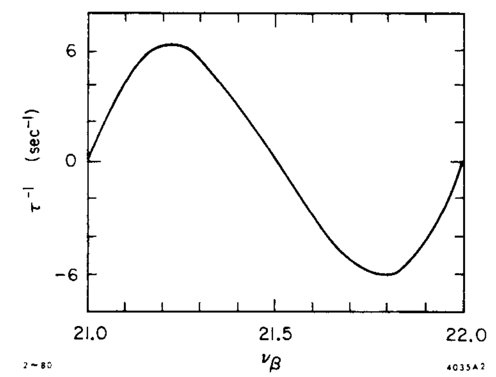

Fig. 2. Instability growth rate (damping rate if negative) versus the unperturbed betatron tune for the rigid bunch mode m = 0 with a 14.5 GeV, 1 ma single-bunch beam. The radiation damping rate at 14.5 GeV is 110 sec^{-1}.

The dependence on the Q-value of the impedance peaks is somewhat sensitive. In the range of Q = 5000 to 10000, our results may vary by about a factor of 2. Our results therefore is accurate only to within such a factor.

Experiments[10] were performed in PEP to obtain the coherent betatron tune shift per 1 ma increase in beam current. Weak (< 5 ma) single-bunch beams were used in these experiments. It was found that

$$\Delta\nu_x/\mathrm{ma} \approx -0.002/\mathrm{ma}$$

$$\Delta\nu_y/\mathrm{ma} \approx -0.003/\mathrm{ma} \quad .$$

These values are in reasonable agreement with the calculated value of ~ -0.002/ma.

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