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THE APPARENT DOUBLEVALUEDNESS OF THE ELECTRIC CHARGE IN QED*

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ABSTRACT

Two straightforward and well-known evaluations of Q, the spatial integral of the electric charge density, are shown to yield differing results. It is argued that the ambiguity arises because the definition of Q, if it is to yield an operator, must be supplemented with domain considerations. The correct domain considerations for physical applications are considered.

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1. Introduction

Quantum electrodynamics (QED) contains a familiar yet unresolved paradox. As usual, let J^μ and Z_3 denote respectively the electric current and the photon field renormalization, and define $Q(x^o) \equiv \int d\vec{x}\, J^o(x)$. Let L, the lepton number, denote the number of electrons minus the number of positrons. It is well-established [1] that $L = Z_3^{-1}Q$ despite arguments based on the canonical commutation relations (CCR's) which seem to imply $L = Q$. The reasons for the failure of the CCR arguments are generally held to be understood. We now argue that in fact they are not.

To distinguish between the various difficulties involved, we begin by considering QED with the interaction spatially cut off. In this case, the CCR arguments are correct; L <u>does</u> equal Q. One might then ask how the factor Z_3^{-1} arises when the spatial cutoff is removed. The answer to this question is given (with an easily corrected error) by J. Bernstein [2] among others. However, a difficulty remains: some of the CCR arguments appear to be every bit as valid in the absence of a spatial cutoff (where they yield the wrong answer) as in the presence of a spatial cutoff (where they yield the correct answer). Bernstein deals with this also. Specifically, he considers a particular CCR argument which crucially assumes the equation $d/dt\, Q(t) = 0$. He then correctly maintains that in fact $d/dt\, Q(t) \neq 0$ so that the argument fails. This might seem to settle the matter, but two difficulties remain. First, $d/dt\, Q = 0$ should hold (and does!) by classical correspondence if for no other reason. How then can Bernstein be correct in maintaining that $d/dt\, Q \neq 0$? Second, the CCR argument considered by Bernstein can be trivially modified to avoid any assumption regarding d/dt Q. The result is a straightforward and compelling derivation of the incorrect result L = Q.

These two difficulties comprise the unresolved paradox to be dealt with in this paper. The resolution is rather technical. We shall see that the integral $Q = \int d\vec{x}\, J^o$ exists only in a very weak sense and is somewhat ambiguous. Indeed, in the presence of an ultraviolet (UV) cutoff, any of the equations d/dt Q = 0, d/dt Q $\neq$ 0, $L = Z_3^{-1}Q$ or L = Q may be viewed as true by appropriately resolving the ambiguity. Nevertheless, the equations d/dt Q $\neq$ 0 and L = Q will be shown to be in some sense unnatural, with the latter being impossible without the UV cutoff. This will resolve the paradox while leaving us with the desired results d/dt Q = 0 and $L = Z_3^{-1}Q$.

Henceforward we shall focus our attention on the difficulty associated with the apparent doublevaluedness of Q, leaving the difficulty with d/dt Q to be dealt with in passing. The remainder of the paper proceeds as follows. In section 2 we supply the details of the paradox we wish to resolve, presenting arguments which yield both $Q = Z_3 L$ and Q = L. In section 3 we obtain useful insights by examining the meaning of Q in free field theory. In section 4 these insights are applied to QED and the paradox is resolved. Conclusions are presented in section 5.

2. The paradox

Before presenting the paradox we establish notation and deal with a technicality.

Our notation is as follows. We consider QED in the Coulomb gauge [3] using dimensional regularization (d $\neq$ 4) as a UV cutoff. All renormalized quantities carry the subscrept ren. $\psi_{ren} = Z_2^{-\frac{1}{2}}\psi$. $A^\mu_{ren} = Z_3^{-\frac{1}{2}}A^\mu$.

$e_{ren} = Z_3^{\frac{1}{2}}e$. $J^\mu = \bar{\psi}\gamma^\mu\psi$. $\partial_\mu F^{\mu\nu} = eJ^\nu$. The lepton number

$$L \equiv \int \overline{dk}\left[b^\dagger_{out}(k)\; b_{out}(k) - d^\dagger_{out}(k)\; d_{out}(k)\right] \quad ,$$

where b_{out} and d_{out} respectively create final state electrons and positrons. $\overline{dk} = d\vec{k}(2\pi)^{-d+1}\;(2k^o)^{-1}$ where $k^o = (\vec{k}^2 + m^2)^{\frac{1}{2}}$. Helicity indices and sums are suppressed.

At this point we have no motivation for being particularly technical. Nevertheless it will save some time if we now carefully consider the meaning of Q. Q(t) was defined as $\int d\vec{x}\; J^o(t,\vec{x}) = \int d^dx\; \delta(x^o-t)\; J^o(x)$. However, like all local quantum fields, J^o is expected to be an operator-valued distribution on space-time; that is, $\int d^dx\; f(x)\; J^o(x)$ is a well-defined operator provided that f(x) is smooth and rapidly decreasing. But $\delta(x^o-t)$ is neither smooth in x^o nor rapidly decreasing in $\vec{x}$ so that, a priori, Q(t) is ill-defined.

This difficulty can be remedied in a number of ways. The following is the most suitable for our purposes. First, we note that the UV cutoff $d \neq 4$ eliminates the divergences associated with the lack of smoothness in x^o, so that J^o need be viewed as a distribution only in the spatial variables. Thus $\int d\vec{x}\; f(\vec{x})\; J^o(t,\vec{x})$ is an operator if $f(\vec{x})$ is smooth and rapidly decreasing. We now define $\lambda_n(\vec{x})$ to be the Fourier transform of $\delta_n(\vec{k}) \equiv (n/\pi)^{(d-1)/2}\; e^{-n\vec{k}^2}$. $\lambda_n(\vec{x})$ is smooth and rapidly decreasing and, as $n \to \infty$, $\delta_n(\vec{k})$ approaches the Dirac δ-function and $\lambda_n(\vec{x}) \to 1$. It therefore makes sense and is natural to give meaning to Q by defining

$$Q(x^o) \equiv \lim_{n\to\infty} Q_n(x^o) \equiv \lim_{n\to\infty} \int d\vec{x}\; \lambda_n(\vec{x})\; J^o(x) \quad .$$

We now proceed with the derivations of $Q = Z_3L$ and $Q = L$. $Q = Z_3L$ is motivated by experiment and classical correspondence: the total electric charge of a system is clearly $e_{ren}L$. The electric charge density is $\vec{\nabla}\cdot\vec{E}_{ren}$ by Gauss' law. But

$$\vec{\nabla}\cdot\vec{E}_{ren} = Z_3^{-1/2}\,\vec{\nabla}\cdot\vec{E} = Z_3^{-1/2}\, e\, J^o = Z_3^{-1}\, e_{ren}\, J^o \ .$$

Thus, we expect $e_{ren}L = \int d\vec{x}\, Z_3^{-1}\, e_{ren}\, J^o$ or $Q = Z_3 L$.

To prove these results we resort to perturbation theory. Let $|\alpha\ \text{out}\rangle$, $|\beta\ \text{in}\rangle$ denote arbitrary multiparticle momentum eigenstates in the out and in scattering representations (e.g., if α refers to m electrons with momenta $k_1, \ldots, k_m$ and n positrons with momenta $\ell_1, \ldots, \ell_n$, then

$$|\alpha\ \text{out}\rangle = b^\dagger_{out}(k_1)\ldots b^\dagger_{out}(k_m)\ d^\dagger_{out}(\ell_1)\ldots d^\dagger_{out}(\ell_n)\ |0\rangle\ .)$$

Let $\phi_a(\alpha)$ and $\phi_b(\beta)$ be arbitrary smooth and rapidly decreasing functions of the momentum variable α and β and set $|a\ \text{out}\rangle = \int d\alpha\, \phi_a(\alpha)\ |\alpha\ \text{out}\rangle$ and $|b\ \text{in}\rangle = \int d\beta\, \phi_b(\beta)\ |\beta\ \text{in}\rangle$. In other words, let $|a\ \text{out}\rangle$ and $|b\ \text{in}\rangle$ be arbitrary states with smooth wave-functions respectively in the out and in representations [4]. Then a perturbative proof is given in the appendix that

$$\lim_{n\to\infty} \left\langle a\ \text{out}\right|\left[Z_3^{-1}\, Q_n - L\right]\left|b\ \text{in}\right\rangle = 0 \ , \qquad (1)$$

which would seem to imply $Q = Z_3 L$.

Next we present the CCR augument [5] which yields $Q = L$. In what follows, all fields are to be evaluated at time $t = 0$. The equal time CCR's imply

$$[J^o(\vec{x}),\bar{\psi}(\vec{y})] = \bar{\psi}(\vec{y})\ \delta(\vec{x}-\vec{y})$$

$$[J^o(\vec{x}),\psi(\vec{y})] = -\psi(\vec{y})\ \delta(\vec{x}-\vec{y})$$

$$[J^o(\vec{x}),A^\mu(\vec{y})] = [J^o(\vec{x}),\dot{A}^\mu(\vec{y})] = 0\ .$$

Multiplying by $\lambda_n(\vec{x})$, integrating over $\vec{x}$, and letting $n \to \infty$ yields

$$[Q,\bar{\psi}] = \bar{\psi}, \quad [Q,\psi] = -\psi,$$
$$[Q,A^\mu] = [Q,\dot{A}^\mu] = 0 \ .$$

Perturbation theory may be used to show that these equations remain true if Q is replaced with L. Thus Q-L commutes with all the fundamental fields and hence with any product of fundamental fields. We shall use the letter $\mathcal{O}$ to denote an arbitrary product of fundamental fields evaluated at time t = 0. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two such products. Then, as just shown, $[Q(0)-L, \mathcal{O}_2] = 0$. Sandwiching this result between $\langle 0|\ \mathcal{O}_1$ and $|0\rangle$ and using $L|0\rangle = 0$ yields

$$\langle 0|\ \mathcal{O}_1[Q(0) - L]\ \mathcal{O}_2|0\rangle = \langle 0|\ \mathcal{O}_1\ \mathcal{O}_2\ Q(0)|0\rangle \ .$$

The right hand side vanishes, but the proof of this is nontrivial and is not presented here. (Why the proof is nontrivial will be clear from the counterexample in section 3.) Thus, recalling the definition of Q, we have

$$\lim_{n\to\infty} \langle 0|\ \mathcal{O}_1[Q_n(0) - L]\ \mathcal{O}_2|0\rangle = 0 \ , \qquad (2)$$

which would seem to imply Q(0) = L.

Our paradox may now be expressed as follows: equation (1) implies that $\lim_{n\to\infty} Q_n = Z_3 L$ (in some sense), while eq. (2) implies that $\lim_{n\to\infty} Q_n(0) = L$ (in some sense). How can this be? The answer lies in the meaning of "in some sense". Indeed, the convergence of Q_n is rather peculiar even in free field theory to which, for the purposes of illustration, we now turn.

3. $\lim_{n\to\infty} Q_n$ in free field theory

In free field theory the above paradox disappears since $Z_3 = 1$. Nevertheless, most of the ideas needed to resolve the paradox in QED can rigorously be displayed in the simpler context of free field theory. We therefore keep our earlier notations (with obvious modifications) but turn off the interaction. The in and out representations become the same, which we call the standard representation. To be completely rigorous, we let the dimension of space-time equal four. Consequently, we must smear $Q_n(t)$ in time; that is, we define

$$Q(f) \equiv \lim_{n\to\infty} Q_n(f) \equiv \lim_{n\to\infty} \int d^4x\ f(x^0)\ \lambda_n(\vec{x})\ J^0(x) ,$$

where f is smooth and rapidly decreasing. Furthermore, we shall require that $\int_{-\infty}^{\infty} dt\ f(t) = 1$.

As every graduate student knows, the limit Q(f) exists and equals L. In particular, we have the free field analog of eq. (1),

$$\lim_{n\to\infty} \langle a|[Q_n(f) - L]|b\rangle = 0 , \qquad (3)$$

where $|a\rangle$ and $|b\rangle$ are states with smooth wave-functions in the standard representation. Such states do not comprise the entire Hilbert space, but they do comprise a dense vector subspace; i.e., a domain. Thus, eq. (3) states that the sequence $Q_n(f)$ converges weakly to L on a dense domain; i.e., that $\lim_{n\to\infty} Q_n(f) = L$ holds when sandwiched between states from a dense vector subspace. Can we strengthen this; that is, can we show that $Q_n(f)$ converges to L weakly on the entire Hilbert space (or, rather, on the intersection of the domains on which the various $Q_n(f)$'s are self-adjoint)? The answer is no. Here is a counterexample which will be important to us in section 4.

Set $|k\ell\rangle = b^{\dagger}(k)d^{\dagger}(\ell)|0\rangle$. Let $g(\vec{k},\vec{\ell})$ be smooth and rapidly decreasing and define $|\chi\rangle$ to be the two-particle state whose wave-function is given by

$$\langle\chi|k\ell\rangle = g(\vec{k},\vec{\ell}) \frac{\bar{v}(\vec{\ell})\gamma^{o}u(\vec{k})}{(\vec{k}+\vec{\ell})^{2}} , \qquad (4)$$

where our Dirac spinors have the normalization $\bar{u}u = -\bar{v}v = 2m$. Note that $|\chi\rangle$ does not have a smooth wave-function but nevertheless that $\langle\chi|\chi\rangle < \infty$. If $\hat{f}$ denotes the Fourier transform of f, then

$$\langle\chi|[Q_n(f) - L]|0\rangle$$

$$= \int \overline{dk}\ \overline{d\ell}\ \langle\chi|k\ell\rangle\ \langle k\ell|Q_n(f)|0\rangle$$

$$= \int \overline{dk}\ \overline{d\ell}\ g(\vec{k},\vec{\ell}) \frac{\bar{v}(\vec{\ell})\,\gamma^{o}\,u(\vec{k})\,\bar{u}(\vec{k})\gamma^{o}\,v(\vec{\ell})}{(\vec{k}+\vec{\ell})^{2}}\ \hat{f}(-k^{o}-\ell^{o})\ \delta_n(\vec{k}+\vec{\ell}) .$$

The suppressed helicity sums yield the factor

$$Tr[\gamma^{o}(\gamma\cdot k+m)\ \gamma^{o}(\gamma\cdot\ell-m)] = 2\vec{\varepsilon}^{2} - 2(\vec{\varepsilon}\cdot\vec{k})^{2}(k^{o})^{-2} + \mathcal{O}\left(|\vec{\varepsilon}|^{3}\right) ,$$

where $\vec{\ell} = -\vec{k}+\vec{\varepsilon}$. It follows that , as $n \to \infty$, the integral reduces to

$$\int \frac{d\vec{k}}{(2\pi)^{6}(2k^{o})^{2}}\ g(\vec{k},-\vec{k})\ \hat{f}(-2k^{o}) \left[2 - \frac{2}{3}\ \frac{\vec{k}^{2}}{k^{o2}}\right] ,$$

which doesn't vanish if g is appropriately chosen. Therefore

$$\lim_{n\to\infty} \langle\chi|[Q_n(f)-L]|0\rangle \neq 0.$$

This counterexample demonstrates that the convergence of the sequence $Q_n(f)$ is indeed neither strong nor weak, but instead something weaker than either; i.e., weak convergence on a dense domain.

Conceivably, then, there could exist another domain on which $Q_n(f)$ converged to something other than L. While this may or may not be so, we claim that something quite similar is in fact the case in QED with a UV cutoff. This will resolve our paradox: $Q = Z_3 L$ will hold weakly on one domain while $Q(0) = L$ will hold weakly on another domain.

4. $\lim_{n\to\infty} Q_n$ in QED

We now reconsider eqs. (1) and (2). Equation (1) states that $Q = Z_3 L$ holds weakly on the dense domain comprised of states with smooth wave-functions in either the out or in representations. The following two claims, if true, resolve the paradox.

1. Unless $L\,\mathcal{O}|0\rangle = 0$, the state $\mathcal{O}|0\rangle$ does not have a smooth wave-function in either the out or in representation. Furthermore, the singularities in the wave-function of $\mathcal{O}|0\rangle$ resemble the singularity in the wave-function of $|\chi\rangle$ discussed in section 3.

2. Because of the singularities in the wave-function of $\mathcal{O}|0\rangle$, it is wrong to conclude from eq. (1) that $Q = Z_3 L$ holds between $\langle 0|\,\mathcal{O}_1$ and $\mathcal{O}_2|0\rangle$. In fact the singularities conspire to produce the result

$$\langle 0|\,\mathcal{O}_1[Q(0) - L]\,\mathcal{O}_2|0\rangle = 0\ .$$

If the above claims are true, then $Q = Z_3 L$ will hold weakly on the domain of states with smooth wave-functions while $Q(0) = L$ will hold on the domain consisting of finite linear combinations of states of the form $\mathcal{O}|0\rangle$. These statements are not at all incompatible, and the paradox will be resolved. (This will also take care of the d/dt Q difficulty; it will be easy to show that $d/dt\,Q(t) = 0$ holds on the former domain and that $d/dt\,Q(t) \neq 0$ holds on the latter.)

Although the above claims (which we believe and attempt below to verify) resolve the paradox, an important question remains. If Q has different values on different domains, which is the correct domain for real physics? Several considerations yield the same answer.

First, the fundamental fields are unobservable, while scattering experiments are actually performed; so it is experimentally more natural to consider a basis of scattering states rather than states of the form $\mathcal{O}|0\rangle$. And given a scattering representation, it is hard to see any reason to expect singularities in the observed wave-functions. Thus, it is natural to expect states found in nature to have smooth wave-functions in a scattering representation [6].

The same conclusion comes from classical correspondence which yields as previously noted, $L = Z_3^{-1} Q$ and $d/dt\ Q(t) = 0$. These equations hold weakly between states with smooth wave-functions and do not hold between states of the form $\mathcal{O}|0\rangle$.

Also, the domain spanned by the $\mathcal{O}|0\rangle$ is unnatural in that the fundamental fields in $\mathcal{O}$ are evaluated at time $t = 0$, which distinguishes one time from all the others. Had we evaluated the fields at some other time t we would have obtained a different domain on which $L = Q(t)$ would weakly hold, but on which $Q(0)$ would be a mess.

Finally, it is expected that interacting fields in four dimensions cannot be evaluated at sharp times. We were able to consider the state $\mathcal{O}|0\rangle$ only because we were using a UV cutoff. As the cutoff is released, states of the form $\mathcal{O}|0\rangle$ probably diverge.

Thus we expect states which occur in nature to be smoothly expressible in terms of scattering representations [6].

We now attempt to verify our claims. For each claim, we shall proceed by examining a special case. This will involve no loss of generality in the first claim, since the necessary generalizations will be obvious. This will not be so for the second claim; nevertheless, our example should make the second claim seem very plausible.

The special case of our first claim which we consider is $\mathcal{O} = \psi^\dagger(0,\vec{x})$. The wave-functions of $\mathcal{O}|0\rangle$ (or, rather, its complex conjugate) in the in-representation is then $\langle 0|\psi(0,\vec{x})|\beta \text{ in}\rangle$. This wave-function will of course have singularities associated with soft photon emission, but these have no significance in the present context. As we expand in powers of e, the first singularity not associated with soft photon emission to be encountered is in the Feynman diagram shown in fig. 5, which equals

$$-e^2\bar{v}(\ell)\ \gamma^\mu\ u(k)\ D_{\mu\nu}(k+\ell)\ S(p+k+\ell)\ \gamma^\nu\ u(p)\ e^{i(\vec{p}+\vec{k}+\vec{\ell})\cdot\vec{x}}\ .$$

Since we are working in the Coulomb gauge, the photon propagator $D_{\mu\nu}(k+\ell)$ has several terms proportional to $1/(\vec{k}+\vec{\ell})^2$. Most of these are eliminated by current conservation; but one survives, and for $\vec{k}+\vec{\ell}$ near zero the diagram equals

$$-ie^2\ \frac{\bar{v}(\ell)\ \gamma^o\ u(k)}{(\vec{k}+\vec{\ell})^2}\ S(p^o+2k^o,\vec{p})\ \gamma^o\ u(p)\ e^{i\vec{p}\cdot\vec{x}} \tag{5}$$

plus nonsingular terms. Note that the singular factor $\bar{v}(\ell)\ \gamma^o\ u(k)/(\vec{k}+\vec{\ell})^2$ occurs in both eqs. (4) and (5), which is the resemblance referred to in the first claim.

The special case of our second claim which we consider is $\mathcal{O}_1 = \psi(0,\vec{x})$ and $\mathcal{O}_2 = \bar{\psi}(0,\vec{y})$, where we expand to leading nontrivial order in e. We wish to verify eq. (2) which may be written

$$\lim_{n\to\infty} \langle 0|\psi(0,\vec{x})Q_n(0)\bar{\psi}(0,\vec{y})|0\rangle = \langle 0|\psi(0,\vec{x})\bar{\psi}(0,\vec{y})|0\rangle \ .$$

In order to consider this equation in perturbation theory, we insert a complete set of scattering states between each pair of adjacent operators, yielding

$$\lim_{n\to\infty} \int d\alpha d\beta \ \langle 0|\psi(0,\vec{x})|\beta \text{ in}\rangle \ \langle \beta \text{ in}|Q_n(0)|\alpha \text{ out}\rangle \ \langle \alpha \text{ out}|\bar{\psi}(0,\vec{y})|0\rangle$$

$$= \int d\alpha \ \langle 0|\psi(0,\vec{x})|\alpha \text{ out}\rangle \ \langle \alpha \text{ out}|\bar{\psi}(0,\vec{y})|0\rangle \ .$$

This equation, expressed in an obvious way in terms of cut diagrams, is shown in fig. 6.

We now expand in powers of e to the leading nontrivial order (which will be $\mathcal{O}(e^2)$). To $\mathcal{O}(e^0)$, fig. 6 reduces to fig. 7. It is simple to check that both sides equal $\Delta \equiv \int \overline{dp}\ (\gamma\cdot p+m)\ e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$.

To $\mathcal{O}(e^2)$, several diagrams contribute to fig. 6. Since we were interested in Z_3 considerations, we shall consider only those diagrams with a closed fermion loop. It is left to the reader to show that the remaining diagrams cancel among themselves. Furthermore, since the fermion loop is proportional to $k^2 g_{0\nu} - k_0 k_\nu$ and since any physical photon polarization ε satisfies $(k^2 g_{0\nu} - k_o k_\nu)\varepsilon^\nu = 0$, any graph vanishes if a photon line connected to the fermion loop is cut. Thus, fig. 6 to $\mathcal{O}(e^2)$ reduces to fig. 8.

The first graph of fig. 8 is easily computed to be $\Pi(0)\cdot\Delta$. $\Pi(0)$ is defined in fig. 9 and equals

$$-\frac{4}{3}\ \frac{e^2}{(4\pi)^{d/2}}\ m^{d-4}\ \Gamma\left(2-\frac{d}{2}\right) \ .$$

The second and third graphs of fig. 8 naively seem to vanish, since the middle third of each vanishes as $n \to \infty$. However, the top third of the second graph and the bottom third of the third graph contain singularities which invalidate the naive result, and in fact the second and third graphs exactly cancel the first.

To see this, we consider the second graph in more detail, as shown in fig. 10. The singular part of the top third (which is all that survives the $n \to \infty$ limit) is given by eq. (5). The middle and bottom thirds respectively equal

$$(2\pi)^{d-1}\ 2p^o\ \delta(\vec{p}-\vec{q})\ \bar{u}(k)\ \gamma^o\ v(\ell)\ (2\pi)^{d-1}\ \delta_n(\vec{k}+\vec{\ell})$$

and $\bar{u}(q)\ e^{-i\vec{q}\cdot\vec{y}}$. If we combine the three factors, do the trivial $\overline{dq}$ integral, note that the helicity sum $\sum u(p)\ \bar{u}(p) = \gamma\cdot p+m$, and note that

$$S(p^o+2k^o,\vec{p})\ \gamma^o(\gamma\cdot p+m) = \frac{i}{2k^o}\ (\gamma\cdot p+m)$$

we obtain

$$\Delta e^2 \int \overline{dk}\ \overline{d\ell}\ \frac{\bar{v}(\ell)\ \gamma^o\ u(k)\ \bar{u}(k)\ \gamma^o\ v(\ell)}{(\vec{k}+\vec{\ell})^2}\ \frac{(2\pi)^{d-1}}{2k^o}\ \delta_n(\vec{k}+\vec{\ell})\ .$$

The suppressed helicity sums and the $n \to \infty$ limit proceed exactly as in the counterexample of section 3, yielding

$$\Delta e^2 \int \frac{d\vec{k}}{(2\pi)^{d-1}\ (2k^o)^3}\left[2 - \frac{2\vec{k}^2}{(d-1)\ (k^o)^2}\right]\ .$$

The $d\vec{k}$ integration involves standard d-1 dimensional integrals, and the final result is $-\frac{1}{2}\Delta\Pi(0)$.

The third graph of fig. 8 equals the second. Thus, the three contributions to fig. 8 sum to zero as they must, but only because the wave-functions of $\langle 0|\psi(0,\vec{x})$ and $\bar{\psi}(0,\vec{y})|0\rangle$ possess the appropriate singularities. This establishes the second claim for the special case considered.

5. Conclusions

Our results may be sketchily summarized as follows. Both eqs. (1) and (2) are correct (although eq. (2) is grossly misleading) and the apparent contradiction between the two is resolved by the fact that a sequence of operators can converge to different results on different domains. (Although these conclusions may seem reasonable at this point, they were obtained by means of rather technical considerations. Since technical considerations are at best time consuming, it is perhaps worth noting that the paradox arises only when we consider the operator $\psi(x)$, which is nonlocal, noncovariant, and unobservable. Such considerations lead to the misleading eq. (2). On the other hand, a consideration of true observables such as $F^{\mu\nu}_{ren}$, J^{ν}_{ren}, and scattering states leads to eq. (1), which is not misleading.)

Appendix

We now derive eq. (1). We proceed by evaluating $\lim_{n\to\infty} \langle a \text{ out}|Q_n(t)|b \text{ in}\rangle$ and showing that it equals $\langle a \text{ out}|Z_3 L|b \text{ in}\rangle$.

By the usual LSZ reduction procedure, it can be shown that

$$\langle \alpha \text{ out}|J^\mu(x)|\beta \text{ in}\rangle = e^{iq\cdot x}\, T^\mu_{\alpha\beta} ,$$

where q is the difference between the incoming and outgoing momenta and $T^\mu_{\alpha\beta}$ is given with the usual Feynman rules in fig. 1. What we wish to compute may be expressed as

$$\lim_{n\to\infty} \int d\alpha\, d\beta\, \phi_a^*(\alpha)\, \phi_b(\beta)(2\pi)^{d-1}\, e^{iq^o t}\, T^o_{\alpha\beta}\, \delta_n(\vec{q}) . \qquad (6)$$

We assume that the limit exists and depends continuously on ϕ_a and ϕ_b; that is, that $\lim_{n\to\infty} T^o_{\alpha\beta}\, \delta_n(\vec{q})$ exists in the sense of distributions in the variables α and β. By current conservation $q_\mu T^\mu_{\alpha\beta} = 0$. Thus, $0 = \lim_{n\to\infty} q_\mu\, T^\mu_{\alpha\beta}\delta_n(\vec{q}) = q_0 \lim_{n\to\infty} T^o_{\alpha\beta}\, \delta_n(\vec{q})$. It follows that $\lim_{n\to\infty} T^o_{\alpha\beta}\delta_n(\vec{q})$ contains a factor of $\delta(q^o)$. If α and β are single-particle states, the factor of $\delta(q^o)$ is contained in $\delta(\vec{q})$. This case is left to the reader.

Otherwise a factor of $\delta(q^o)$ resides in $T^o_{\alpha\beta}\big|_{\vec{q}=0}$. It therefore suffices in computing $T^o_{\alpha\beta}\big|_{\vec{q}=0}$ to discard diagrams which are nonsingular as $q \to 0$ and to extract the $\delta(q^o)$ pieces from the singular diagrams. The singular diagrams may be found by means of the Landau rules [7] and are typically of the form shown in fig. 2a. (The usual infrared singularities, which have nothing to do with the J^o insertion, do not yield $\delta(q^o)$ contributions.) Near $q = 0$, these diagrams may collectively be replaced by diagrams of the form shown in fig. 2b, where we have used $Z_1^{-1}Z_2 = 1$.

As a particular case, consider the two contributions shown in fig. 3. Evaluating the propagator near the J^0 insertion and using $\vec{q} = 0$ yields that fig. 3 = fig. 4. (The momentum q shown in fig. 4 should flow into the vertex to which the indicated external line is attached.) The sum of the two terms proportional to the principle value of $1/q^0$ is nonsingular at $q^0 = 0$ and may be discarded. The $\delta(q^0)$ terms contribute the term $Z_3\langle a \text{ out}|b \text{ in}\rangle$ to (6). Had we considered a positron line rather than an electron line, the contribution to (6) would have been $-Z_3\langle a \text{ out}|b \text{ in}\rangle$. Lines with an electron at one end and a positron at the other contribute zero. Thus, the sum over all contributions to (6) is simply $Z_3\langle a \text{ out}|L|b \text{ in}\rangle$, the desired result.

Footnotes

1. D. Lurie, Particles and Fields (1968), eq. 8(37) for example.
2. J. Bernstein, Elementary Particles and their Currents (1968), pp. 42-45. The error is that the observed electric charge is obtained from Gauss' law with an unrenormalized electric field, so that the right hand side of eq. (4.13) lacks a factor of $Z_3^{-\frac{1}{2}}$. Consequently, "$\sqrt{Z}$ " should be replaced with "Z_3" everywhere on p. 45.
3. Any physical gauge could be used, but we do not here consider operator realizations of gauge fields in unphysical gauges.
4. We regulate the infrared divergences associated with QED scattering theory by requiring $d > 4$. (It is unfortunate that the states $|a\ \text{out}\rangle$ and $|b\ \text{in}\rangle$ as defined here diverge as $d \to 4$. Presumably when a rigorous QED scattering theory is developed, it will be possible to generalize our results to include the case $d = 4$.)
5. Similar arguments may be found in the references cited in footnotes 1 and 2, as well as in S. Fubini and G. Furlan, Phys. 1, 229 (1965).
6. Here "scattering representation" refers to the as yet undiscovered scattering representation associated with a rigorous QED scattering theory.
7. L. D. Landau, Nucl. Phys. 13, 181 (1959).

   J. D. Bjorken, doctoral dissertation, Stanford University (1959).

Figure Captions

1. $\overleftarrow{T}^{\mu}_{\alpha\beta}$

2. (a) Typical singular contribution to $T^{o}_{\alpha\beta}\big|_{\vec{q}} = 0$.

   (b) Same as (a) evaluated near $q^{o} = 0$.

3. Two contributions to $T^{o}_{\alpha\beta}\big|_{\vec{q}} = 0$.

4. Fig. 3 re-expressed.

5. A contribution to $\langle 0|\psi(0,\vec{x})|k\ell p\ \mathrm{in}\rangle$

6. Graphical expression of a special case of eq. 2.

7. Fig. 6 to $O(e^{o})$.

8. Fig. 6 to $O(e^{2})$.

9. Definition of $\Pi(0)$.

10. Second term of fig. 8.

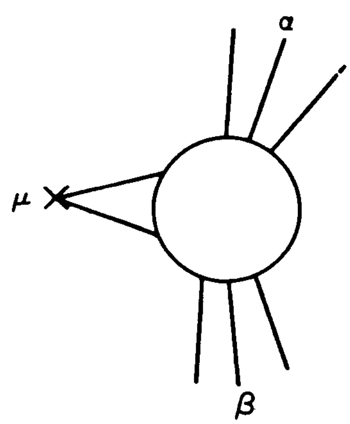

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Fig. 1

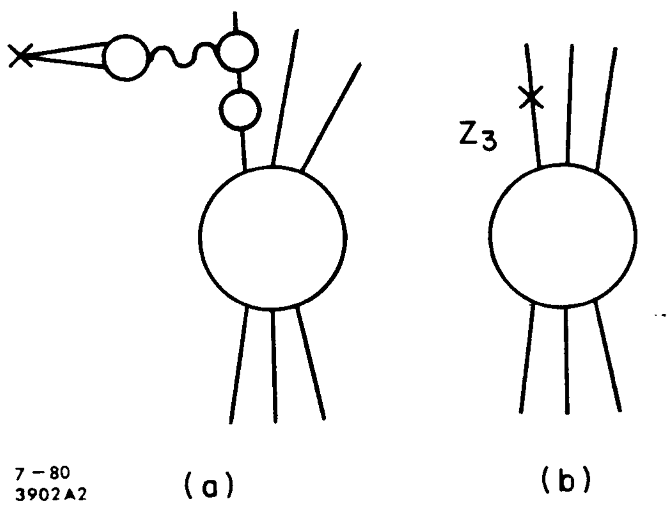

Fig. 2

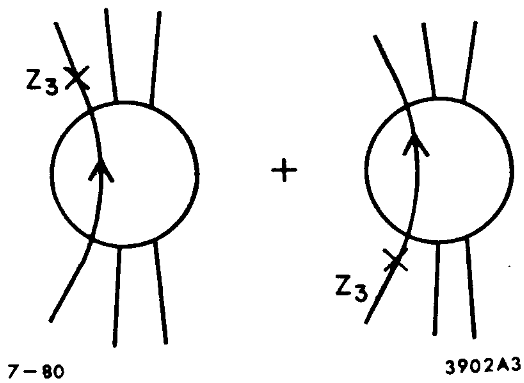

Fig. 3

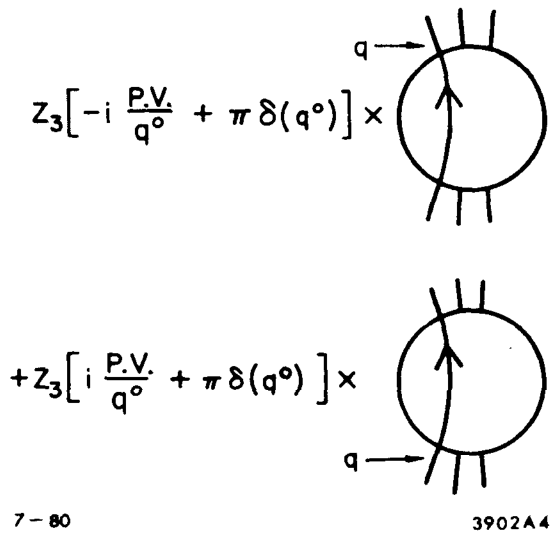
q
$Z_3\left[-i\,\frac{P.V.}{q^0}+\pi\delta(q^0)\right]\times$
$+Z_3\left[i\,\frac{P.V.}{q^0}+\pi\delta(q^0)\right]\times$
q
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Fig. 4

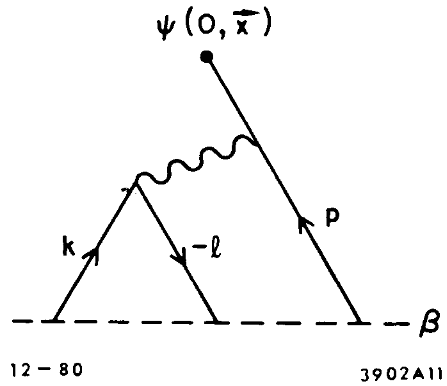

Fig. 5

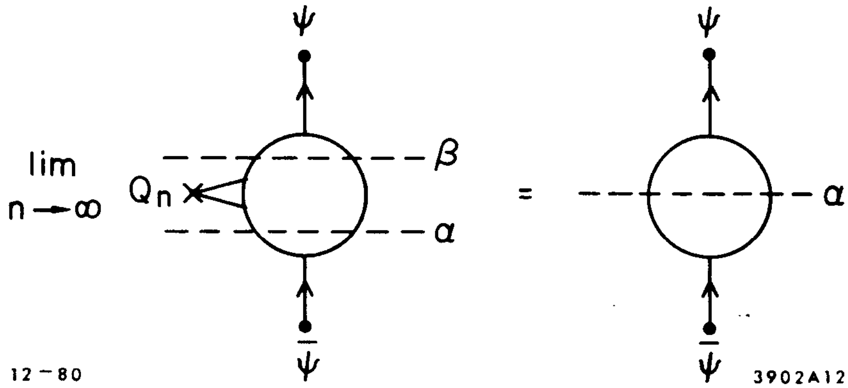

Fig. 6

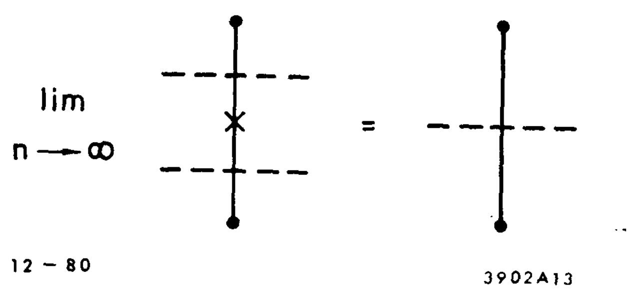

Fig. 7

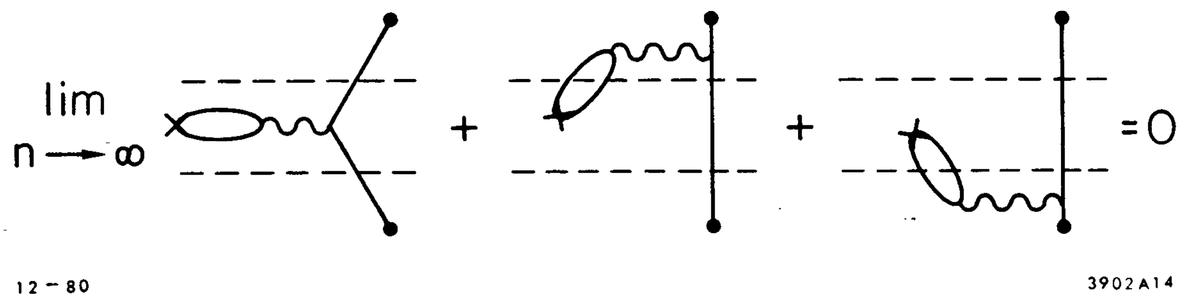

Fig. 8

$$i\,\Pi\,(k^2)\left[g_{\mu\nu}\,k^2 - k_\mu\,k_\nu\right] =$$

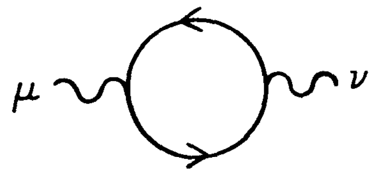

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Fig. 9

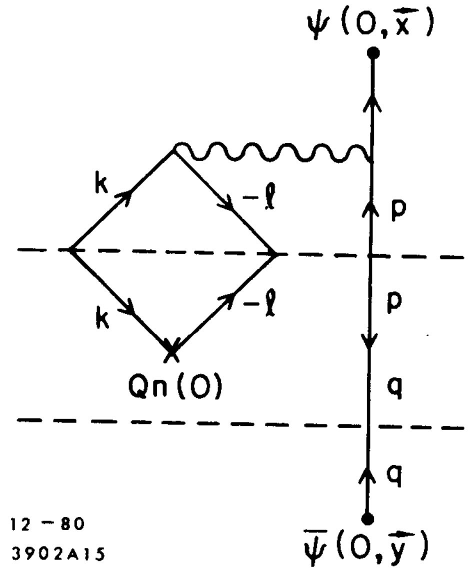

Fig. 10