SLAC-PUB-2668
January 1981
(T)

FEYNMAN PROPAGATOR-BASED EXPANSIONS AT STRONG COUPLING*

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ABSTRACT

With the objective of making direct contact with the usual weak coupling Feynman propagator expansions for quantum field theory, we present a formulation of strongly coupled renormalizable fields based on Feynman propagator expansions. As an example, the theory of the scalar field with quartic self-coupling is considered in detail (in 4-dimensional Minkowsky space). In agreement with an original result of Wilson, we find that this renormalized strongly quartically self-coupled field theory consists of a free particle two-point 1PI Green's function, and nothing else! Comparison is made with the recent work of Bender et al. and Willey, and possible physical implications in connection with $SU_2 \times U_1$ Higgs theory are noted.

Submitted to Physical Review D

* Work supported by the Department of Energy, contract DE-AC03-76SF00515.

I. INTRODUCTION

The problem of strongly coupled renormalizable fields has become of utmost importance because of the apparent reasonable success of the QCD theory[1,2] of "strong" interactions at short distances[3,4] in the Wilson short distance framework.[4] Indeed, if QCD is to be verified as the theory of strong interactions, then its strongly interacting large distance sector must be calculated in some detail.

Toward this latter objective, we will note two apparently complementary approaches.[5,6,7,8] More precisely, we call attention to the lattice theory of Wilson,[5] wherein one attempts to probe the large-distance behavior of QCD by cutting off the short-distance part of the theory already at the Lagrangian level, with a gauge invariant lattice formalism. A key triumph of this approach is the celebrated Wilson area law for the negative of the logarithm of the vacuum expectation value of the Wilson loop operator — this area law signals the desired confinement behavior for QCD. Considerable work toward understanding the predictions of the Wilson lattice for the details of hadron dynamics has been done.[6]

The complementary approach[7,8] attempts to maintain manifest Lorentz invariance in all end results in treating the detailed dynamics of strongly interacting systems. Hence, it can hope to answer any questions which may arise concerning the lack of manifest Lorentz invariance in the widely popular Wilson framework.

More specifically, the complementary approach, as developed in Refs. 7 and 8, attempts to isolate a small part of the Lagrangian as the respective coupling or couplings become large. While this isolation may be accomplished in several ways, the net result has always been an expansion in inverse powers of the appropriately large couplings. Furthermore, it has also been true that most of these inverse coupling constant expansions have always resulted in expansions in the kinetic part of the respective Lagrangian. The possibility of exceptions to this last remark may be found in Ref. 7. One thing is quite manifest: to the extent that one expands in the kinetic part of the Lagrangian one should only differ, in principle, from the Wilson approach to strong coupling by the regulator which one uses. Indeed, as one can see from the work of Bender *et al.*,[8] if one uses the kinetic part of the Lagrangian as the expansion operator, the use of a Wilson lattice to regulate the respective theory, with an appropriate procedure for extrapolation to zero lattice spacing, leads to well-known results in problems such as the anharmonic oscillator—where there exist independent methods of calculation.[9] We consider the consistency of the work of Bender *et al.* and the results in Ref. 9 to be a positive test for both the Wilson approach and the complementary approach.[10]

However, we do wish to emphasize that, in the complementary apporach, in particular, there is no reason that the kinetic part of the respective Lagrangian has to be used as the expansion operator at strong coupling. Indeed, it was emphasized in Ref. 7 that no particular assumption has been made about the relative sizes of the kinetic and large coupling parts of the Lagrangian at strong coupling. Rather, this relative size was argued to have been left to dynamics.

The manner in which this particular part of the strong coupling dynamics manifests itself has remained unclear, unfortunately, primarily because the formulation of the complementary approach presented in

Ref. 7 is somewhat tedious to use. Indeed, the formulation in Ref. 7 is very difficult to relate to the respective conventionally normalized weak coupling theory. This difficulty stems primarily from the indirect use of the kinetic part of the Lagrangian as the expansion operator. To reiterate, such a use is entirely unnecessary in the complementary approach. It will be our main purpose to illustrate this last remark in some detail in the present communication.

More precisely, we shall develop the analoga of Feynman's expansions[11] for weak coupling-Feynman propagator-based expansions at strong coupling. The four-momentum structure of the expansion will then, in general, be determined by the usual Feynman propagator. Contact with weak-coupling theory will therefore be much more immediate. Consequently, the application of these Feynman expansions to QCD and other strongly interacting systems[12] would appear to be facilitated greatly. Such systems will be taken-up elsewhere.[13]

Our work is organized as follows. In the next section, Sect. II, we recapitulate the relevant aspects of the complementary approach to strong coupling and analyze the quartically self-coupled scalar field. In Sect. III, we treat the renormalization of our Feynman expansion in detail. Sect. IV contains a comparison with the work of Bender et al. and Willey,[8] all of whom have considered this renormalized strongly, quartically self-coupled scalar field using the lattice regulator-based computation scheme of Bender et al.[8] Finally, Sect. V contains some concluding remarks with regard to implications of our work for the Higgs sector of the $SU_2 \times U_1$ model.[14]

II. LORENTZ INVARIANT APPROACH TO STRONG COUPLING

Referring to Ref. 7, we recall that the starting point for the Lorentz invariant formulation of strongly coupled fields is the generating functional Z(J) for the connected Green's functions of the respective theory: for example, using the theory of the quartically self-coupled scalar field in 4-Minkowsky dimensions as a prototype, we have the Lagrangian

$$\mathscr{L}(x) = \frac{1}{2}\left(\partial_\mu\phi\ \partial^\mu\phi - m^2\phi^2\right) - g\phi^4 \tag{1}$$

so that

$$\exp\left\{iZ(J)\right\} = \int \mathscr{D}\phi\ \exp\left\{i\int d^4x\left[\mathscr{L} + J\phi\right]\right\} \quad . \tag{2}$$

In (1), g is the coupling constant and m is the mass parameter. In (2), J is the usual external source. The functional derivatives of Z(J) with respect to J, evaluated at $J=0$, are the connected Green's functions, as is well-known. We wish to study (1), as it is represented by (2), for large g.

More precisely, for large g, we use[7] auxiliary fields σ and ρ to write

$$\exp\left\{iZ(J)\right\} = \int \mathscr{D}\phi\,\mathscr{D}\sigma\,\mathscr{D}\rho\,\mathscr{D}\kappa\ \exp\left\{i\int d^4x\left[\mathscr{L}_0(\kappa) + \rho(\kappa-\phi) + J\kappa + \sigma^2 + 2\sqrt{g}\,\sigma\,\phi^2\right]\right\} \tag{3}$$

for an appropriate normalization of the functional integrals. Here,

$$\mathscr{L}_0(\kappa) \equiv \frac{1}{2}\left(\partial_\mu\kappa\ \partial^\mu\kappa - m^2\kappa^2\right) \tag{4}$$

is the usual free scalar field Lagrangian for mass m. It is the expression (3) that we shall analyze.

Before proceeding further, let us note that in arriving at (3), we have used the results (up to unimportant constant factors)

$$\exp\left\{ i \int d^4x \; F(\phi)\right\} = \int \mathscr{D}\rho \, \mathscr{D}\kappa \; \exp\left\{ i \int d^4x \left[F(\kappa) + \rho(\kappa-\phi)\right]\right\} \tag{5}$$

$$\exp\left\{ - i \int d^4x \; g\phi^4 \right\} = \int \mathscr{D}\sigma \; \exp\left\{ i \int d^4x \left[\sigma^2 + 2\sqrt{g} \; \sigma \; \phi^2 \right]\right\} \tag{6}$$

where F(y) is some function of y. The result (5) obtains because the integration over $\mathscr{D}\rho$ produces the delta functional $\delta(\kappa-\phi)$ so that the subsequent integration over $\mathscr{D}\kappa$ simply sets $\kappa = \phi$. The result (6) is understood by completing the square

$$\sigma^2 + 2\sqrt{g} \; \sigma \; \phi^2 = \left(\sigma + \sqrt{g} \; \phi^2\right)^2 - g\phi^4 \tag{7}$$

and shifting σ by $-\sqrt{g} \; \phi^2$ before integrating over $\mathscr{D}\sigma$, as is well known from the work of Feynman.[15] With these explanatory remarks, we now return to the general development.

As in Ref. 7, we decouple κ and ϕ with the shift

$$\phi \rightarrow \phi + \rho\Big/\left(4\sqrt{g} \; \sigma\right) \quad . \tag{8}$$

There results, up to an unimportant constant factor,

$$iZ(J) = \ell n \int \mathscr{D}\phi \, \mathscr{D}\sigma \, \mathscr{D}\rho \, \mathscr{D}\kappa \; \exp\left\{ i \int d^4x \left[\mathscr{L}_0(\kappa) + \rho\kappa + J\kappa + \sigma^2 + 2\sqrt{g} \; \sigma \; \phi^2 - \rho^2\Big/(8\sqrt{g} \; \sigma)\right]\right\} \quad . \tag{9}$$

The methods of Ref. 7 may now be used to evaluate (9). Indeed, on scaling $\rho \rightarrow \rho/g^\gamma$, $\gamma > 0$, and expanding in the operators

$$\kappa\rho/g^\gamma \qquad \text{and} \qquad \rho^2\Big/\left(8g^{2\gamma+\frac{1}{2}}\sigma\right) \quad , \tag{10}$$

we find, for an appropriate normalization of the functional integrals,

$$iZ(J) = \ell n \int \mathscr{D}\phi\, \mathscr{D}\sigma\, \mathscr{D}\rho\, \mathscr{D}\kappa\, \exp\left\{i \int d^4x \left[\mathscr{L}_0(\kappa) + \rho\kappa/g^\gamma + J\kappa + \sigma^2 + 2\sqrt{g}\,\sigma\,\phi^2 - \rho^2\Big/\left(8g^{2\gamma+\frac{1}{2}}\,\sigma\right)\right]\right\}$$

$$= \ell n \int \mathscr{D}\phi\, \mathscr{D}\sigma\, \mathscr{D}\rho\, \mathscr{D}\kappa \sum_{m=0}^{\infty} \frac{i^m}{m!} \prod_{\ell=1}^{m} \int d^4x_\ell \frac{\delta}{g^\gamma i\delta H(x_\ell)} \frac{\delta}{i\delta J(x_\ell)}$$

$$\times \sum_{n=0}^{\infty} \frac{i^n}{n!} \prod_{j=1}^{n} \int d^4x_j \frac{(-)\delta^2}{i^2\delta^2 H(x_j)} \frac{1}{8g^{2\gamma+\frac{1}{2}}}$$

$$\times \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j + i\varepsilon} \int_{-\infty}^{\infty} \frac{d\alpha_j}{2\pi} \exp\left\{ i \sum_j \alpha_j\left(\beta_j - \sigma(x_j)\right)\right.$$

$$\left. + i \int d^4x \left[\mathscr{L}_0(\kappa) + J\kappa + H\rho + \sigma^2 + 2\sqrt{g}\,\sigma\,\phi^2\right]\right\}\Bigg|_{H=0} \quad . \tag{11}$$

This result (11) already illustrates our basic result. We see that, for g large, we can develop iZ(J) in a Feynman propagator expansion, since

$$\int \mathscr{D}\kappa \exp\left\{i \int d^4x \left[\mathscr{L}_0(\kappa) + J\kappa\right]\right\} = \exp\left\{-\frac{i}{2} \int d^4x\, d^4y\; J(x)\, \Delta_F(x-y)\, J(y)\right\} \tag{12}$$

as is well-known, where Δ_F is Feynman's propagator,[11] and the remaining functional integrals in (11) can be defined by using, for example, the following (for an appropriate normalization of $\mathscr{D}\rho$):

$$\int \mathscr{D}\rho \exp\left\{i \int d^4x H\rho\right\} \equiv \lim_{a\downarrow 0} \int \mathscr{D}\rho \exp\left\{i \int d^4x \left[H\rho + \frac{a}{2}\left(\partial_\mu\rho\,\partial^\mu\rho - m^2\rho^2\right)\right]\right\}$$

$$= \lim_{a\downarrow 0} \exp\left\{-\frac{i}{2a} \int d^4x\, d^4y\, H(x)\, \Delta_F(x-y)\, H(y)\right\} \tag{13}$$

and, from Ref. 7,

$$\prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j + i\varepsilon} \int_{-\infty}^{\infty} \frac{d\alpha_j}{2\pi}$$

$$\times \frac{\int \mathscr{D}\sigma\, \mathscr{D}\phi \;\; \exp\left\{i \int d^4x\left(\sigma^2 + 2\sqrt{g}\;\sigma\;\phi^2\right) + \sum_{j=1}^{n} \alpha_j\left(\beta_j - \sigma\left(x_j\right)\right)\right\}}{\int \mathscr{D}\sigma\, \mathscr{D}\phi \;\; \exp\left\{ i \int d^4x \left(\sigma^2 + 2\sqrt{g}\;\sigma\;\phi^2\right)\right\}}$$

$$= \prod_{j=1}^{n} \left(-i^{\frac{1}{2}}\; (\Delta x)^{\frac{1}{2}}\; \sqrt{2\pi}\right) \quad . \tag{14}$$

Here, we recall that, in arriving at (14), we have (in Ref. 7) used a uniform covering $\mathscr{O}$ of space-time by sets of measure Δx with center points $\{x_j\}$ so that the left-hand side of (14) is the same as

$$\left(\prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j + i\varepsilon} \int_{-\infty}^{\infty} \frac{d\alpha_j}{2\pi}\right)$$

$$\times \frac{\left(\prod_{\ell=1}^{\infty} \int_{-\infty}^{\infty} d\sigma_\ell \int_{-\infty}^{\infty} d\phi_\ell\right) \exp\left\{i\left[\Delta x \sum_{\ell}\left(\sigma_\ell^2 + 2\sqrt{g}\,\sigma_\ell\phi_\ell^2\right) + \sum_{j=1}^{n} \alpha_j\left(\beta_j - \sigma_j\right)\right]\right\}}{\left(\prod_{\ell=1}^{\infty} \int_{-\infty}^{\infty} d\sigma_\ell \int_{-\infty}^{\infty} d\phi_\ell\right) \exp\left\{i\left[\Delta x \sum_{\ell}\left(\sigma_\ell^2 + 2\sqrt{g}\,\sigma_\ell\phi_\ell^2\right)\right]\right\}} \tag{15}$$

$$= \prod_{j=1}^{n} \int_{-\infty}^{\infty} \frac{d\beta_j}{\beta_j + i\varepsilon} \int_{-\infty}^{\infty} \frac{d\alpha_j}{2\pi}\; \frac{2i^{3/4}}{\Gamma(1/4)} \sqrt{\frac{\pi}{\alpha_j}}\; (\Delta x)^{1/4}\; e^{-i\alpha_j^2/(4\Delta x)}$$

$$= \prod_{j=1}^{n} \left(-i^{1/2}\; (\Delta x)^{1/2}\; \sqrt{2\pi}\right) \quad ,$$

in agreement with (14).

To be more precise, if we introduce (12), (13) and (14) into (11), we have

$$
\begin{aligned}
iZ(J) = \ln \sum_{m=0}^{\infty} \frac{i^m}{m!} \prod_{\ell=1}^{m} \int d^4x_\ell \; \frac{\delta}{ig^\gamma \delta H(x_\ell)} \; \frac{\delta}{i\delta J(x_\ell)} \\
\times \sum_{n=0}^{\infty} \frac{i^n}{n!} \prod_{j=1}^{n} \int d^4x_j \; \frac{1}{8g^{2\gamma + \frac{1}{2}}} \; \frac{\delta^2}{i^2 \delta H^2(x_j)} \; \frac{(\Delta x)^{\frac{1}{2}} \sqrt{2\pi}}{i^{-\frac{1}{2}}} \\
\times \exp \left\{ i \left[-\frac{1}{2} \int d^4x \, d^4y \; J(x) \, \Delta_F(x-y) \, J(y) \right. \right. \\
\left. \left. - \frac{1}{2a} \int d^4x \, d^4y \; H(x) \, \Delta_F(x-y) \, H(y) \right] \right\} \Bigg|_{H=0} \quad . \qquad (16)
\end{aligned}
$$

The theory represented by (16) has the Feynman rules which follow:

(a) For each ρ^2 vertex a factor of

$$
-i\mu^2 = \frac{i}{4g^{2\gamma + \frac{1}{2}}} \; \frac{(\Delta x)^{\frac{1}{2}} \sqrt{2\pi}}{i^{-\frac{1}{2}}} \quad . \qquad (17)
$$

(b) For each ρ-κ vertex, a factor of

$$
i\xi = i/g^\gamma \quad . \qquad (18)
$$

(c) For each ghost ρ propagator, a factor of

$$
i \Big/ \left[a \left(k^2 - m^2 + i\varepsilon \right) \right] = \frac{i}{a} \, \Delta_F(k) \qquad (19)
$$

(d) For each κ propagator, a factor of

$$
\frac{i}{\left(k^2 - m^2 + i\varepsilon \right)} = i\Delta_F(k) \quad . \qquad (20)
$$

The respective diagrams are shown in Fig. 1. These rules may now be used to solve the theory, in the standard manner.

To illustrate this solution, we first observe that the ghost ρ has no external legs (since $H = 0$) for the Green's functions of interest. The (physical) connected Green's functions for Z, the Green's functions of interest, may be considered in turn, where these Green's functions are defined by

$$G_n\left(x_1, \ldots, x_n\right) = \frac{\delta^n}{i^n \delta J\left(x_1\right) \ldots \delta J\left(x_n\right)} iZ \Bigg|_{J=0} \quad . \tag{21}$$

For $n = 1$, we have

$$G_1 \equiv 0 \quad , \tag{22}$$

by inspection.

For n=2, we have to sum the various contributions to the two-point connected κ Green's function, G_2. The series can be considered most simply by considering simultaneously the ghost ρ two-point connected Green's function $\mathcal{G}_2$. Suppose the complete contribution of the ρ^2 vertex to $\mathcal{G}_2$ is known. Then, representing $\tilde{\mathcal{G}}_2$, the sum of the graphs for this contribution and the free propagator, by the blob in Fig. 2, we see that the complete G_2 is given by the diagrammatic expansion in Fig. 3. One finds

$$G_2(k) = \frac{i}{k^2 - m^2 + i\xi^2 \tilde{\mathcal{G}}_2(k) + i\varepsilon} \quad . \tag{23}$$

We therefore have to determine $\tilde{\mathcal{G}}_2(k)$.

The interactions for $\tilde{\mathcal{G}}_2(k)$, as shown in Fig. 2, are the familiar effects of a mass insertion on a propagator. One finds readily

$$\tilde{\mathcal{G}}_2(k) = \frac{i}{a\left(k^2 - m^2\right) + \dfrac{(\Delta x)^{\frac{1}{2}} \sqrt{2\pi}}{4g^{2\gamma + \frac{1}{2}} i^{-\frac{1}{2}}} + i\varepsilon} \quad . \tag{24}$$

Hence, we have

$$G_2(k) = \frac{i}{k^2 - m^2 - \dfrac{1}{a\, g^{2\gamma}\left(k^2 - m^2\right) + \dfrac{(\Delta x)^{\frac{1}{2}} \sqrt{2\pi}}{4\sqrt{g}\; i^{-\frac{1}{2}}}} + i\varepsilon} \quad . \tag{25}$$

The limit $a \downarrow 0$ now gives

$$G_2(k) = \frac{i}{k^2 - m^2 - \sqrt{8g}\; i^{-\frac{1}{2}} \big/ \left(\pi \Delta x\right)^{\frac{1}{2}}} \quad . \tag{26}$$

For all odd n greater than or equal to three, we have

$$G_n \equiv 0 \tag{27}$$

by inspection, since $Z(J)$ is invariant under

$$J \to -J \quad . \tag{28}$$

For n even and greater than or equal to two, the value of G_n is also trivial. The reason is that, since there are only two-point vertices, these vertices can always be viewed as corrections to either a ρ propagator, a κ propagator, or a κ-ρ propagator, i.e., corrections to either $\mathcal{G}_2$, G_2 or the complete connected κ-ρ propagator. But, we have already included all the corrections to G_2 (we are not interested in the complete ghost propagator $\mathcal{G}_2$ or the complete ghost-κ propagator since $\{G_n\}$ have no external ghost lines). Hence, there are no remaining interactions to generate G_{2n}, for $2n > 2$. Thus, as is well-known, a theory with the Feynman rules (a)-(d) above corresponds to simple mass squared shifts.

Indeed, the result that only G_2 is non-trivial can also be seen by simply substituting (14) alone into (11). One finds

$$iZ(J) = \ell n \int \mathscr{D}\kappa \mathscr{D}\rho \sum_{n=0}^{\infty} \frac{i^n}{n!} \prod_{j=1}^{n} \int d^4x_j \frac{\rho^2(x_j)}{2\sqrt{8g}} \frac{(\Delta x)^{\frac{1}{2}}\sqrt{\pi}}{i^{-\frac{1}{2}}}$$

$$\times \exp\left\{i \int d^4x \left[\mathscr{L}_0(\kappa) + J\kappa + \rho\kappa\right]\right\}$$

$$= \ell n \int \mathscr{D}\kappa \mathscr{D}\rho \exp\left\{i \int d^4x \left[\mathscr{L}_0(\kappa) + J\kappa + \rho\kappa + \rho^2 (\Delta x)^{\frac{1}{2}}\sqrt{\pi}/(-32gi)^{\frac{1}{2}}\right]\right\}$$

$$= \ell n \int \mathscr{D}\kappa \exp\left\{i \int d^4x \left[\mathscr{L}_0(\kappa) + J\kappa - \frac{(-8ig)^{\frac{1}{2}}\kappa^2}{2\sqrt{\pi}(\Delta x)^{\frac{1}{2}}}\right]\right\}$$

$$= \ell n \int \mathscr{D}\kappa \exp\left\{i \int d^4x \left[\frac{1}{2}\left(\partial_\mu\kappa\,\partial^\mu\kappa - \left[m^2 + \frac{(-8ig)^{\frac{1}{2}}}{\sqrt{\pi\Delta x}}\right]\kappa^2\right) + J\kappa\right]\right\} \tag{29}$$

for an appropriate normalization of the functional integrals. The use of (12) then gives

$$iZ(j) = -\frac{1}{2} \int d^4x\, d^4y\, J(x)\, G_2(x-y)\, J(y) \tag{30}$$

where $G_2(x-y)$ is the Fourier transform of (26)

$$G_2(x-y) = i \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k^2 - \left[m^2 + \left(-8ig/(\pi\Delta x)\right)^{\frac{1}{2}}\right] + i\varepsilon} . \tag{31}$$

This result (29) corroborates our results for $\{G_n\}$. What we see is that, for (1), we can sum all of the terms in this Feynman propagator-based inverse coupling expansion directly. We wish to emphasize, however, the complete generality of the procedure leading to (11)-(31).

Indeed, the manipulations represented in (12), (13) and (14) are sufficient to develop the large coupling limit of any renormalizable theory in terms of Feynman propagators. For, using the methods in Ref. 7, as illustrated in (5)-(9) here, appropriate small parts of renormalizable Lagrangians can be isolated in the large coupling limit. Consider a field variable A in such a Lagrangian $\hat{\mathscr{L}}$ in which appropriate small parts have been isolated. Then either A has a kinetic term in $\hat{\mathscr{L}}$ of the form

$$\hat{\mathscr{L}}_0 = \frac{1}{2}\left(\partial_\mu A\ \partial^\mu A - m_A^2\ A^2\right) \quad , \tag{32}$$

or it does not. Here m_A is the respective mass parameter. If it has the term (32), then its propagation from vertex to vertex in the expansion of the theory represented by $\hat{\mathscr{L}}$ in powers of the small parts of $\hat{\mathscr{L}}$ will be given by the Feynman propagator

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4}\ \frac{e^{-ik\cdot x}}{k^2 - m_A^2 + i\varepsilon} \quad . \tag{33}$$

On the other hand, if no such term as (32) exists, then using

$$1 \equiv \lim_{a\downarrow 0}\ \exp\left\{i\int d^4x\left[\frac{a}{2}\left(\partial_\mu A\ \partial^\mu A - m_A^2\ A^2\right)\right]\right\} \tag{34}$$

as we did in (13), we can introduce such a term into $\hat{\mathscr{L}}$:

$$\hat{\mathscr{L}} \to \hat{\mathscr{L}} + \frac{a}{2}\left(\partial_\mu A\ \partial^\mu A - m_A^2\ A^2\right) \quad , \tag{35}$$

where we take $a\downarrow 0$, just as Feynman's ε in (33) is understood to be $\varepsilon\downarrow 0$. The propagator of A will then involve the Feynman propagator

$$\Delta_F'(x) = \frac{1}{a}\ \Delta_F(x) \quad . \tag{36}$$

Thus in both cases the propagation of A in the expansions of $\hat{\mathscr{L}}$ in terms of its small parts in the large coupling limit will be given by a Feynman-type propagator. We have illustrated this result here for the case of $\hat{\mathscr{L}}$ equal to the Lagrangian $\mathscr{L}$ in (1), where the respective small part of $\mathscr{L}$ is

$$\rho\kappa/g^{\gamma} - \rho^{2}\Big/\left(8g^{2\gamma+\frac{1}{2}}\,\sigma\right) \quad . \tag{37}$$

To repeat, other such illustrations will appear elsewhere.[13]

In reality what we have given so far is only a concrete example of the use of unrenormalized Feynman-propagator based strong coupling expansions; we must turn next to the detailed question of renormalization. For, only if we can consistently interpret the results in this section in the spirit of the usual renormalization program, can we claim a complete discussion of the respective large coupling limit.

III. RENORMALIZATION

Having solved for the unrenormalized Green's functions for the strong coupling limit of (1), we now turn to the renormalization of our unrenormalized solution. We proceed in a familiar manner.[16]

More precisely, we write

$$\begin{aligned}\mathscr{L}(x) &= \frac{1}{2}\left(\partial_{\mu}\phi\ \partial^{\mu}\phi - m^{2}\phi^{2}\right) - g\phi^{4}\\ &= \frac{1}{2}\left(\partial_{\mu}\phi_{R}\ \partial^{\mu}\phi_{R} - m_{R}^{2}\phi_{R}^{2}\right) - g_{R}\phi_{R}^{4} + \frac{1}{2}\left(Z_{3}-1\right)\left(\partial_{\mu}\phi_{R}\ \partial^{\mu}\phi_{R} - m_{R}^{2}\phi_{R}^{2}\right)\\ &\qquad - \left(Z_{1}-1\right)g_{R}\phi_{R}^{4} + \frac{1}{2}\,\Delta m^{2}\ Z_{3}\phi_{R}^{2}\end{aligned} \tag{38}$$

where Z_1 is the usual vertex renormalization, Z_3 is the usual wavefunction renormalization, $\Delta m^2 = m_R^2 - m^2$ is the usual mass counter-term, and m_R^2 is the renormalized mass. To renormalize (26), we must choose appropriate normalizations by which we can determine Z_1, Z_3 and Δm^2.

To this end, we choose to normalize the 1PI two-point function $\Gamma_2(p) = G_2^{-1}(p)$ at $p^2 = m_R^2$ and the 1PI four-point function $\Gamma_4(p_1,p_2,p_3,p_4)$

(which is trivial) at $p_1 = p_2$, $p_3 = p_4$, $p_1^2 = m_R^2$, $p_3^2 = m_R^2$, and $p_1 = (m_R,\vec{0})$, $p_3 = (-m_R,\vec{0})$. (For definiteness we take $\{p_i : i=1, \ldots, 4\}$ all to be directed into the vertex Γ_4.)

Turning first to $G_2^{-1}(p)$ we have

$$Z_3 \frac{d}{d(p^2)} G_2^{-1}(p) \Big|_{p^2=m_R^2} = -i \quad . \tag{39}$$

This forces

$$Z_3 = 1 \quad . \tag{40}$$

Further, we require that G_2^{-1} vanishes at $p^2 = m_R^2$. This gives

$$\begin{aligned} 0 &= m_R^2 - \left[m_R^2 - \Delta m^2 + \left(-8iZ_1 g_R\right)^{\frac{1}{2}} \big/ (\pi\Delta x)^{\frac{1}{2}}\right] \\ &= \Delta m^2 - 2(1-i) \left(Z_1 g_R\right)^{\frac{1}{2}} \big/ (\pi\Delta x)^{\frac{1}{2}} \end{aligned} \tag{41}$$

or

$$\Delta m^2 = 2(1-i) \left(Z_1 g_R\right)^{\frac{1}{2}} \big/ (\pi\Delta x)^{\frac{1}{2}} \quad . \tag{42}$$

Now, in arriving at (41) we wish to note that we have substituted (38) for (1) in our calculations in Section II and, hence, have represented g in terms of g_R by the usual relation $g_R = Z_3^2 g/Z_1$.

The result (42) then gives

$$m_R^2 = m^2 + 2(1-i) \left(Z_1 g_R\right)^{\frac{1}{2}} \big/ (\pi\Delta x)^{\frac{1}{2}} \quad . \tag{43}$$

The renormalized 1PI two-point function is thus

$$\Gamma_2(p) = G_2^{-1}(p) = -i\left(p^2 - m_R^2 + i\varepsilon\right) \quad . \tag{44}$$

The 1PI four-point vertex is easily seen to vanish since G_4 is trivial. Thus there is, at strong coupling, no condition on Z_1 as it is of no consequence, i.e., Z_1 is arbitrary. Since there are no further 1PI vertices, the remaining Γ_n are also trivial because $\{G_n : n > 4\}$ are all trivial), this completes the renormalization of the strongly coupled quartically self-coupled scalar field theory. The triviality of this particular large coupling limit in Minkowsky space (4-dimensions) was also found by Wilson.[17]

This problem of the renormalization of the quartically self-coupled scalar field in the strong coupling limit has also been discussed by Bender *et al.*,[8] and by Willey,[8] using the lattice regulator-based computation scheme of Bender *et al.*, where the lattice spacing is ultimately taken to zero. In the next section, we wish to compare our work with these lattice regulator-based results.

IV. COMPARISON WITH THE LATTICE REGULATOR-BASED RESULTS

Bender *et al.* and Willey have all used the quartically self-coupled scalar field at strong coupling to investigate the renormalization properties of the Bender *et al.* - lattice regulator-based computation scheme. We wish to compare their work with ours. In making this comparison we shall effectively discuss the respective two sets of results in turn. We do this by considering the work of Willey first, for pedagogical reasons.

The basic solution of Willey for the renormalized strong coupling limit of the quartically self-coupled field theory is in general agreement with the results in the preceding section. Namely, the theory, according to our work, in this limit, possesses only one non-trivial 1PI vertex, the two-point 1PI vertex. Willey finds, in general, for comparison, that Γ_4 is an arbitrary finite constant as well as that Γ_2 is a simple free inverse propagator in 4-dimensional Minkowsky space, with Γ_n trivial for $n > 4$. However, our results for the free particle nature of this propagator and for the value of Γ_4 do not depend on the dimension d of space-time. While Willey's result for Γ_2 is independent of d, his results for Γ_n, $n > 2$, are in fact extremely dependent on the dimension of space-time. We therefore wish to discuss these apparent points of departure.

More specifically, in order to understand the results of Willey in our context, we must identify the key ingredients in his work insofar as it relates to our work. As in any renormalizable quantum field theory, these ingredients are the expansion operators, the cut-off, and the normalization conditions. In the example under consideration, the simplicity of the solution (a two-point 1PI vertex and, perhaps, an arbitrary four-point 1PI vertex) makes this identification extremely simple. Namely, Willey has also found a finite Z_3, and an arbitrary value of $(Z_1 - 1)$. This would indicate the similarity of the normalization conditions. It further suggests that only in the treatment of expansion operators can our two solutions differ physically. We do not entertain the idea that the lattice regulator could produce, by itself, a physically distinct solution from our Lorentz invariant regulator. For, both regulators are allowed to approach their respective limiting values after renormalization.

Indeed, Willey considers a particular relationship between his cut-off, the lattice spacing a, his bare mass μ_0, and his bare coupling g_0. This relationship is

$$\mu_0^2 a^2 \propto \sqrt{g_0} \Big/ a^{(d/2-2)} \tag{45}$$

where d is the dimension of space-time. This relationship (45), taken together with the expansion of Green's functions in powers of the kinetic part of (1), then leads Willey to the conclusion that (1) has all Γ_{2n}, $n \geq 3$ trivial for $d > 3$, that only Γ_6 of $\{\Gamma_{2n}: n \geq 3\}$ is non-trivial for $d = 3$ (Γ_6 is a constant here), and that Γ_6 is infinite for $d < 3$. For Γ_4, Willey finds a constant for all $d > 2$ and infinity for $d < 2$. For Γ_2, Willey's result agrees with (44). Thus, to summarize, Willey finds that (1) has no canonical renormalizable strong coupling expansion for $d < 3$.

To compare with our results, first note that for $d \geq 4$, Willey's results and our work coincide for Γ_2 and all Γ_n with $n \neq 4$; for Γ_4, the two results appear to differ by a constant. In fact, only for $d \leq 3$ do the two treatments radically differ. This difference may be traced to the expansion operators in the two approaches. In our approach, the expansion operators are Feynman propagators. In Willey's treatment, the expansion operators are, effectively, inverse Feynman propagators [with the relation (45)]. Thus, whereas the terms in our expansions of Green's functions do not increase in degree of divergence with decreasing space-time dimension, the terms in Willey's expansions become more divergent with decreasing dimension. From (26)-(44), we see that, instead of concluding that (1) does not have a conventional renormalizable large g canonical $1/\sqrt{g}$ expansion for $d < 3$, Willey could just as easily have concluded that the representation of the $1/\sqrt{g}$ expansion in inverse Feynman propagators, as used by him, is not useful for conventional renormalization for $d < 3$. It is in this way that we reconcile our results with Willey's work. In other words, we attribute all differencees between Willey's work and ours to the difference in space-time expansion operators.

Consistent with our interpretation of Willey's work are the results obtained by Bender et al. [8] in their study of the renormalized effective potential V_{eff} for the theory (1). Using their lattice regulator calculational techniques, these authors find that, for $d < 4$, the respective zero lattice spacing extrapolants[8] for the renormalized coupling constant and the higher point vertices V_6 and V_8 (using their notation) all decrease toward zero as the order of the respective perturbative calculation increases. Here, V_{2n} is the coefficient of $\phi^{2n}_{R,\text{Classical}}$ in the expansion of $V_{eff}(\phi_{R,\text{Classical}})$ in powers of the classical field $\phi_{R,\text{Classical}}$. Thus, we should be tempted to take these decreases to be consistent with our result that these renormalized quantities are actually zero. We are and we do.

The reader may rightfully wonder, "Why are the results of Bender et al. not inconsistent with Willey's results?" The answer to this question appears to lie in the method used to take the limits g large, $a \to 0$, where a is the lattice spacing in the regulator scheme of Bender et al. Willey takes the limits in the order $a \downarrow 0$ first, then $g \to \infty$. Bender et al.[8], on the other hand, take $g \to \infty$ first, then $a \downarrow 0$. The two procedures need not agree — apparently, they don't.

V. DISCUSSION

What we have accomplished here is the following. The strong coupling limit of the renormalized quartically self-coupled scalar field has been obtained by using Feynman propagator expansions. The resulting expansion has yielded an extremely simple result — the respective strongly coupled renormalized theory is a field theory consisting of a free particle 1PI two-point vertex of mass squared m_R^2 and nothing else! Here, m_R^2 is the renormalized position of the pole in the respective connected propagator. We wish now to discuss the physical implications of this simple result.

We consider the possible implications of (44) for the Higgs sector of the $SU_2 \times U_1$ model. As we showed in Ref. 18, when the Higgs theory is strongly interacting, one has the effective theory (1) for the Higgs sector with

$$m^2 = m_H^2 \qquad \text{and} \qquad h/4 = g \quad , \tag{46}$$

using the standard notation[14,18] for the Higgs mass m_H and the quartic Higgs self-coupling h. In Ref. 18, we considered the possibility that the physical Higgs particle was a composite of heavy color fields.[12] Our basic result for the respective scenario was that such a Higgs field was unstable, with a calculated width in the strong coupling limit of

$$\Gamma_C = \sqrt{2}\left[\left(m^4 + \frac{4m^2}{\sqrt{\pi}}\, g^{\frac{1}{2}}\, \Lambda_{HC}^2 + \frac{8g}{\pi}\, \Lambda_{HC}^4\right)^{\frac{1}{2}} - m^2 - \frac{2g^{\frac{1}{2}}}{\sqrt{\pi}}\, \Lambda_{HC}^2\right]^{\frac{1}{2}} \quad , \tag{47}$$

where Λ_{HC} is the respective heavy color scale parameter. The phenomenological consequences of (47) were discussed in Ref. 18. (For example, a 383 GeV Higgs particle has a width of .663 TeV.) Here, we may entertain the alternative possibility — namely, that the Higgs field is in fact elementary. Indeed, the result (44) shows that this renormalized elementary field is stable against its strong self-interactions, the strong self-interactions of this elementary field simply produce a free particle. The immediate implication is that all of the decay characteristics of the elementary Higgs field are determined by its "weak" couplings to fermions and vector bosons in the limit that h is large. (The respective calculations have been discussed by others.[19]) In this way the elementary Higgs field with h large, i.e., with

$$h^{\frac{1}{2}} > \tfrac{1}{2} \tag{48}$$

according to Ref. 18, will behave substantially differently from the heavy color composite Higgs field of comparable mass. This difference may ultimately be used to distinguish between the two types of Higgs particle. We will elaborate upon this possibility elsewhere.[13]

In closing, we should like to emphasize that nature may very well have utilized this elementary, renormalized, quartically self-coupled scalar field. For this reason, we feel that the Feynman propagator approach to its theory at large coupling may be of more than academic interest. Clearly, the techniques used in our analysis of this large coupling limit are of general applicability. As such, it is of primary importance to apply them to the other strongly coupled renormalizable systems. To repeat, such applications will be taken up elsewhere.

ACKNOWLEDGMENTS

The author is greatly indebted to Professor S. D. Drell for the continued hospitality of the SLAC Theory Group. This work was supported by the Department of Energy under contract number DE-AC03-76SF00515.

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FIGURE CAPTIONS

1. Diagrammatic representation of the theory in (16).

2. The μ^2 - mass insertion to the ρ-propagator in (16).

3. Sum of the ρ-κ interaction contributions to G_2 in the theory in (16).

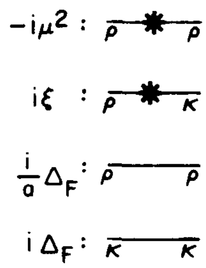

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Fig. 1

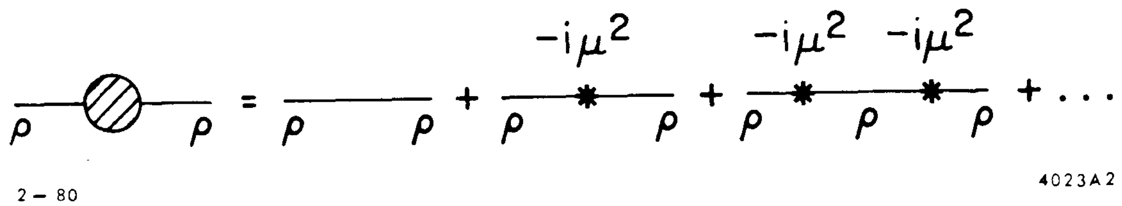

Fig. 2

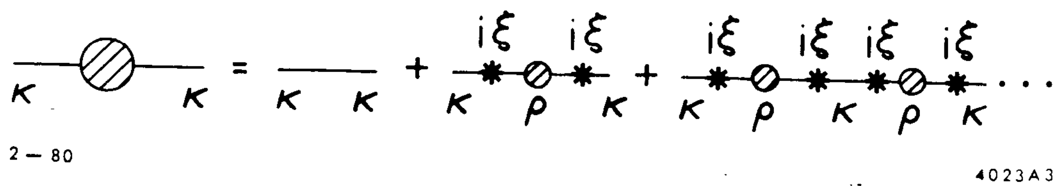

Fig. 3